Problem Books in Mathematics

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The Stair-Step Approach in Mathematics

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Even a small simple step brings us closer to our success. We never know how far can we go in Mathematics, until we keep moving forward. If it is too hard to solve all the problems at once, just take one step at a time.

Foreword

This book is intended as a teacher's manual of mathematics and a self-study handbook for high school or college students and mathematical competitors. It only consists of new problems created by authors with author-prepared solutions. It is based on a traditional teaching philosophy and a non-traditional writing approach that authors call **stair-step approach**. The main idea of this approach is to start from relatively easy problems and step by step increase the level of difficulty of the next problems in order to most effectively maximize students' learning potential. This aids to develop self-mastery with a slow but continuous rise and to develop creativity for solving mathematical problems. This book is arranged by topics and difficulty level. Moreover, in order to support and promote independent learning and for the reader's convenience, the authors, besides providing the solutions, simultaneously provide a separate table of answers at the end of the book. This book gives a broad view of mathematics and goes beyond the typical elementary mathematics by providing deeper treatment of the following topics: Geometry and trigonometry, Number theory, Algebra, Calculus and Combinatorics.

Keywords: Geometry and trigonometry, Number theory, Algebra, Calculus, Combinatorics, problem book of mathematics, self-study of mathematics, mathematical Olympiad, mathematical competition problems and solutions.

Preface

This book was generally aimed to be a guidebook for mathematics teachers, high school students for the self-study of mathematics, and for the self-mastery of mathematical competitors. It is a useful handbook to develop mathematical skills and cover the main topics of pure mathematics when all you have is basic mathematical skills from school. We call our approach the **stair-step approach**, as the main idea is to start from relatively easy problems and step by step increase the level of difficulty of the next problems.

This approach can be compared to **one kilo rule** in weightlifting; of course, 10 kg Personal Record is more desirable than 1 kg Personal Record and on the other hand, 1 kg Personal Record is more desirable than continually failing 10 kg Personal Record. The key point is to make smaller but more frequent improvements rather than hoping to make big improvements infrequently, as the continuous progress not only exhibits the effectiveness of the learning process, but also keeps motivated to continue. There is a wide range of the literature with problem sets of increasing difficulty. In this book, we have tried to minimize the leap between the difficulty levels to provide smooth and uniform passages from one problem to the next one. In other words, we have tried to follow the above-mentioned one kilo rule.

This book covers five main topics of (pure) mathematics and consists of seven chapters; the first five chapters are devoted to the following topics:

1. **Geometry and trigonometry,**
2. **Number theory,**
3. **Algebra,**
4. **Calculus,**
5. **Combinatorics.**

As it is unusual to include a separate chapter dedicated to Calculus in Chapter 6 are provided some hints for the hardest problems of the chapter Calculus. In Chapter 7 are presented author-prepared solutions. Moreover, in order to promote independent learning and for the reader's convenience, besides providing the solutions in Chapter 7, we provide a separate table of answers in Chapter 8. Every topic consists of 15 sets of problems. The first 11 sets of problems consist of nine

problems of increasing difficulty, and the last four sets of problems consist of twelve problems of increasing difficulty. We provide these additional three problems in the last four sets in order to allow the reader to focus more on advanced level problems instead of intermediate level problems.

This book is arranged by topics and difficulty level. It is intended to improve the mathematical knowledge through solving these 15 sets of problems of each topic, that is 147 problems of each topic. Altogether, these 15 sets include 735 problems; all problems are new problems created by the authors and with author-prepared solutions. Some problems in this book are inspired by the previous works of the authors [1], [2], [3]. We hope that this book will help the interested readers to improve their mathematical skills to a truly advanced level.

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Hayk Sedrakyan
Nairi Sedrakyan

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To Margarita,
a wonderful wife and a loving mother.

To Ani,
a wonderful daughter and a loving sister.

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About the Authors

Hayk Sedrakyan is an IMO medal winner, Professor of mathematics in Paris, France, and a professional Mathematical Olympiads Coach in Greater Boston area, Massachusetts, USA. He has defended his PhD thesis in mathematics in UPMC-Sorbonne University, Paris, France. Hayk is a Doctor of Mathematical Sciences in USA, France and Armenia. He has received three master's degrees in mathematics from Germany, Armenia, Austria and has spent a small part of his PhD studies in Italy. Hayk has authored several scientific mathematical research articles and several books on the topic of problem solving and Olympiad style mathematics published in USA and South Korea.

Nairi Sedrakyan is involved in national and international Olympiads of mathematics, having been the President of Armenian Mathematics Olympiads and IMO problem selection committee member. He is the author of one of the hardest problems ever proposed in the history of International Mathematical Olympiad (5th problem of 37th IMO). He has been the Leader of Armenian IMO Team, jury member of IMO, jury member and problem selection committee member of Zhautykov International Mathematical Olympiad (ZIMO), jury member and problem selection committee member of International Olympiad of Metropolises, the President of International Mathematical Olympiad Tournament of the Towns in Armenia. He is also the author of a large number of problems proposed in these Olympiads and has authored several books on the topic of problem solving and Olympiad style mathematics published in the USA, Russia, Armenia and South Korea. For his outstanding teaching, Nairi Sedrakyan received the title of *the best teacher* of the Republic of Armenia and was awarded by a special gift from the Prime Minister.

Chapter 1
Geometry and Trigonometry

Introduction. In the chapter Geometry and trigonometry, the majority of problems can be treated either by classical proof techniques and properties provided in the regular school program or by using some additional proof techniques and theorems that are considered to be outside of the scope of the regular school program. The classical proof techniques, theorems, properties or mathematical laws used in the proofs included in this chapter are the properties of congruence of triangles, properties of similarities of triangles, the Pythagorean theorem, the properties of the right-angled triangle, law of sines and law of cosines, the properties of an area, the properties of inscribed and circumscribed quadrilaterals, trigonometric methods, the method of coordinates and applications of geometric inequalities (for more applications, see [1]). A large number of problems are related with the maxima and minima properties, such problems can be treated either by the different methods of estimations or by transformations and simplifications of the trigonometric expressions. In some cases, when we need to find the sum of the given trigonometric expressions, the main trick is to represent each of the summands as the difference of two expressions and then perform the corresponding simplifications in order to find the required sum. For the readers convenience, we list and provide the formulations of all used theorems and proof techniques that are outside of the scope of the regular school program.

Theorem 1.1. *Let AA_1 be a bisector of angle BAC of triangle ABC. Given a point I on segment AA_1. Prove that, I is the incenter of ABC, if and only if*

$$\angle BIC = 90^\circ + \frac{1}{2}\angle BAC.$$

Theorem 1.2. *Let AA_1, BB_1, CC_1 be the altitudes of triangle ABC, O its circumcenter and A_2 the midpoint of side BC. Let H be the intersection point of lines AA_1 and BB_1. Prove that*

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$$OA_2 = \frac{1}{2}AH.$$

Theorem 1.3 (Ceva's Theorem). *Let ABC be a triangle and D, E, F be points on lines BC, CA, AB, respectively. Lines AD, BE, CF (called cevians) are concurrent (intersect in a single point) if and only if*

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Theorem 1.4. *Let ABC be a triangle and A_1, B_1, C_1 be points on lines BC, CA, AB, respectively. Prove that the perpendiculars to sides BC, CA, AB from points A_1,B_1,C_1 intersect in a single point, if and only if*

$$A_1B^2 - BC_1^2 + C_1A^2 - AB_1^2 + B_1C^2 - CA_1^2 = 0.$$

Theorem 1.5 (Ptolemy's Theorem). *For any cyclic quadrilateral $ABCD$, the sum of the products of the two pairs of opposite sides equals the product of the diagonals*

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

Remark 1.1 (***Ptolemy's Inequality***). For any quadrilateral $ABCD$, it holds true

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD,$$

with equality if and only if $ABCD$ is cyclic.

1.1 Problem Set 1

Problem 1. Let M be a given point in rectangle $ABCD$, such that $\angle AMD = 90^\circ$, $BC = 2MD$, $CD = AM$. Find $16\cos^2\angle CMD$.

Problem 2. Let ABC be a given triangle, such that $AD = 3$, $DE = 5$, $EC = 24$ and $\angle ABE = 90^\circ$, $\angle DBC = 90^\circ$, where D and E are some points on AC. Find $3AB$.

Problem 3. Let $ABCD$ be a convex quadrilateral, such that $AB = BC$ and $\angle ADC = \angle BAD + \angle BCD$. Find $\frac{BD}{BC}$.

Problem 4. Given that

$$\cos\sqrt{(\sin x + \cos x)(1 - \sin x \cos x)} = \sqrt{\cos(\sin x + \cos x) \cdot \cos(1 - \sin x \cos x)}.$$

Find $\sin^5 x + \cos^5 x$.

Problem 5. Let I be the incenter of triangle ABC and $AC = 10$. A circle passes through the points I and B and intersects AB and BC correspondingly at points M and N. Find $AM + CN$.

Problem 6. Find the minimum value of the expression

$$\left(\sin x - \frac{1}{\sin^2 y} - 2\right)^2 + \left(\frac{\sin x}{\sin^2 y} + 1\right)^2.$$

Problem 7. Let $ABCD$ be a quadrilateral with a circle inscribed in it. Denote by O the intersection point of the diagonals of $ABCD$. Assume that $\angle AOB < 90^\circ$. Let AA_1, BB_1, CC_1 and DD_1 be, respectively, the altitudes of triangles AOB and COD. Find the perimeter of quadrilateral $A_1B_1C_1D_1$, if $A_1B_1 + C_1D_1 = 27$.

Problem 8. Evaluate the expression

$$\cos^2 \frac{\pi}{18} + \frac{\cos^2 \frac{\pi}{18}}{\left(4\cos\frac{\pi}{18} + \sqrt{3}\right)^2}.$$

Problem 9. Let $ABCD$ be a convex quadrilateral, such that $\angle ACB = \angle DBC = \angle BAM$, where M is the intersection point of rays CB and DA. Moreover, $CD = 6$ and $MB = BC$. Find AD.

1.2 Problem Set 2

Problem 1. Let two circles be mutually externally tangent at point A. Let a be a tangent line to those circles at points B and C. Given that $AB = 20$ and $AC = 21$. Find BC.

Problem 2. Find the maximum value of the expression $\cos x + \cos y + \sin x \sin y$.

Problem 3. The incircle of triangle ABC with the incenter I is tangent to sides AB and BC at points C_1 and A_1, respectively. The lines AI and A_1C_1 intersect at the point M. Given that $AC = 68$ and $\angle A = 30^\circ$. Find the area of triangle AMC.

Problem 4. Evaluate the expression

$$\left(\frac{\sqrt{3}}{\sin\frac{\pi}{9}} + \frac{1}{\cos\frac{\pi}{9}}\right)\sec\frac{2\pi}{9}.$$

Problem 5. Let I be the incenter of triangle ABC. Given that the perimeter of ABC is equal to 25 and $AC = 10$. A circle passes through the points I and B and intersects the continuation of the side AB and the side BC at points M and N, respectively. Find $BN - BM$.

Problem 6. Let $ABCD$ be a convex quadrilateral. Given that $BC = 3$, $AD = 5$ and $MN = 1$, where the points M and N are midpoints of the diagonals AC and BD. Find the ratio of the area of triangle ABC to the area of triangle BCD.

Problem 7. Evaluate the expression

$$\frac{1}{2^{13}}(1+\tan 1^\circ)(1+\tan 2^\circ)\cdot\dots\cdot(1+\tan 44^\circ).$$

Problem 8. Consider a point M inside of an isosceles triangle ABC ($AB=BC$). Given that $\angle MBA = 10^\circ$, $\angle MBC = 30^\circ$ and $BM = AC$. Find $36\sin^2\angle MCA$.

Problem 9. Consider a quadrilateral $ABCD$, such that $\angle A = \angle C = 60^\circ$ and $\angle B = 100^\circ$. Let the angle between the lines AI_2 and CI_1 be equal to n°, where I_1 and I_2 are the incenters of triangles ABD and CBD, respectively. Find n.

1.3 Problem Set 3

Problem 1. Let H be the intersection point of the altitudes of acute triangle ABC. Given that $AH=1$, $AB=15$, $BC=18$. Find CH.

Problem 2. The incircle of triangle ABC with the incenter I is tangent to sides AB and BC at points C_1 and A_1, respectively. The lines AI and CI intersect the line A_1C_1 at the points M and N, respectively. Given that $AC=12$ and $\angle B = 60^\circ$. Find MN.

Problem 3. Evaluate the expression

$$\cos^2\frac{5\pi}{18} + \frac{\cos^2\frac{5\pi}{18}}{\left(4\cos\frac{5\pi}{18} - \sqrt{3}\right)^2}.$$

Problem 4. Let $ABCDE$ be a pentagon inscribed in a circle. Given that $AB=CD$, $BC=2AB$, $AE=1$, $BE=4$, $CE=14$. Find DE.

Problem 5. Evaluate the expression

$$32\sin\frac{\pi}{22}\sin\frac{3\pi}{22}\sin\frac{5\pi}{22}\sin\frac{7\pi}{22}\sin\frac{9\pi}{22}.$$

Problem 6. Given that the sines of the angles of a quadrilateral are the terms of a geometric progression. Find the number of all the possible values of the common ratio of that geometric progression.

Problem 7. Find the greatest value of the following expression

$$4\sin x + 48\sin x\cos x + 3\cos x + 14\sin^2 x.$$

Problem 8. Consider a quadrilateral $ABCD$, such that $\angle A = \angle C = 60^\circ$ and $\angle B = 100^\circ$. Let O_1 and O_2 be the circumcenters of triangles ABD and CBD, respectively. Given that the angle between the lines AO_2 and CO_1 is equal to n°. Find n.

Problem 9. Consider an obtuse triangle ABC with non-equal sides, circumcenter O and radius R. Given that the segment CO intersects the side AB at the point E and the radiuses of inscribed circles of triangles ACE and BCE are equal. Find $\frac{10R}{AC+BC}$.

1.4 Problem Set 4

Problem 1. Given that $3\sin\alpha + 4\cos\alpha - 7\cos\beta = 12$. Find the value of the expression $3\sin\beta - 4\cos\beta + 25\cos\alpha$.

Problem 2. Given that a circle intersects the sides of an angle at four points. Let M and N be the midpoints of the arcs (of the circle) which are outside of the angle. Denote by n° the angle between the line MN and the bisector of the given angle. Find n.

Problem 3. Let ABC be a triangle. Given that the median (of vertex B) is equal to 27. Consider a point N on that median, such that $BN = 24$ and $\angle ANC = 180^\circ - \angle ABC$. Find AC.

Problem 4. Consider a triangle ABC. Let the sides AB and BC be tangent of a circle ω at points E and F, respectively. Given that ω intersects the side AC and M is such a point on the side AC that $AE : CF = AM : MC$. Assume that the line FM intersects ω at point K. Given also that $AE = 14$. Find AK.

Problem 5. Given that the points $M(-1,5), N(2,6), K(4,4), P(0,1)$ are on the district sides of a square. Find the area of the square.

Problem 6. Given that $\phi = \frac{\pi}{12} - \frac{1}{2}\arccos\frac{5\sqrt{3}+1}{10}$. Find the value of the following expression

$$\frac{1}{\sin\phi \cdot \sin\left(\phi + \frac{\pi}{6}\right)} + \frac{1}{\sin\left(\phi + \frac{\pi}{6}\right) \cdot \sin\left(\phi + \frac{2\pi}{6}\right)} + \cdots + \frac{1}{\sin\left(\phi + \frac{4\pi}{6}\right) \cdot \sin\left(\phi + \frac{5\pi}{6}\right)}.$$

Problem 7. Let in triangle ABC a line, which passes through its incenter and is parallel to the side AC, intersects sides AB and BC at points E, F, respectively. Denote by D the midpoint of the side AC, by M the intersection point of rays ED and BC, by N the intersection point of rays FD and BA. Given that the parameter of ABC is equal to 22 and $AC = 10$. Find MN.

Problem 8. Evaluate the expression

$$4096\sqrt{3}\sin\frac{\pi}{27}\sin\frac{2\pi}{27}\sin\frac{4\pi}{27}\sin\frac{5\pi}{27}\sin\frac{7\pi}{27}\sin\frac{8\pi}{27}\sin\frac{10\pi}{27}\sin\frac{11\pi}{27}\sin\frac{13\pi}{27}.$$

Problem 9. Let all the vertices of a triangle ABC are on parabola $y = x^2$. Given that line AB is parallel to axes Ox and that the abscissa of point C is located in between of the abscissas of points A and B. Let CH be the altitude of ABC and $\tan\angle ACB = 0.01$. Find $\frac{CH-1}{AB}$.

1.5 Problem Set 5

Problem 1. The medians of triangle ABC intersect at point G. Given that $\angle AGB = 90^\circ$ and $AB = 20$. Find the length of the median corresponding to vertex C.

Problem 2. Let $ABCD$ be a trapezoid. A circle is inscribed inside the trapezoid and touches the bases AD and BC at points M and N, respectively. Given that $AM = 9$, $MD = 12$ and $BN = 4$. Find NC.

Problem 3. Find the greatest value of the expression

$$2\cos 3x - 3\sqrt{3}\sin x - 3\cos x.$$

Problem 4. Consider a trapezoid, such that the base angles (corresponding to the longer base) are equal to 15° and 75°. Given that the length of the segment connecting the midpoints of its diagonals is equal to 20. Find the altitude of the trapezoid.

Problem 5. Let ABC be an acute-angled triangle. Denote by H the intersection point of the altitudes AA_1 and BB_1. Let M be a random point on the side AC and ω be the circumcircle of triangle MA_1C. Let MN be a diameter of ω. Denote by n° the angle between the lines NH and MB. Find n.

Problem 6. Evaluate the expression

$$11\sin^2\frac{3\pi}{11} + \left(2\sin\frac{\pi}{11} - \sin\frac{3\pi}{11} - 2\sin\frac{5\pi}{11}\right)^2.$$

Problem 7. Let A and B be points on the parabola $y = x^2$. Let C be a point on that parabola, such that the tangent line to a parabola passing through point C is parallel to AB. Given that the median corresponding to vertex C of triangle ABC is equal to 81. Find the area of triangle ABC.

Problem 8. Let

$$2\cos^2\left(2\phi + \frac{5\pi}{6}\right) = \sqrt{3}\left(\sin\left(2\phi + \frac{5\pi}{6}\right) - \cos\left(2\phi + \frac{5\pi}{6}\right)\right).$$

Find the value of the following expression

$$\frac{\cos\left(\phi + \frac{\pi}{6}\right)}{\sin\phi\sin\left(\phi + \frac{\pi}{6}\right)\sin\left(\phi + \frac{2\pi}{6}\right)} + \frac{\cos\left(\phi + \frac{2\pi}{6}\right)}{\sin\left(\phi + \frac{\pi}{6}\right)\sin\left(\phi + \frac{2\pi}{6}\right)\sin\left(\phi + \frac{3\pi}{6}\right)} +$$

$$\frac{\cos\left(\phi + \frac{3\pi}{6}\right)}{\sin\left(\phi + \frac{2\pi}{6}\right)\sin\left(\phi + \frac{3\pi}{6}\right)\sin\left(\phi + \frac{4\pi}{6}\right)} + \frac{\cos\left(\phi + \frac{4\pi}{6}\right)}{\sin\left(\phi + \frac{3\pi}{6}\right)\sin\left(\phi + \frac{4\pi}{6}\right)\sin\left(\phi + \frac{5\pi}{6}\right)}.$$

Problem 9. Let ABC be an isosceles triangle. Denote by O the centre of a circle that intersects sides AB and BC at points M, N and P, Q, respectively, such that point M is between points A, N and point Q is between points P, C. Given that the distance of point O and the altitude BH of triangle ABC is equal to $\sqrt{6} + \sqrt{2}$ and $\angle B = 30^\circ$. Find $|AM + BP - CQ - BN|$.

1.6 Problem Set 6

Problem 1. Let AA_1 and BB_1 be the altitudes of an acute triangle ABC. Let O be the circumcenter of triangle ABC. Given that $\angle C = 60^\circ$ and that the distance of point O and line AA_1 is equal to 15. Find the distance of point O and line BB_1.

Problem 2. Let α, β, γ be the angles of triangle ABC. Given that $4\sin\alpha + 4\sin\beta = 9 + 8\cos\gamma$. Find $64(\cos^2\gamma - \sin^2\alpha - \sin^2\beta)$.

Problem 3. Consider a right-angled triangle ABC. Let O be the centre of a semicircle that is inscribed to ABC, such that O is on the hypotenuse AB. Let D be a point of that semicircle, such that $BD = BC$. Given that $\cos\angle A = \frac{12}{13}$. Find $13\cos\angle BOD$.

Problem 4. Let acute triangle ABC is inscribed to a circle. Let l be a tangent line passing through point C. Given that the altitude CC_1 is equal to 29. Denote by a the distance of point A and line l. Denote by b the distance of point B and line l. Find the value of the product ab.

Problem 5. Let $ABCD$ be inscribed and circumscribed quadrilateral. Let the inscribed circle touches the sides AB and CD at points M and N. Given that $CN = 14$, $DN = 8$, $AM = 12$. Find BM.

Problem 6. Given that

$$\cos\alpha + \cos\beta + \cos\gamma + \cos\alpha\cos\beta\cos\gamma = \frac{1273}{845},$$

$$\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\alpha\cos\gamma = \frac{2532}{4225}.$$

Find the value of the following expression

$$\frac{4225}{3}|\sin\alpha\sin\beta\sin\gamma|.$$

Problem 7. Let quadrilateral $ABCD$ be, such that $\angle BAC = 55^\circ$, $\angle DAC = 10^\circ$, $\angle BCA = \angle DCA = 25^\circ$ and $\angle ADB = n^\circ$. Find n.

Problem 8. Let M be the smallest number, such that the following inequality

$$8\cos x\cos y\cos z(\tan x + \tan y + \tan z) \leq M,$$

holds true for any positive numbers x, y, z, where $x + y + z = \frac{\pi}{2}$. Find M.

Problem 9. Consider a triangle ABC. Let the segments AA_1, BB_1, CC_1 intersect at the same point. The segments AA_1, B_1C_1 intersect at point P and the segments CC_1, A_1B_1 intersect at point Q. Given that $\angle ABP = 75^\circ$, $\angle PBB_1 = 30^\circ$, $\angle QBB_1 = 15^\circ$ and $\angle ABC = n^\circ$. Find n.

1.7 Problem Set 7

Problem 1. The altitudes of triangle ABC intersect at point H. Given that $\angle C = 45^\circ$ and $AB = 20$. Find CH.

Problem 2. Find the greatest value of the following equation

$$(|\sin\alpha - \cos\beta| + |\cos\alpha + \sin\beta|)^2.$$

Problem 3. Let $ABCDEFPK$ be a octagon created by the intersection of two equal rectangles. Given that $AB + CD + EF + PK = 23$. Find the perimeter of $ABCDEFPK$.

Problem 4. Let CD be a bisector of triangle ABC. Given that $AB = 1.5CD$, $\angle A = 1.5\angle C$. Assume $\angle B = n^\circ$. Find n.

Problem 5. Let A_1, A_2, A_3, A_4 be points on one of the sides of angle A and B_1, B_2, B_3, B_4 be points on the other side of angle A, such that triangle AA_iB_i is covered by triangle $AA_{i+1}B_{i+1}$, for $i = 1,2,3$. Given that $A_iB_iB_{i+1}A_{i+1}$, $i = 1, 2, 3$, are simultaneously inscribed and circumscribed quadrilaterals. Let $A_1B_1 = 1$ and $A_4B_4 = 8$. Find A_3B_3.

Problem 6. Evaluate the expression

$$\frac{1 + 4\cos\frac{\pi}{7} + 2\cos\frac{2\pi}{7}}{\cos^3\frac{\pi}{7}}.$$

Problem 7. Let $ABCD$ be inscribed and circumscribed quadrilateral. Given that the radiuses of the inscribed and circumscribed circles are equal to 7 and 12, respectively. Find the product of the diagonals of $ABCD$.

Problem 8. Let $A = \sin\frac{\pi}{81}\cdot\sin\frac{2\pi}{81}\cdot\dots\cdot\sin\frac{80\pi}{81}$ and $B = \sin\frac{\pi}{27}\cdot\sin\frac{2\pi}{27}\cdot\dots\cdot\sin\frac{26\pi}{27}$. Find the value of the following expression $\sqrt[7]{\frac{3B}{A}}$.

Problem 9. Let AB and BC be the arcs constructed outside of triangle ABC, such that the sum of their angle measures is equal to 360°. Let M, N, K be given points on sides AB, BC, AC, respectively, such that $MK \parallel BC$ and $NK \parallel AB$. Given that E and F are such points on arcs AB and BC, respectively, that $\angle AME = \angle CNF = 60^\circ$. Assume $\angle EKF = n^\circ$. Find n.

1.8 Problem Set 8

Problem 1. Consider a trapezoid with bases equal to 3 and 7. Given that the sum of the squares of its diagonals is equal to 100. Find the sum of the squares of its legs.

Problem 2. Evaluate the following expression

$$\sqrt{2}\left(\frac{\cos 44^\circ}{\cos 1^\circ}+\frac{\cos 43^\circ}{\cos 2^\circ}+\cdots+\frac{\cos 1^\circ}{\cos 44^\circ}\right)-\tan 1^\circ-\tan 2^\circ-\cdots-\tan 44^\circ.$$

Problem 3. Let ABC be a triangle, such that $BC=5\sqrt{6}+5\sqrt{2}$, $\angle A=30^\circ$ and $AB>BC$, $AC>BC$. Consider points M and N on sides AB and AC, respectively, such that $MB=BC=CN$. Find the distance of the midpoints of CM and BN.

Problem 4. Find the minimum value of the expression

$$\frac{64}{\sin^6 2\alpha}-\frac{1}{\sin^6\alpha}-\frac{1}{\cos^6\alpha}.$$

Problem 5. Let D be a point chosen on the side AC of triangle ABC, such that $\angle ABD=90^\circ$, $\angle DBC=18^\circ$, $CD=10$. Given that $AC=10\sqrt{5}$. Find BD.

Problem 6. Let $ABCD$ be a cyclic quadrilateral, such that the bisectors of the angles A and D intersect on the side BC. Given that $AB=3$, $CD=7$. Find BC.

Problem 7. Let $ABCD$ be a convex quadrilateral, such that $\angle BAC+\angle ADB=\angle BCA+\angle CDB$ and $\angle BAC\neq\angle CDB$. Given that $AB=3$, $BC=4$, $CD=8$. Find AD.

Problem 8. Evaluate the expression

$$\frac{1}{\sqrt{2}\cos 18^\circ}\left(\frac{1}{\sin^3 9^\circ}-\frac{1}{\cos^3 9^\circ}\right)-40\sqrt{5}.$$

Problem 9. Let M, N, K be points chosen on the sides AB, BC, AC of triangle ABC, respectively. Given that triangle MNK is an acute triangle and that $h_m+h_n+h_k-\min(h_m,h_n,h_k)=\min(h_a,h_b,h_c)$, where h_a, h_b, h_c and h_m, h_n, h_k are the altitudes of triangles ABC and MNK, respectively. Find all possible values of the ratio $\frac{(ABC)}{(MNK)}$, where we denote by (ABC) the area of triangle ABC.

1.9 Problem Set 9

Problem 1. Let BH be an altitude of acute triangle ABC and O be its circumcenter. Given that the line HO passes through the midpoint of BC side and $BC=20\sqrt{2}$, $AC=35$. Find AB.

Problem 2. Find the smallest value of the following expression

$$\tan^2\alpha+\frac{4}{\cos\alpha}+40.$$

Problem 3. Let $ABCDE$ be a convex pentagon, such that its perimeter is equal to 25, $DE=5$ and $EB=BD$. Given that the inscribed circles of triangles ABE, BCD touch diagonals EB, BD at points M, N, respectively, such that $BM=BN$. Find $AE+BC$.

Problem 4. Let the lines including the legs AB and CD of trapezoid $ABCD$ intersect at point M. Given that $AB = 14$, $CD = 16$, $\angle AMD = 60^\circ$. Find the distance between the midpoints of the bases of trapezoid $ABCD$.

Problem 5. Evaluate the expression

$$\frac{\cos 2^\circ \cos^2 89^\circ}{\cos 3^\circ}\left(\frac{1}{\cos 2^\circ \cos 3^\circ}+\frac{1}{\cos 3^\circ \cos 4^\circ}+\cdots+\frac{1}{\cos 88^\circ \cos 89^\circ}\right).$$

Problem 6. Let $ABCD$ be a circumscribed quadrilateral and its diagonals intersect at point M. Given that $\frac{BC}{AD}=\frac{1}{2}$, $\frac{BM}{DM}=\frac{1}{3}$, $\frac{AM}{MC}=\frac{5}{3}$. Find $\frac{AB}{CD}$.

Problem 7. Given that the circumradius of triangle ABC is equal to the length of one of its bisectors. Given that the circumcenter of triangle ABC is inside of the inscribed circle of ABC. Let the greatest angle of triangle is equal to n°. Find n.

Problem 8. Let $\alpha_1+\alpha_2+\cdots+\alpha_{2015}=\frac{\pi}{2}$, where $\alpha_1>0$, $\alpha_2>0,\ldots,$ $\alpha_{2015}>0$. Calculate the following expression

$$\frac{\sin\alpha_1}{\sin\alpha_1+\cos\alpha_1}+\frac{\sin\alpha_2}{(\sin\alpha_1+\cos\alpha_1)}\times\frac{1}{(\sin(\alpha_1+\alpha_2)+\cos(\alpha_1+\alpha_2))}+\cdots+$$

$$+\frac{\sin\alpha_{2014}}{(\sin(\alpha_1+\alpha_2+\cdots+\alpha_{2013})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{2013}))}\times$$

$$\times\frac{1}{(\sin(\alpha_1+\alpha_2+\cdots+\alpha_{2014})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{2014}))}+$$

$$+\frac{\sin\alpha_{2015}}{\sin(\alpha_1+\alpha_2+\cdots+\alpha_{2014})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{2014})}.$$

Problem 9. Let $ABCDEF$ be a convex hexagon, such that $\angle A = 90^\circ$, $\angle C = 120^\circ$, $\angle E = 150^\circ$ and $\angle CDE = 2\angle BDF$. Given that $AB - BC = 10$ and $CD - DE = 20$. Find $\sqrt{3}(EF - FA)$.

1.10 Problem Set 10

Problem 1. Find the greatest value of the expression

$$2(\sin x + 2\cos x)(\cos x + 2\sin x).$$

Problem 2. Let ABC be a right triangle, such that the radius of the inscribed circle is equal to 10. Given that CD is the altitude to the hypotenuse. Find

$$(AD + CD - AC)^2 + (BD + CD - BC)^2.$$

Problem 3. Let ABC be a triangle, such that $AC = 195$, $\angle B = 120^\circ$. Let BD be a bisector. Given that $AB \cdot CD = 10920$. Find BD.

Problem 4. Let M and N be the midpoints of sides AD and BC of quadrilateral $ABCD$. Given that $AB = 6$, $BC = 4$, $CD = 8$, $AD = 14$. Find the possible greatest value of MN.

Problem 5. Find the value of the expression

$$\left(3\cos^2\frac{\pi}{82} - \sin^2\frac{\pi}{82}\right)\left(3\cos^2\frac{3\pi}{82} - \sin^2\frac{3\pi}{82}\right)\left(3\cos^2\frac{9\pi}{82} - \sin^2\frac{9\pi}{82}\right)\cdot$$
$$\cdot\left(3\cos^2\frac{27\pi}{82} - \sin^2\frac{27\pi}{82}\right).$$

Problem 6. Let N be a given point on side CD of quadrilateral $ABCD$ and M be a given point inside of triangle ABD. Given that $\angle BND \neq 90^\circ$ and $\angle BDC = 40^\circ$, $\angle BMN = 80^\circ$, $MN = MD$. Let $\angle MBN = n^\circ$. Find n.

Problem 7. Find the value of the expression

$$\frac{\sin\frac{\pi}{16}}{\sin\frac{\pi}{16} + \cos\frac{\pi}{16}} + \frac{\sin\frac{\pi}{16}}{\left(\sin\frac{\pi}{16} + \cos\frac{\pi}{16}\right)\left(\sin\frac{2\pi}{16} + \cos\frac{2\pi}{16}\right)} + \cdots$$
$$\cdots + \frac{\sin\frac{\pi}{16}}{\left(\sin\frac{6\pi}{16} + \cos\frac{6\pi}{16}\right)\left(\sin\frac{7\pi}{16} + \cos\frac{7\pi}{16}\right)} + \frac{\sin\frac{\pi}{16}}{\sin\frac{7\pi}{16} + \cos\frac{7\pi}{16}}.$$

Problem 8. Let ABC be an acute triangle, such that the circumradius and inradius are equal 26 and 10, respectively. Given that $\angle A = 60^\circ$. Let S be the area of the triangle that has vertices at the midpoints of minor arcs AB, AC and major arc BC. Find $\sqrt{3}S$.

Problem 9. Let $DABC$ be a tetrahedron, such that $\angle DAC = \angle DBC$, $\angle ADB = \angle ACB$ and $AC + AD = 25$. Find $BC + BD$.

1.11 Problem Set 11

Problem 1. Let M be the midpoint of the leg CD of the trapezoid $ABCD$. Given that $AB = BD$, $\angle ACB = 30^\circ$ and $\angle MBC = n^\circ$. Find n.

Problem 2. Given that

$$\frac{1}{\sin\frac{\pi}{12}} - \frac{1}{\cos\frac{\pi}{12}} = \sqrt{n}.$$

Find n.

Problem 3. Let $ABCD$ be a cyclic quadrilateral. Given that $\angle BAC = \angle BCA = 75^\circ$ and $BD = 20$. Find the area of $ABCD$.

Problem 4. Consider a triangle ABC, such that $\angle A = 52^\circ$, $\angle B = 109^\circ$ and $(ABC) = 100$. Let BD be the altitude of triangle ABC. Find $AB \cdot CD$.

Problem 5. Consider triangle ABC and point T. Given that $\angle ATB = \angle BTC = \angle ATC = 120^\circ$, $AC = 3$, $BC = 4$, $\angle ACB = 90^\circ$. Find $\frac{9BT+7CT}{AT}$.

Problem 6. Find the possible smallest value of the expression

$$20 + 16\sin\left(\frac{\pi}{6} - \alpha\right) \cdot \sin\alpha \cdot \sin\left(\frac{\pi}{6} + \alpha\right).$$

Problem 7. Let $ABCD$ be a convex quadrilateral. Given that $BC = CD$, $\angle BAD = 30^\circ$, $AC = 20$ and $AB + AD = 10\sqrt{2}(\sqrt{3}+1)$. Find the area of $ABCD$.

Problem 8. Find the value of the expression

$$\frac{\sin\frac{\pi}{40}}{\cos\frac{3\pi}{40}} + \frac{\sin\frac{3\pi}{40}}{\cos\frac{9\pi}{40}} + \frac{\sin\frac{9\pi}{40}}{\cos\frac{27\pi}{40}} + \frac{\sin\frac{27\pi}{40}}{\sin\frac{81\pi}{40}}.$$

Problem 9. Let $ABCDEF$ be a cyclic hexagon, such that $AB = DE$, $BC = EF$, $CD = AF$, $AB \neq BC$. Given that the minor arc AF is equal to 60°. For how many points M of the minor arc AF can it hold true the following equation?

$$MC + MD = MA + MB + ME + MF.$$

1.12 Problem Set 12

Problem 1. Let ABC be an isosceles triangle, such that $\angle B = 120^\circ$. Let M be a point on base AC, such that $\angle MBC = 30^\circ$. Given that the inradius of triangle ABM is equal to $15 + 5\sqrt{3}$. Find the value of the inradius of triangle BMC.

Problem 2. Let $0 \le \alpha < \frac{\pi}{6}$. Find the smallest possible value of the expression

$$\tan\alpha + \tan(2\alpha) - \tan(3\alpha).$$

Problem 3. Let the inscribed circle of quadrilateral $ABCD$ be tangent to sides AB and CD at points M and N, respectively. Given that $AB + CD = 25$ and $AM \cdot MB = CN \cdot ND = 36$. Find the area of quadrilateral $ABCD$.

Problem 4. Let the distance between the centres of circles ω_1, ω_2, with radiuses equal to 10 and 17, be equal to 21. Given that circles ω_1, ω_2 intersect at points A and B. Let l be a line passing through point B and intersecting circles ω_1, ω_2 at points M and N. Find the greatest possible value of the area of triangle AMN.

Problem 5. Find the greatest possible value of the expression

$$\sin x \sin y + \cos y \cos z + \sin z \sin x.$$

Problem 6. Let O be the circumcenter of quadrilateral $ABCD$. We denote by M the intersection point of the diagonals of quadrilateral $ABCD$. Given that $AD = 20$, $BD = 21$ and $AB = AM = 13$. Let the circumcircle of triangle AMD intersects line segments AB and CD at points N, K, respectively. We denote by S the area of pentagon $ONBCK$. Find the value of $\frac{507}{700}S$.

Problem 7. Find the value of the sum

$$\frac{1}{\sin\frac{\pi}{2^{2016}-1}}\cdot\sum_{k=0}^{2015}\left(\sin\frac{2^k\pi}{2^{2016}-1}\cdot\left(8\cos^3\frac{2^k\pi}{2^{2016}-1}-8\cos\frac{2^k\pi}{2^{2016}-1}+1\right)\right).$$

Problem 8. Let $ABCD$ be a convex quadrilateral, such that $R_A\cdot R_C = R_B\cdot R_D$, where R_A, R_B, R_C, R_D are circumradiuses of triangles DAB, ABC, BCD, CDA, respectively. Find the number of possible values of the expression $\frac{R_A}{R_B}$.

Problem 9. Let two circles (considered with their interior parts) with centres O_1, O_2 do not have a common interior point and do not have any point outside the square with side length $15+10\sqrt{2}$. Given that the sum of their diameters is equal to $20+10\sqrt{2}$. Find $(2-\sqrt{2})\cdot\frac{AD}{O_1O_2}$.

Problem 10. Let $\alpha,\beta,\gamma,\delta$ be acute angles, such that

$$4(\tan\alpha+\tan\beta+\tan\gamma+\tan\delta)\cos\alpha\cos\beta\cos\gamma\cos\delta = 3\sqrt{3}.$$

Find the value of the expression

$$(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta)^2.$$

Problem 11. Let $ABCD$ be a cyclic quadrilateral, such that $BC = CD$ and $\angle BAD = 30°$. Let M, N be points on sides AB, AD, respectively, such that $MN = BM + DN$. Given that $\angle MCN = n°$. Find n.

Problem 12. Let $ABCD$ be a tetrahedron that does not have points outside a unit cube and $AB\cdot CD\cdot d = 2$, where d is the distance between lines AB and CD. Find the number of possible values of the total surface area of $ABCD$.

1.13 Problem Set 13

Problem 1. Let there exist n $(n\geq 4)$ line segments, such that any four among them are the sides of a quadrilateral and any three of them are not the sides of a triangle. Find the greatest possible value of n.

Problem 2. Find the smallest possible value of the expression

$$2\sin x\cos y+2\sin y\cos z+2\sin z\cos x+10.$$

Problem 3. Let the area of convex quadrilateral $ABCD$ be equal to 100 and M be the intersection point of its diagonals. Denote by AA_1, DD_1, BB_1, CC_1 the altitudes of triangles AMD and BMC. Given that $\angle AMD = 60^\circ$. Find the area of quadrilateral $A_1B_1C_1D_1$.

Problem 4. Let angles α, β, γ be such that

$$|\sin\alpha| = \sin\beta\cos\gamma, \quad |\cos\alpha| = \sin\gamma\cos\beta.$$

Find

$$|\sin\alpha| + |\cos\alpha| + 2|\sin\beta| + 2|\cos\beta| + 4|\sin\gamma| + 4|\cos\gamma|.$$

Problem 5. Let N be a point on median BM of triangle ABC, such that $\angle ABM = \angle MNC$. Find the ratio of the median of triangle MNC passing through point M to the median of triangle NBC passing through point N.

Problem 6. Let $ABCD$ be a given rhombus and M be such a point that $MA = 27$, $MB = 12$, $AB = 18$, $\angle MAB = \angle MBC$. Find the smallest possible value of MD.

Problem 7. Let $ABCD$ be a cyclic quadrilateral, such that

$$2\tan\frac{\angle D}{2} = \tan\frac{\angle A}{2}.$$

Given that $CD - AB = 5$. Find $AD + BC$.

Problem 8. Find the value of the following expression

$$4096\cos\frac{\pi}{17}\cos\frac{2\pi}{17}\cos\frac{3\pi}{17}\cos\frac{4\pi}{17}\cos\frac{5\pi}{17}\cos\frac{6\pi}{17}\cos\frac{7\pi}{17}\cos\frac{8\pi}{17}.$$

Problem 9. Let $ABCDE$ be a convex pentagon, such that $AB = AE$, $BC = 3$, $CD = 5$, $DE = 4$ and $\angle BAE = 2\angle CAD$. Find $(ABCD) + (ACDE) - 1.5(ABCDE)$.

Problem 10. Let $ABCD$ be a convex quadrilateral, such that

$$\frac{CD - AB}{BC + AD} = \frac{\tan\frac{\angle A}{2} - \tan\frac{\angle D}{2}}{\tan\frac{\angle A}{2} + \tan\frac{\angle D}{2}}.$$

Given that $\angle A + \angle C = n^\circ$. Find n.

Problem 11. Let a, b be real numbers, such that the following inequality

$$\sum_{i=0}^{n}|\sin(2^i x)| \le a + bn,$$

holds true for any x and for any non-negative integer n. Find the smallest possible value of $20a^2 + 16b^2$.

Problem 12. Let two balls with centres O_1, O_2 have no common inner point. Given that the sum of the diameters is equal to $15 + 5\sqrt{3}$ and that none of them has a point outside the cube $ABCDA_1B_1C_1D_1$ having a side length $10 + 5\sqrt{3}$. Find

$$(3-\sqrt{3})\cdot\frac{AB}{O_1O_2}.$$

1.14 Problem Set 14

Problem 1. Let $ABCD$ be a trapezoid with bases BC and AD, such that $BC=4$ and $AD=9$. Given that $\frac{AB}{CD}=\frac{2}{3}$. Find AC.

Problem 2. Let $SABC$ be a tetrahedron. Given that $SA=BC=8.5$, $SB=AC=5$, $SC=AB=\frac{1}{2}\sqrt{261}$. Find the volume of $SABC$.

Problem 3. Let ABC be a triangle, such that $\cos\angle A=0.8$, $\cos\angle B=0.28$. Given that points M, N are on sides AB, AC, respectively, such that $AM=MN=NC=8$. Find BC.

Problem 4. Find the greatest value of the expression

$$3\tan x-\frac{5}{\cos x}+15,$$

where $x\in\left(0,\frac{\pi}{2}\right)$.

Problem 5. Let $ABCD$ be a cyclic quadrilateral, such that $\angle A=30^\circ$, $AC=5(\sqrt{6}-\sqrt{2})$ and $BC=CD$. Find $AB+AD$.

Problem 6. Find the value of the expression

$$\frac{\sin^2\frac{2\pi}{3^{100}-1}}{\cos\frac{2\cdot 3\pi}{3^{100}-1}}+\frac{\sin^2\frac{2\cdot 3\pi}{3^{100}-1}}{\cos\frac{2\cdot 3^2\pi}{3^{100}-1}}+\cdots+\frac{\sin^2\frac{2\cdot 3^{99}\pi}{3^{100}-1}}{\cos\frac{2\cdot 3^{100}\pi}{3^{100}-1}}.$$

Problem 7. Consider a quadrilateral $ABCD$, such that $AB=4$, $AC=6$, $AD=9$ and $\angle BAC=\angle DAC=30^\circ$. Find the square of the circumradius of triangle BCD.

Problem 8. Let $ABCD$ be a circumscribed quadrilateral, such that $\angle BAC=16^\circ$, $\angle DAC=44^\circ$, $\angle ADC=32^\circ$. Given that $\angle ACB=n^\circ$. Find n.

Problem 9. Given that tetrahedron $ABCD$ does not have any point outside $3\times4\times5$ rectangular parallelepiped. Find the greatest possible value of the expression

$$DA\cdot BC+DB\cdot AC+DC\cdot AB.$$

Problem 10. Find the value of the expression

$$\tan\frac{\pi}{9}\tan\frac{2\pi}{9}+\tan\frac{2\pi}{9}\tan\frac{4\pi}{9}-\tan\frac{\pi}{9}\tan\frac{4\pi}{9}.$$

Problem 11. Let $ABCD$ be a quadrilateral, such that $\angle A=75^\circ$, $\angle B=45^\circ$, $\angle C=120^\circ$. Given that lines AB, CD intersect at point E and lines AD, BC intersect at

point F. Denote by M, N, P, Q the intersection points of the altitudes of triangles EAD, EBC, FCD, FAB, respectively. Find $\left(\frac{MN}{PQ}-3\right)^2$.

Problem 12. Let $ABCD$ be a parallelogram, such that $\angle A$ is an acute angle. Denote by M, N the midpoints of sides AD and BC. Let CE be the altitude of parallelogram $ABCD$, such that $E \in AD$ and K be a point on line MN, such that ray EB is the bisector of angle AEK. Given that $DE = 6$ and $EK = 15$. Find AK.

1.15 Problem Set 15

Problem 1. Let AB be the hypotenuse of a right triangle ABC. Given that $AB = 200$ and $\angle B = \frac{5\pi}{24}$. Consider the altitude CH and the median CM of triangle ABC. Find the distance from the point H to the line CM.

Problem 2. Find the greatest possible value of the expression

$$5\tan x + \frac{13}{\cos x} + 30,$$

where $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Problem 3. Let the diagonals of a convex quadrilateral $ABCD$ intersect at point M. Given that $(ABC) = 25$, $(BCD) = 16$, $(ACD) = 15$. Find (AMB).

Problem 4. Let M, N be the intersection points of two circles with centres O_1, O_2 and radiuses 25, 60, respectively. Given that $O_1O_2 = 65$. Let line l passes through the point M, such that points O_1 and O_2 are on the same side of line l. Given also that the distance from the point O_1 to the line l is equal to 15. Find the distance from the point O_2 to the line l.

Problem 5. Let ABC be an acute triangle, such that $\angle C \le \arccos(\sqrt{5}-2)$ and $AB = 5(\sqrt{5}+1)$. Consider the altitudes AA_1 and BB_1. Given that AB_1A_1B is a circumscribed quadrilateral. Find the perimeter of quadrilateral AB_1A_1B.

Problem 6. Let M be a chosen point on the incircle of an isosceles triangle ABC and N be the intersection point of that circle with the line segment BM. Given that $\angle B = 120^\circ$ and $\frac{BN}{BM} = \frac{2-\sqrt{3}}{2}$. Let $\angle AMC = n^\circ$. Find n.

Problem 7. Let the greatest value of the expression $|\sin x(1-\cos y)| + |\sin y(1-\cos x)|$ be equal to M. Find $4M^2$.

Problem 8. Let $ABCD$ be a cyclic quadrilateral and $AB = 20$, $BC = 18$, $CD = 45$. Given points M, N on the diagonal AC, such that $\angle ABM = \angle CBN$ and $\angle ADM = \angle CDN$. Find AD.

Problem 9. Evaluate the expression

$$\frac{\cos\frac{2\pi}{3^{100}+1}}{\sin\frac{3\pi}{3^{100}+1}} + \frac{\cos\frac{2\cdot 3\pi}{3^{100}+1}}{\sin\frac{3^2\pi}{3^{100}+1}} + \cdots + \frac{\cos\frac{2\cdot 3^{99}\pi}{3^{100}+1}}{\sin\frac{3^{100}\pi}{3^{100}+1}}.$$

Problem 10. Given that a tetrahedron $ABCD$ does not have any points outside a cube with a side length of $4\sqrt[4]{3}$. Find the greatest possible value of the total surface area of tetrahedron $ABCD$.

Problem 11. Let $ABCDE$ be a convex pentagon, such that $AB = AE$, $\angle ABC + \angle AED = 180^\circ$ and $\angle ACB = \angle ACD$. Given that $BC = 12$ and $DE = 13$. Find CD.

Problem 12. Given the points M, N, P, Q on the diagonal AC of a cyclic quadrilateral $ABCD$, such that $\angle ABM = \angle CBN$, $\angle ADM = \angle CDN$, $\angle ABP = \angle CBQ = 100^\circ$ and $\angle ADP = 30^\circ$. Let $\angle PBQ + \angle PDQ = n^\circ$. Find n.

Chapter 2
Number Theory

Introduction. In the chapter Number theory, the majority of problems can be treated either by classical proof techniques and properties provided in the regular school program or by using some additional proof techniques and theorems that are considered to be outside of the scope of the regular school program. The classical proof techniques, theorems, properties or mathematical laws used in the proofs included in this chapter are the divisibility rules for 2, 3, 5, 9, 11, the properties of divisibility, the fundamental theorem of arithmetic (also called the unique-prime-factorization theorem), the properties of the integer and the fractional parts, the properties of the least common multiple (LCM) and the greatest common divisor (GCD). Note that the problems devoted to the prime numbers and to the Diophantine equations can be handled by considering the divisibility by some number. There are several examples of the Diophantine equations that are treated using the factorization method or Vieta's theorem. In some divisibility problems, it is useful to considering the least prime divisor. For the readers convenience, we list and provide the formulations of all used theorems and proof techniques that are outside of the scope of the regular school program.

Theorem 2.1. *Let a, b be any two integers, such that $b \mid a$. Prove that either $a = 0$ or $|a| \geq |b|$.*

Theorem 2.2 (Bézout's Identity). *Let a, b be any two integers, then there exist integers u, v, such that*

$$au + bv = GCD(a,b),$$

where $GCD(a,b)$ is the greatest common divisor of a and b.

Theorem 2.3. *Let $n > 1$ be a positive integer and p be a primer number. Prove that*

$$v_p(n!) = [\frac{n}{p}] + [\frac{n}{p^2}] + \dots,$$

H. Sedrakyan and N. Sedrakyan, *The Stair-Step Approach in Mathematics*,
Problem Books in Mathematics, https://doi.org/10.1007/978-3-319-70632-0_2

where $v_p(k)$ is equal to the biggest integer m such that p^m divides k, if $k \neq 0$ and is equal to $+\infty$, if $k = 0$.

Theorem 2.4 (Fermat's Little Theorem). *Let a be any integer and p be a prime number, such that a is not divisible by p, then $p \mid a^{p-1} - 1$.*

Theorem 2.5. *Prove that if $x^2 + y^2 = z^2$, where x, y, z are positive integers and x is even, then there exist positive integers m, n, k, such that $m > n$ and $x = 2mnd$, $y = (m^2 - n^2)d$, $z = (m^2 + n^2)d$.*

Theorem 2.6. *Let a, b, c, d be positive integers, such that $a^2 + b^2 = c^2 + d^2$, where $a < c \leq d < b$. Prove that $a^2 + b^2$ is a composite number.*

Theorem 2.7. *Let $a > 1$ be a positive integer. Prove that for any positive integers m, n it holds true*

$$(a^m - 1, a^n - 1) = a^{(m,n)} - 1.$$

2.1 Problem Set 1

Problem 1. Given that $p, q, p^2 + q^3, p^3 + q^2$ are prime numbers, where p and q are positive integers. Find the value of the sum $p + q + p^2 + q^2 + p^3 + q^3$.

Problem 2. A positive integer is called "interesting", if the sum of its digits is a square of a positive integer. What is the possible maximum number of consecutive "interesting" positive integers?

Problem 3. What is the maximal length of a geometric progression consisting of distinct prime numbers?

Problem 4. Find the total number of all positive integers n, such that $n(n+1)(n+4)(n+5)+4$ is the fourth power of a positive integer.

Problem 5. Let x, y, z be positive integers, such that $\frac{1}{x} + \frac{1}{y} - \frac{1}{z} = 1$. Find the value of the expression $\frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{z^2}$.

Problem 6. Let a, b, c, d be positive integers, such that $a! = b! + c! + d! + 1$ (for a positive integer n, we define $n! = 1 \cdot 2 \cdot \dots \cdot n$). Find the value of the sum of a, b, c, d.

Problem 7. Find the sum of all the solutions of the equation $\{2\{3\{4x\}\}\} = x$, where $\{a\}$ is a fractional part of a.

Problem 8. Let n and k be positive integers, $n \neq k$, such that $n + 1 \vdots k + 1$, $n + 2 \vdots k + 2, \dots, n + 11 \vdots k + 11$. Find the possible minimum value of $n - 27000$.

Problem 9. The sum of three positive integers is equal to 2014. Denote by L the possible maximum value of the least common multiple of that numbers. Find the value of $\frac{L}{449570}$.

2.2 Problem Set 2

Problem 1. The entries of 3×3 table are integers from 1 to 9. Consider the row and column sums of numbers in the table. How many of those six sums at maximum can be prime numbers?

Problem 2. Let x, y, z be positive integers, such that $\sqrt{x+2\sqrt{2015}} = \sqrt{y}+\sqrt{z}$. Find the possible smallest value of x.

Problem 3. Given that infinite arithmetic progression with a positive common difference includes finitely many prime numbers (at least one term). Find the number of those primes.

Problem 4. Given that the sum of the squares of four positive integers is equal to 9×2^{2015}. Find the ratio of the greatest and smallest numbers among them.

Problem 5. Find the number of all positive integers n, such that for each of them $n(n+1)(n+2)$ is a product of six consecutive positive integers.

Problem 6. Given that p, q, r are prime numbers, such that $p(p-5)+q(q-5) = r(r+5)$. Find the value of the product pqr.

Problem 7. Let x, y be rational numbers. Find the number of (x, y) couples, such that $x+\frac{2}{y}$ and $y+\frac{2}{x}$ are positive integers.

Problem 8. Let a and b be positive integers. Find the number of all positive integers, such that any of them is not possible to represent in the unique way in the following form $\frac{a^2+b}{ab^2+1}$.

Problem 9. Given that there exist n distinct positive integers, such that we cannot choose four numbers a, b, c, d among them in a way that $ab-cd$ is divisible by n. Find the greatest positive integer n satisfying this condition.

2.3 Problem Set 3

Problem 1. Given that $16y(x^2+1) = 25x(y^2+1)$, where x, y are positive integers. Find the possible minimum value of xy.

Problem 2. Let x, y, z, t be positive integers. Given that $68(xyzt+xy+zt+xt+1) = 157(yzt+y+t)$. Find the value of the product $xyzt$.

Problem 3. The entries of 3×3 table are integers from 1 to 9. Consider the row, column and diagonal sums of numbers in the table. At most, how many of those eight sums can be prime numbers?

Problem 4. Let n be a positive integer and p be a prime number. Find the number of solutions of the following equation $n^5 - p = 5p^2 - n^2$.

Problem 5. Let a, b be positive integers. Given that b is an odd number. Find the number of solutions (in the set of integer numbers) of the following equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{x-y} = \frac{a}{b}.$$

Problem 6. Consider three pairwise distinct positive integers. Given that the sum of the greatest common divisor and the least common multiple of any two numbers among those three numbers is divisible by the third one. Denote by D the ratio of the greatest number to the smallest one. Find the possible values of D.

Problem 7. Given that $xy^2-(3x^2-4x+1)y+x^3-2x^2+x=0$, where x,y are integer numbers. Find the possible maximum value of the sum $|x|+|y|$.

Problem 8. Let n be a positive integer, which is a multiple of 2015. Given that any divisor of n (except itself) is equal to the difference of any two other divisors of n. Find the sum of the digits of the smallest value of n.

Problem 9. Let m,n be positive integers, such that $m<301$ and $n<301$. Find the number of (m,n)-couples, such as $\frac{m^3+1}{mn+1}$ is a positive integer.

2.4 Problem Set 4

Problem 1. Consider k consequent positive integers, such that their sum is equal to 2015. Find the greatest value of k.

Problem 2. Let a and b be positive divisors of 720. Given that $720^2(b-a)=a^2(721+a)$. Find the product of a and b.

Problem 3. Find the number of integer solutions of the following equation

$$(x^2+y^2+z^2)(x^4+y^4+z^4+x^3y+y^3z+z^3x-xy^3-yz^3-zx^3)=201520152015.$$

Problem 4. A positive integer is called "interesting", if we can choose some of its digits and write a new number with those digits which is divisible by 7, otherwise its called "uninteresting" number. Find the ratio of the greatest "uninteresting" number to 1001.

Problem 5. Find the number of all positive integers n, such that for any n the number 2^n+5^n-65 is a square of some positive integer.

Problem 6. Let $p_1<p_2<\cdots<p_{11}$ be prime numbers. Given that $210 \mid p_1^{12}+p_2^{12}+\cdots+p_{11}^{12}$. Find $p_1p_2p_3p_4$.

Problem 7. Let (u_n) be a sequence of positive integers, such that $u_{n+1}=u_n^3+2015$, $n=1,2,\ldots$ Let k be the number of terms, which are squares of some positive integers. Find the greatest value of k.

Problem 8. Find the number of all positive integers n, such that for any of them $2^{2^{n^{2015}}-1}-1$ is divisible by n.

Problem 9. Let m and n be relatively prime numbers, such that $m+n=1000$. Consider the couples (a,b) of positive integers a and b, such that for any of them $m \mid a+nb$ and $n \mid a+mb$. Denote by $f(s)=\frac{mn}{m+n}s-(m+n-1)\left\{\frac{m^2s}{m+n}\right\}$, where $\{x\}$ stands for the fractional part of a real number x. Find the difference of $\min(a+b)$ and $\min(f(1),f(2),\ldots,f(m+n))$.

2.5 Problem Set 5

Problem 1. Find the smallest possible value of a positive integer n, such that at least one of the digits of the number $25n+1$ is equal to 8 or 9.

Problem 2. How many zeros does $(1^2+1)(2^2+1)\cdot\dots\cdot(100^2+1)$ end with?

Problem 3. Let x, y be positive integers. Given that $y \leq 2015$ and

$$(x^2+x+1)(x^2+3x+3) = y^2+y+1.$$

Find the greatest possible value of $\frac{y}{2}$.

Problem 4. Let a, b be positive integers. Given that $(a,b)+[a,b] = a+b+1874$. Find the smallest possible value of the sum $a+b$.

Problem 5. Find the number of integer solutions of the following equation

$$x^3(y-z)^3+y^3(z-x)^3+z^3(x-y)^3 = 36.$$

Problem 6. Let a, b be positive integers. Given that the value of the following expression

$$\frac{a^3-ab+1}{a^2+ab+2}$$

is an integer number. Find the greatest possible value of the sum $a+b$.

Problem 7. Let x, y be positive integers. Find the number of all (x,y) pairs, such that $x+y \leq 100$ and the value of the following expression

$$\frac{x^3+y^3-x^2y^2}{(x+y)^2}$$

is an integer number.

Problem 8. Find the number of all positive integers n, such that for any of them the number $2014^{n^{2015}-1}+1$ is divisible by n.

Problem 9. Let x, y, z be positive integers. Find the number of all triples (x,y,z), such that for any of them there exist three-digit numbers m and n with the following property

$$(x+y+z)^m = (x^2+y^2+z^2)^n.$$

(For different triples (x,y,z), the pairs (m,n) can be different.)

2.6 Problem Set 6

Problem 1. Find the smallest positive integer n, such that $25 \mid n+3$ and $49 \mid n+8$.

Problem 2. Find the possible smallest value of positive integer n, such that one of the digits of $125n+3$ is equal to 9.

Problem 3. Find the number of integer solutions of the following equation

$$(100-x)^2+(100-y)^2=(x+y)^2.$$

Problem 4. Let p,q,r be prime numbers. Given that $p^3r+p^2+p=2qr+q^2+q$. Find pqr.

Problem 5. Let x,y be positive integers. Find the number of all couples (x,y), such that $x\leq 2015$ and $(x-y)^3=x+2y$.

Problem 6. Let a,b be positive integers. Given that $ab-\sqrt{a^2-b^2}=53$. Find a.

Problem 7. Let m,n be positive integers. Given that $\frac{m^2+n^2}{mn-5}$ is a positive integer. Find the possible greatest value of $\frac{m^2+n^2}{mn-5}$.

Problem 8. Let x,y be positive integers. Find the number of all couples (x,y), such that $\frac{x^5+y}{x^2+y^2}$ and $\frac{y^5+x}{x^2+y^2}$ are positive integers.

Problem 9. Find the smallest value of a positive integer n, if it is known that, for any division of numbers $1,2,\dots,n$ into two groups, in one of the groups there are three numbers that generate a geometric progression.

2.7 Problem Set 7

Problem 1. Find the smallest positive integer n, such that n and $n+2015$ are squares of some integers.

Problem 2. Find all positive integers n, such that $9^{2^{n-1}}+3^{2^{n-1}}+1$ is a prime number.

Problem 3. Let n be a positive integer. Denote by S the sum of all the digits of $2n^2+44n+443$. Find the possible minimal value of S.

Problem 4. Find the number of all positive integers n, such that $3^{n-1}+7^{n-1}\mid 3^{n+1}+7^{n+1}$.

Problem 5. Let $n>1$ and $a_1,\dots,a_n$ be integer numbers. Given that one of the roots of the polynomial $p(x)=x^n+a_1x^{n-1}+\cdots+a_n$ and $p(2015)$ are prime numbers. Find the value of $p(2015)$.

Problem 6. Let x,y,z be positive integers. Find all the possible triples (x,y,z), such that $x+y+z\leq 1000$ and $xy^2+y^2z^4=5x+4z+x^2y^2z^2$.

Problem 7. Let x be a positive integer. Given that for any even value of x the quadratic expression $p(x)$ is a square of a positive integer. Find the value of $4p(2015)$, if $50<p(2015)<60$.

Problem 8. Let x,y be positive integers. Find all couples (x,y), such that $x^2+y^2\mid x^{101}+y$ and $x^2+y^2\mid x+y^{101}$.

Problem 9. Find the number of all triples (x,y,z), where x,y,z are odd positive integers, such that

$$x+y+z\leq 2015$$

and

$$3x^4 - 3x^2y^2 + y^4 = z^2.$$

2.8 Problem Set 8

Problem 1. Let p and q be prime numbers. Given that the equation $x^{2015} - px^{2014} + q = 0$ has an integer root. Find the sum $p+q$.

Problem 2. Find the number of all positive integer solutions of the equation

$$55x(x^3y^3 + x^2 + y) = 446(xy^3 + 1).$$

Problem 3. Let x and y be positive integers, such that $x < 2^{10}$, $y < 2^{20}$ and the value of the expression

$$\frac{2^{10}}{x} + \frac{2^{20}}{y}$$

is a positive integer. Find the number of all possible value of $\frac{x}{y}$.

Problem 4. Let n be a positive integer, such that the sum of the digits of n^2 is equal to the sum of the digits of $(n+2015)^2$. Find the smallest possible value of n.

Problem 5. Find the sum of the last 24 digits of the number $4 \cdot 6 \cdot (4!)^2 + 5 \cdot 7 \cdot (5!)^2 + \cdots + 50 \cdot 52 \cdot (50!)^2$.

Problem 6. Given that the sum of five positive integers is equal to 595. At most, with how many zeros can end the product of those numbers?

Problem 7. Let x, y, z, t be positive integers. Find the number of all quadruples (x,y,z,t), such that $x+y+zt = xy+z+t$ and $x+y+z+t = 245$.

Problem 8. Find the maximal number of summands, such that any summand is a positive number greater than 1, any two summands are mutually prime and the sum of all summands is equal to 2015.

Problem 9. Find the number of all positive integers n, such that $2^{n!}+1$ has a divisor less than or equal to $2n+1$.

2.9 Problem Set 9

Problem 1. Let n be a positive integer, such that $(n, n+20) = 20$ and $(n, n+15) = 15$. Find $(n, n+60)$.

Problem 2. Find the number of all couples (p,q) of prime numbers p, q, such that $p^4 + q$ and $q^4 + p$ are prime numbers.

Problem 3. Find the number of integer solutions of the equation

$$x(x-1) = y(y-2).$$

Problem 4. Let a, b be rational numbers, such that

$$a^2 - 47b^2 - 18ab - 4a + 4b + 2 = 0.$$

Find $a - b$.

Problem 5. Find the smallest positive integer n, such that the following equation

$$(2x+3y-5z)^3 + (2y+3z-5x)^3 + (2z+3x-5y)^3 = n,$$

has a solution in the set of integer numbers.

Problem 6. Find the greatest positive integer n, such that the numbers n and $8n+15$ have the same prime divisors.

Problem 7. Consider a sequence (a_n) of positive integers, such that for any $i, j \in \mathbb{N}, i \neq j$ it holds true

$$(a_i, a_j) = (i^2 + 3i + 2, j^2 + 3j + 2).$$

Find the number of all possible values of the 2015th term of this sequence.

Problem 8. Find the number of all positive integers, such that any of them is possible to represent, in more than one way, in the following form $\frac{x^2+y}{xy+1}$, where x, y are positive integers.

Problem 9. Let x, y be rational numbers. Find the number of (x,y) couples, such that $x + \frac{2}{y}$ and $y + \frac{3}{x}$ are positive integers.

2.10 Problem Set 10

Problem 1. Find the smallest positive integer, such that the sum of its digits is equal to 19 and it is divisible by 11.

Problem 2. Find the number of all positive integers, such that any of them is impossible to represent as $\frac{1+xy}{x+y}$, where x, y are positive integers.

Problem 3. Let p, q be prime numbers, such that the following equation

$$x^4 - (q+3)x^3 + (3q+2)x^2 - 2qx - 225 - p = 0,$$

has an integer root. Find $p + q$.

Problem 4. Let a, b, c be positive integers greater than 1, such that $c+1 \mid a+b$, $a+1 \mid b+c$, $b+1 \mid c+a$. Find the greatest possible value of $a+b+c$.

Problem 5. Find the number of integer solutions of the equation

$$x^5 + y^5 = 2^{2016}.$$

Problem 6. Let x, y, z be positive rational numbers, such that $x+\frac{1}{y}$, $y+\frac{2}{z}$, $z+\frac{3}{x}$ are integer numbers. Find the sum of all possible values of xyz.

Problem 7. Find the number of integer solutions of the equation

$$x^{10}+y^{10}=x^6y^5.$$

Problem 8. Let a, b be positive integers, such that

$$80 \mid 144a^3-32a^2b+92ab^2+b^3.$$

Find the possible smallest value of $16a^2+b^2$.

Problem 9. Find the number of integer solutions of the equation

$$2x^2-y^6=1.$$

2.11 Problem Set 11

Problem 1. Find the number of all positive integers, such that any of them is not possible to represent as $\frac{1+xy}{x-y}$, where x and y are positive integers.

Problem 2. How many integer solutions does the following equation have?

$$xy^2-2xy+3y^2+2y-8=0.$$

Problem 3. Find the smallest positive integer n, such that

$$0<\{\sqrt{n}\}<0.05,$$

where we denote by $\{x\}$ the fractional part of a real number x.

Problem 4. How many positive integer solutions does the following equation have?

$$(x^y-1)(z^t-1)=2^{200}.$$

Problem 5. Find the smallest positive integer, such that it is possible to represent as $\frac{m^3-n^3}{2016}$, where m and n are positive integers.

Problem 6. Consider 8×8 grid square. One needs to write in its each square a positive integer, such that all written numbers are different and any two numbers written in the squares with a common side are not mutually prime numbers. Find the smallest possible value of the greatest number among those numbers.

Problem 7. Given that x, y and $\frac{x^{2016}+y^{2016}}{x^{1009}\cdot y^{1008}}$ are integer numbers. Find the greatest possible value of $\frac{x^{2016}+y^{2016}}{x^{1009}\cdot y^{1008}}$.

Problem 8. Let a, b, c, d be positive integers. Find the number of quadruples (a,b,c,d), such that for any of them it holds true $ad+bc<bd$, $b+d\leq 132$ and

$$(9ac+bd)(ad+bc)=a^2d^2+10abcd+b^2c^2.$$

Problem 9. Given that the numbers 1, 2, ..., 100 are written in a line (in a random way), such that we obtain the numbers a_1, a_2, ..., a_{100}. Let $S_1=a_1$, $S_2=a_1+a_2,\ldots,S_{100}=a_1+a_2+\cdots+a_{100}$. At most, how many square numbers can be among the numbers S_1, $S_2,\ldots,S_{100}$?

2.12 Problem Set 12

Problem 1. Given that the age difference between father and son is not more than 40. At most, how many times can the age of the father be greater than the age of the son?

Problem 2. Let the entries of 3×3 grid square be the numbers $1,2,\ldots,9$, such that in any square is written only one number. Consider row and column sums. At most, how many numbers among those 6 numbers (sums) can be a square number?

Problem 3. Find the greatest three-digit number that is divisible by the product of its digits.

Problem 4. Find the number of all positive integers a, such as for each of them there exists a positive integer b, such that $b>a$ and the sum of the remainders after dividing 2016 by a and dividing 2016 by b is equal to 2016.

Problem 5. Find the number of integer solutions of the equation

$$x^3-3x^2y+4y^3=2016.$$

Problem 6. Find the number of all positive integers n, such that

$$n^8+n^6+n^4+n^2+1,$$

is a prime number.

Problem 7. Find the number of couples (m,n), where m, n are positive integers, such that it holds true the following equation

$$m!+136=n^2.$$

Problem 8. Find the number of all three-digit numbers n, such that the following system

$$\begin{cases} x+y+z=2^n, \\ (x-y)^2+(y-z)^2+(z-x)^2=2^n, \end{cases}$$

has an integer solution.

Problem 9. A positive integer is called "interesting", if there are two divisors among its divisors, such that their difference is equal to 2. Find the number of all "interesting" numbers of the form $n^{12}+n^{11}+\cdots+n+1$, where n is a positive integer.

Problem 10. Find the greatest value of C, such that the following inequality

$$(a-b)^3 > Cab,$$

holds true for any positive integers a, b satisfying the conditions $a>b$ and $a^2+ab+b^2 \mid ab(a+b)$.

Problem 11. Given that positive integer n is not divisible by 2016 and one divides n by 2016 using the long division method. At most, how many numbers 9 can there be after the decimal point?

Problem 12. Find the number of all positive integers less than or equal to 2016, such that each of them is possible to represent as

$$\frac{m^3+n^2}{m^2n^2+1},$$

where m, n are positive integers.

2.13 Problem Set 13

Problem 1. Let m, n be positive integers, such that

$$m(m,n)+n^2[m,n]=m^2+n^3-330.$$

Find $m+n$.

Problem 2. Let m, n be positive integers, such that

$$m!+n!=(m+n+3)^2.$$

Find mn.

Problem 3. Find the number of all couples (m,n) of positive integers m, n, such that for any of them $n \mid 12m-1$ and $m \mid 12n-1$.

Problem 4. How many zeros does the expression

$$(4^3+1)(5^3+1)\cdot\cdots\cdot(2017^3+1)+(3^3-1)(4^3-1)\cdot\cdots\cdot(2016^3-1),$$

end with?

Problem 5. Find the number of integer solutions of the equation

$$(x+2015y)(x+2016y)=x.$$

Problem 6. Find the number of all integers n, such that $n^5 - 2n^3 - n - 1$ is a prime number.

Problem 7. Let $p(x) = x^2 + x - 70$. A couple (m,n) of positive integers m, n, is called "rare", if $n > m$, $n \mid p(m)$ and $n+1 \mid p(m+1)$. Find the number of all "rare" couples.

Problem 8. At most, in how many ways the product of two distinct prime numbers can be represented as the sum of squares of two positive integers?

Problem 9. Find the smallest positive integer, that can be represented at least in three different ways as the sum of squares of two positive integers.

Problem 10. A positive integer is called "extraordinary", if it is possible to represent as

$$\frac{a+1}{a} \cdot \frac{b+1}{b} \cdot \frac{c+1}{c} \cdot \frac{d+1}{d},$$

where $a,b,c,d \in \mathbb{N}$. Find the sum of all "extraordinary" numbers.

Problem 11. Let a, b, c, d be positive integers, such that

a) $a < b < c < d < 2016$,

b) $ad = bc$,

c) $a + d = 2^k$, $b + c = 2^m$, where $k,m \in \mathbb{N}$.

Find the number of all possible values of a.

Problem 12. For any positive integer n denote by M_n the set of remainders after division of 1^2-1, $2^2-1, \ldots$, n^2-1, 1^2+1, $2^2+1, \ldots$, n^2+1 by n. A positive integer n is called "nice", if the number of elements of set M_n is equal to n. Find the product of all "nice" even numbers.

2.14 Problem Set 14

Problem 1. At most, how many composite numbers can one choose from the numbers $100, 101, \ldots, 500$, such that any two among the chosen numbers are mutually prime?

Problem 2. Let p, q, r be prime numbers, such that

$$\frac{1}{pq} + \frac{1}{qr} + \frac{1}{pr} = \frac{1}{839}.$$

Find the smallest possible value of $p + q + r$.

Problem 3. Find the smallest positive odd number, that is not possible to represent as

$$\frac{2^m - 1}{2^n + 1},$$

where $m,n \in \mathbb{N}$.

Problem 4. Find the smallest number, that is possible to represent at least in twenty different ways as $5m+7n$, where m, n are non-negative integers.

Problem 5. Let x, y be positive integers, such that

$$2201025(x^3y^3+x^2-y^2)=6902413244(xy^3+1).$$

Find $x+y$.

Problem 6. Let m, n, k, s be positive integers, such that

$$(3^m-3^n)^2=2^k+2^s.$$

Find the greatest possible value of the product $mnks$.

Problem 7. Find the smallest positive integer, that is not possible to represent as

$$\frac{a^2+b^2+c^2}{ab+bc+ca},$$

where $a,b,c\in\mathbb{N}$.

Problem 8. Let a, b, c be nonzero integers, such that

$$a^3+b^3+c^3=2.$$

Find the smallest possible value of $|a+b+c|$.

Problem 9. Find the number of triples (x,y,z) of positive integers x, y, z, such that the following conditions hold true

$$x+y+z\leq 100,$$

and

$$x^2+y^2+z^2=2xy+2yz+2zx.$$

Problem 10. A positive integer greater than one is called "uninteresting", if it is not possible to represent as

$$\frac{a+1}{a}\cdot\frac{b+1}{b}\cdot\frac{c+1}{c}\cdot\frac{d+1}{d}\cdot\frac{e+1}{e}\cdot\frac{f+1}{f},$$

where a, b, c, d, $e,f\in\mathbb{N}$. Find the smallest "uninteresting" number.

Problem 11. A positive integer n is called "amazing", if there exist pairwise distinct integers a, b, c, d, such that

$$n^2=a+b+c+d,$$

and any among the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ is a square of an integer number. Find the smallest "amazing" number.

Problem 12. Let $C(k)$ denotes the sum of all different prime divisors of a positive integer k. For example, $C(1) = 0$, $C(2) = 2$, $C(45) = 8$. Find all positive integers n such that $C(2^n + 1) = C(n)$.

2.15 Problem Set 15

Problem 1. Let m, n, k be such integers that

$$4^m \cdot 14^n \cdot 21^k = 2016.$$

Find $m^2 + n^2 + k^2$.

Problem 2. Find the smallest odd positive integer that is not possible to represent as

$$\frac{2^m + 1}{2^n - 1},$$

where $m, n \in \mathbb{N}$.

Problem 3. Let p, q be prime numbers, such that

$$p^2 - 2q^2 = 2801.$$

Find $p + q$.

Problem 4. Find the number of all positive integers n, such that the equation

$$x^z + y^z = 2^n,$$

does not have any solution in the set of positive integers, where $z > 1$.

Problem 5. Find the number of all positive integers, such that any of them is not possible to represent as

$$\frac{xyz}{x + y + z},$$

where x, y, z are positive integers.

Problem 6. Let x, y, z be integer numbers, such that

$$(x - y)^2 + (y - z)^2 + (z - x)^2 = 2016.$$

Find $|x - y| + |y - z| + |z - x|$.

Problem 7. Let a, b, c, d be positive integers, such that $a + b + c + d \leq 100$. Given that the sum of the squares of any two among the numbers a, b, c, d is divisible by the product of the other two numbers. Find the number of all possible values of the product $abcd$.

Problem 8. Find the number of all triples (a,b,c), where $a,b,c \in \mathbb{N}$, such that it holds true

$$a+b+c \leq 100,$$

and

$$(a+b)(b+c)(c+a) = 2^n, n \in \mathbb{N}.$$

Problem 9. Let a, b be distinct positive integers, such that

$$a^2+b^3 \mid a^3+b^2.$$

Find the smallest possible value of the sum $a+b$.

Problem 10. Find the number of all triples (a,b,c), where $a,b,c \in \mathbb{Z}$, such that it holds true

$$a+b+c = 1,$$

and

$$a^5+b^5+c^5 = 31.$$

Problem 11. Let x, y, z be positive integers, such that

$$(x^2+y)(x+y^2) = 3^z.$$

Find the greatest possible value of the product xyz.

Problem 12. Positive integer n is called "wonderful", if there exist positive integers a, b, k, such that

$$3^n = a^k + b^k,$$

where $k > 1$. Find the sum of all "wonderful" numbers smaller than 75.

Chapter 3
Algebra

Introduction. In the chapter Algebra, many problems can be treated using the properties of arithmetic or geometric progressions. There are several problems about finding a sum or a product. The majority of equations or system of equations can be treated by algebraic transformations, by the method of estimation or by using relation between arithmetic mean and geometric mean. There are many problems devoted to polynomials. In some of these problems, one needs to find the roots of a polynomial with integer coefficients. In particular, the following statement is used: if the leading coefficient of a polynomial is equal to 1, then the rational roots of this polynomial are integers. Now, let us list and provide all additional theorems used in the proofs.

Theorem 3.1. *Let sequence (a_n) be an arithmetic progression and positive integers k,l,m,p be, such that $k+l=m+p$. Prove that*

$$a_k + a_l = a_m + a_p.$$

Theorem 3.2. *Let sequence (b_n) be a geometric progression and positive integers k,l,m,p be, such that $k+l=m+p$. Prove that*

$$b_k \cdot b_l = b_m \cdot b_p.$$

Theorem 3.3 (AM-GM Inequality). *Let $n>1$ be a positive integer and $a_1,a_2,\dots,$ a_n be positive real numbers. Prove that*

$$\frac{a_1+\cdots+a_n}{n} \geq \sqrt[n]{a_1\cdot\dots\cdot a_n}.$$

Moreover, the inequality holds true if and only if $a_1=a_2=\cdots=a_n$.

Theorem 3.4. *Let $p(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ be a polynomial with integer coefficients, where $n>1$. Given that the rational number $\frac{m}{k}$ is a root of the polynomial $p(x)$, where $m\in\mathbb{Z}$, $k\in\mathbb{N}$ and $(m,k)=1$. Prove that $k \mid a_0$ and $m \mid a_n$.*

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Theorem 3.5. *Let $p(x)$ be a polynomial with integer coefficients and a, b be distinct integers. Prove that*

$$a-b \mid p(a)-p(b).$$

Theorem 3.6 (Vieta's Theorem). *Let $x_1, x_2, \ldots, x_n$ be the roots of the polynomial $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$. Prove that*

$$x_1 + x_2 + \cdots + x_n = -\frac{a_1}{a_0},$$

$$x_1x_2 + x_1x_3 + \cdots + x_1x_n + \cdots + x_{n-1}x_n = \frac{a_2}{a_0},$$

$$\ldots$$

$$x_1 \cdot x_2 \cdot \cdots \cdot x_n = (-1)^n \frac{a_n}{a_0}.$$

3.1 Problem Set 1

Problem 1. Solve the equation $4^x + 9^x + 49^x = 6^x + 14^x + 21^x$.

Problem 2. Let (b_n) be a geometric progression, such that $b_1 + b_{10} = 9$ and $b_2 + b_3 + \cdots + b_9 = 10$. Find the value of the expression $\frac{10b_2 + b_1b_2 + b_1^2}{b_1}$.

Problem 3. Find the sum of the greatest and smallest solutions of the equation $4^x + 64 = 2^{x^2-5x}$.

Problem 4. Find the sum of all the coefficients of the polynomial $(3x^{2014} - 2x^3 + 1)^9$.

Problem 5. Find the greatest value of the expression $\frac{3\sqrt{x^2+8x+15}+4}{x+4}$.

Problem 6. Find the product of the greatest and the smallest solutions of the inequality $\sqrt{6x-2} \leq 2 + \sqrt[3]{x+5}$.

Problem 7. Find the product of the solutions of the equation $\log_3 x - 1 = \log_2(\sqrt{x} - 1)$.

Problem 8. Consider $n \times n$ chess board, $n > 1$. In each square is written a positive integer. Given that the sum of the numbers of each column and the sum of the numbers of each row make a geometric progression (with $2n$ terms). Find the total number of all possible values of the common ratio of that geometric progression.

Problem 9. Let a, b, c be complex numbers different from 0, such that $a + b + c = a^2 + b^2 + c^2 = a^3 + b^3 + c^3 = a^4 + b^4 + c^4$. Find the sum of a, b, c.

3.2 Problem Set 2

Problem 1. Find the absolute value of the smallest integer solution of the following inequality

$$x \geq \frac{2015}{x}.$$

Problem 2. Let $x \geq 0, y > 0, z > 0$ and $\frac{4x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} = 2$. Find $\frac{x+z}{y}$.

Problem 3. Find the greatest integer solution of the following inequality $4^x - 3^x + 2^x < 2015$.

Problem 4. Let distinct complex numbers a and b be the roots of $x^3 + x^2 - 1$. Find the value of $(a+b)^3 + 2(a+b)^2 + a + b + 1000$.

Problem 5. Given that $\{a\}^2 + [b] = 15 - 6\sqrt{5}$ and $\{b\}^2 + [a] = 11 - 4\sqrt{5}$, where $\{x\}$ and $[x]$ are the fractional and integer parts of a real number x, respectively. Find the value of $a + b$.

Problem 6. Let a, b, c, d be real numbers, such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 0$ and $a^5 + b^5 + c^5 + d^5 = 0$. Find the number of all the possible values of the sum $a^{2015} + b^{2015} + c^{2015} + d^{2015}$.

Problem 7. Find the greatest value of a positive integer n, for which there exist n numbers, such that their pairwise sums are $\frac{n(n-1)}{2}$ consecutive integer numbers.

Problem 8. Find the value of the expression

$$21\sqrt[5]{6} - \sqrt{\frac{2205}{\sqrt[5]{1296} + 2\sqrt[5]{216} + 3\sqrt[5]{36} + 4\sqrt[5]{6} + 5}}.$$

Problem 9. Find the smallest value of the expression

$$(1 + x^2)\sqrt[3]{\frac{1024}{1 + 6x^2 + 9x^4}}.$$

3.3 Problem Set 3

Problem 1. Evaluate the expression

$$4^{\sqrt[3]{\log_2^2 3}} - 9^{\sqrt[3]{\log_3 2}}.$$

Problem 2. Find the number of solutions (in the set of real numbers) of the following equation

$$x^6 - 2x^4 + 4x^3 - 3x^2 - 4x + 4 = 0.$$

Problem 3. Given that in the geometric progression the sum of the first three terms with the odd indexes is equal to 10 and the sum of the first two terms with the even indexes is equal to 5. Find the sum of the squares of the first five terms.

Problem 4. Find the product of the real-valued solutions of the following equation

$$x^2 + 7x - 5 = 5\sqrt{x^3 - 1}.$$

Problem 5. Let x, y, z be nonzero real numbers. Given that $x^2 - yz = 1.5y^2$, $y^2 - xz = 1.5z^2$ and $z^2 - xy = 1.5x^2$. Find the absolute value of the following expression

$$\frac{24xyz}{x^2z + y^2x + z^2y}.$$

Problem 6. At most, how many numbers one can choose from the sequence $1, 2, \ldots, 17$, such that among the chosen numbers there are no three numbers which create an arithmetic progression?

Problem 7. Let x, y, z be real numbers. Given that $2x(y^2 - 1) + 2y(x^2 - 1) = (1 + x^2)(1 + y^2)$ and $4z(1 - y^2) + 4y(1 - z^2) = (1 + z^2)(1 + y^2)$. Find the value of the following expression

$$\left(\frac{2x}{1+x^2} - \frac{2z}{1+z^2}\right)^2 + \left(\frac{1-z^2}{1+z^2} - \frac{1-x^2}{1+x^2}\right)^2.$$

Problem 8. Let f, g be a quadratic expression. Given that f, g has at least one root, $f + g$ has no roots and $f - g$ is a quadratic expression. Find the number of roots of $f - g$.

Problem 9. Let $b_1, b_2, \ldots, b_{15}$ be a geometric progression with a common ratio q. Find the possible values of $\frac{b_1}{q}$, such as for any of the following system of equations has a solution in the set of real numbers

$$\begin{cases} x_1 + x_2 + \cdots + x_{15} = b_1, \\ x_1^2 + x_2^2 + \cdots + x_{15}^2 = b_2, \\ \ldots \\ x_1^{15} + x_2^{15} + \cdots + x_{15}^{15} = b_{15}. \end{cases}$$

3.4 Problem Set 4

Problem 1. Evaluate the expression

$$\log_3^2 45 - \frac{\log_3 15}{\log_{135} 3}.$$

Problem 2. Let a, b, c be real numbers. Given that

$$\frac{a^{2014}}{b+c} + \frac{b^{2014}}{a+c} + \frac{c^{2014}}{a+b} = \frac{a^{2014} + b^{2014} + c^{2014}}{a+b+c}.$$

Find the value of the sum

$$\frac{a^{2015}}{b+c} + \frac{b^{2015}}{a+c} + \frac{c^{2015}}{a+b}.$$

Problem 3. Find the greatest integer solution of inequality $x - \sin x < 6$.

Problem 4. Evaluate the expression $\left(\sqrt[3]{22+10\sqrt{7}}\right)^2 - 2\sqrt[3]{22+10\sqrt{7}}$.

Problem 5. Given that $[x]^3 \cdot \{x\} \geq \frac{16}{3}$, where by $[x]$ and $\{x\}$ we denote, respectively, the integer and fractional parts of a real number x. Find the possible smallest value of $30x$.

Problem 6. Find the sum of all the solutions of the equation $5^{x^3-17x^2+20x-3} = 15 \cdot 3^{-x}$.

Problem 7. Evaluate the expression

$$\left[\frac{2}{5} + \frac{2}{5} \cdot \frac{4}{7} + \cdots + \frac{2}{5} \cdot \frac{4}{7} \cdot \cdots \cdot \frac{100}{103}\right],$$

where we denote by $[x]$ the integer part of a real number x.

Problem 8. Let a,b,c,d be real numbers. Given that $a+b+c+d=0$ and $(ab+ac+ad+bc+bd+cd)^2+12=6(abc+abd+acd+bcd)$. Find the value of the expression $abc+abd+acd+bcd$.

Problem 9. Let M be the greatest number, such that the inequality $a^3+b^3+c^3 \geq 3abc+M|(a-b)(b-c)(c-a)|$ holds true for all non-negative a,b,c. Find the integer part of M.

3.5 Problem Set 5

Problem 1. Find the greatest root of the equation

$$x^3+x=30(x^2-3x+1)^2.$$

Problem 2. Find the number of solutions belonging to $[0,1]$

$$\{x\}+\{2x\}+\{3x\}+\{4x\}=1,$$

where by $\{x\}$ we denote the fractional part of real number x.

Problem 3. Given that 2015 is written as a sum of some three-digit and one one-digit numbers, such that the digits of each three-digit numbers are consecutive terms of an arithmetic progression. Find the possible greatest value of one-digit number.

Problem 4. Let x_1,x_2,x_3 be the roots of the equation $x^3-3x^2+4x-65=0$. Find the value of the expression $x_1^3+3x_2^2+3x_3^2-4x_2-4x_3$.

Problem 5. Let a,b be real numbers. Given that the equation $\cos x+\cos(ax)=b$ has a unique solution. Find the number of solutions of the equation $\cos x+\cos(ax)=-b$.

Problem 6. Let x,y,z be real numbers. Given that

$$x+y^2+z^3=y+z^2+x^3=z+x^2+y^3=0.$$

Let n be the number of possible values of the expression $x^2y+y^2z+z^2x-xyz-x^2y^2z^2$. Find $n^2-3n+100$.

Problem 7. Find the sum of all real-valued solutions of the equation

$$x^5-12x^4+54x^3-108x^2+81x+1=\sqrt{\cos\frac{2\pi x}{3}}.$$

Problem 8. Let x,y,z be positive numbers. Given that

$$\sqrt{x^2+y^2}+\sqrt{x^2+z^2}+y+z=\sqrt[4]{108}\cdot\sqrt{x(y+z)}.$$

Find the value of the expression $\frac{x^2+y^2}{z^2}$.

Problem 9. Find the greatest possible value of M, such that the inequality

$$M|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)|\leq(a^2+b^2+c^2)^2$$

holds true for any non-negative numbers a,b,c.

3.6 Problem Set 6

Problem 1. Find the sum of all integer solutions of the inequality

$$4^x+x\cdot 2^{x-6}-2^{x+11}-32x\leq 0.$$

Problem 2. Given that

$$x^2+y^2-12x-16y+99=0,\ u^2+v^2-20u-10v+44=0,$$

and

$$(x-u)^2+(y-v)^2=225.$$

Find the value of $x+y+2u+2v$.

Problem 3. Find the greatest solution of the inequality

$$27x^9-54x^7+36x^5-10x^3+x\leq 0.$$

Problem 4. Let x,y,z be real numbers, such that

$$x^3+y^3=\frac{100}{x^2+xy+y^2},$$

$$y^3+z^3=\frac{50}{y^2+yz+z^2},$$

$$z^3+x^3=\frac{25}{z^2+zx+x^2}.$$

Find the value of the following expression

$$\frac{30z}{3x-2y+z}.$$

Problem 5. Let a,b,c,d,e,f be real numbers, such that the equation

$$ax^2+bxy+cy^2+dx+ey+f=0$$

has exactly n real-valued solutions. Find n^2-n+50.

Problem 6. Solve the following equation

$$(x-1)^{215}\sqrt{|x-1|+1}+(x-2)^{215}\sqrt{|x-2|+1}+\cdots+$$

$$+(x-215)^{215}\sqrt{|x-215|+1}=0.$$

Problem 7. Let $p(x)$ be a polynomial of the forth degree with real coefficients, such that for any real x its value is non-negative. Given that $p(1)=p(2)=0, p(3)=12$. Find $p(4)+p(5)$.

Problem 8. Let a be a real number, such that the following inequality

$$(x-4)^6+3x^2+3y^2-24x-18y+75-3a\leq(a-9+6y-y^2)^3$$

has a unique solution in the set of real numbers. Find a.

Problem 9. Let $a\geq 0$, $b\geq 0$, $c\geq 0$, $a+b+c=1$ and

$$A=10(\sqrt{(1+a)(1+b)}+\sqrt{(1+b)(1+c)}+\sqrt{(1+c)(1+a)}.$$

Find the sum of all possible values of the integer part of A.

3.7 Problem Set 7

Problem 1. Given that $x-\frac{1}{x}=1$. Find the value of the expression $\sqrt{5}|x^8-\frac{1}{x^8}|$.

Problem 2. Solve the equation

$$\cos 2\pi x-4\cos\pi x+2x^2-24x+75=0.$$

Problem 3. Consider the numbers 200, 201,..., 400. At most, how many numbers one can choose among those numbers, such that they make a geometric progression?

Problem 4. Let A, B, C be pairwise disjoint sets. Let the number of elements of the union of the sets A, B, C is divisible by 80. Given that the pairwise arithmetic means of the sets $A \cup B \cup C, A \cup B, B \cup C, C \cup A$ are equal to 3, -1, 13, 6, respectively. Find the number of elements of the set $A \cup B \cup C$.

Problem 5. Solve the inequality

$$\log_{0.3} \frac{4^{x+10} - 2^{x+12} + 5}{4^{2\sqrt{x}+9} - 2^{2\sqrt{x}+11} + 5} \geq \frac{1}{2^{2\sqrt{x}+9} - 1} - \frac{1}{2^{x+10} - 1}.$$

Problem 6. Let $x > 0, y > 0, z > 0, t > 0$ be real numbers. Find the greatest possible value of the following expression

$$\frac{xyzt(x^3 + yzt)(y^3 + ztx)(z^3 + txy)(t^3 + xyz)}{\sqrt{(x^8 + y^4z^4)(y^8 + z^4t^4)(z^8 + t^4x^4)(t^8 + x^4y^4)}}.$$

Problem 7. Let x, y, z be real numbers, such that $67[y] + 39\{z\} - 5x^2 = 408.3$, $39[z] + 24\{x\} - 5y^2 = -97.85$ and $24[x] + 67\{y\} - 5y^2 = 18.45$. Find the value of the sum $x + y + z$. Here, we denote by $\{x\}$ the fractional part and $[x]$ the integer part of a real number x.

Problem 8. Find the number of the real solutions of the following equation

$$(\sqrt{3} + x)^2 + (1 - \sqrt{3}x)^2 = (\sqrt{3} + x)^2 \cdot (1 - \sqrt{3}x)^2.$$

Problem 9. Let $a > 0$, $b > 0$, $c > 0$, $d > 0$ be real numbers, such that

$$\sqrt{a^2 + b^2 - \sqrt{2}ab} + \sqrt{b^2 + c^2 - \sqrt{2}bc} + \sqrt{c^2 + d^2 - \sqrt{2}cd} = \sqrt{a^2 + d^2 + \sqrt{2}ad}.$$

Find the value of the following expression $\frac{(a+c)(b+d)}{ad}$.

3.8 Problem Set 8

Problem 1. Let a, b, c be the terms of an arithmetic progression. Given that the numbers $\frac{1}{c}$, $\frac{1}{b+1}$, $\frac{1}{a+2}$ are the terms of an arithmetic progression. Find the value of the difference $b - a$.

Problem 2. Let $x^2 + y^2 = \sqrt{3} + xy$. Find the value of the following expression

$$x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4.$$

Problem 3. Consider an increasing sequence of positive integers, such that the first term is equal to 2 and the last term is equal to 9. Given that any three consecutive terms of this sequence make either an arithmetic progression or a geometric progression. At least, how many terms can have this sequence?

Problem 4. Solve the equation

$$x+3^5 = 108\sqrt[4]{x}.$$

Problem 5. Given that $x^2+\frac{1}{x}=3$. Find the value of the following expression

$$9x^2+3x+\frac{1}{x^3}.$$

Problem 6. Consider numbers a, b, c, such that

$$\frac{a^3}{4a^2+2ab+b^2}+\frac{b^3}{4b^2+2bc+c^2}+\frac{c^3}{4c^2+2ca+a^2}=10$$

and

$$\frac{b^3}{4a^2+2ab+b^2}+\frac{c^3}{4b^2+2bc+c^2}+\frac{a^3}{4c^2+2ca+a^2}=12.$$

Find the value of the sum $a+b+c$.

Problem 7. Find the number of solutions of the following equation

$$[x]+2\{x^2\}=[x^3],$$

where by $[a]$ we denote the integer part of a real number a and by $\{a\}$ we denote the fractional part of a.

Problem 8. Consider numbers x, y, z, such that

$$x^3+x(y^2+yz+z^2)=76,$$

$$y^3+y(x^2+xz+z^2)=34,$$

and

$$z^3+z(x^2+xy+y^2)=-29.$$

Find the value of the expression $\frac{30y}{3x+y+2z}$.

Problem 9. Let $p(x)$ be n-th degree polynomial with integer coefficients. Given that $p\left(2^{\frac{1}{5}}+2^{-\frac{1}{5}}\right)=2015$. Find the smallest possible value of n.

3.9 Problem Set 9

Problem 1. Evaluate the expression

$$\frac{(1^2-1\cdot 100+100^2)+(2^2-2\cdot 99+99^2)+\cdots+(50^2-50\cdot 51+51^2)}{50^2}.$$

Problem 2. Given that

$$\frac{x^2}{x+1}+\frac{y^2}{y+2}+\frac{z^2}{z+3}=x+y+z+106.$$

Find the value of the following expression

$$\frac{1}{4x+4}+\frac{1}{y+2}+\frac{9}{4z+12}.$$

Problem 3. Solve the equation

$$\sqrt{x^2-4x+3}+7\sqrt{-x^2+6x-8}=7+\sqrt{x^3-10x^2+31x-30}.$$

Problem 4. Given that

$$a(a^2-9b+9)=(b+2)(b^2-5b+13).$$

Find the total number of the possible values of $a-b$.

Problem 5. Given that $a+b+c=0$, $a^2+b^2+c^2=14$, $a^3+b^3+c^3=21$. Find the value of $a^6+b^6+c^6$.

Problem 6. Evaluate the expression

$$\left[\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{\ldots\sqrt{2014+\sqrt{2015}}}}}}\right].$$

Problem 7. Let a,b,c be pairwise distinct real numbers. Given that

$$(b-c)\sqrt[3]{a^3+1}+(c-a)\sqrt[3]{b^3+1}+(a-b)\sqrt[3]{c^3+1}=0.$$

Find the total number of the possible values of the following expression

$$(a^3+1)(b^3+1)(c^3+1)-(abc+1)^3.$$

Problem 8. Let sequence $x_1, x_2, \ldots, x_{100}$ be such that $x_1 = x_{100} = 2015$ and

$$x_{i+1} = x_i^3 - x_{i-1}^3 + 2x_i - x_{i-1}, i = 2, 3, \ldots, 99.$$

Find the number of all positive integers n, $(1 \leq n \leq 100)$, such that $x_n = 2015$.

Problem 9. Let a, b, c be non-negative numbers, such that $a^2 + b^2 + c^2 \leq 6$. Find the possible greatest value of $\frac{c(a+b)}{ab+3}$.

3.10 Problem Set 10

Problem 1. Solve the equation

$$\frac{1}{x(x-1)} + \frac{2}{(x-1)(x-3)} + \frac{3}{(x-3)(x-6)} + \frac{4}{(x-6)(x-10)} = -0.4.$$

Problem 2. Given that $x^3 - \frac{1}{x} = 2$. Find the value of the following expression

$$-x^4 + 3x^3 + 2x - \frac{3}{x}.$$

Problem 3. Find the number of solutions of the equation

$$\sqrt{[x]} + \sqrt{\{x\}} = x,$$

where $[x]$ is the integer part and $\{x\}$ is the fractional part of a real number x.

Problem 4. Find the value of the expression

$$\left(\frac{1}{1-a+a^2} - \frac{1}{1+a+a^2} - \frac{2a}{1-a^2+a^4} + \frac{4a^3}{1-a^4+a^8}\right) : \frac{a^7}{1+a^8+a^{16}}.$$

Problem 5. Consider a sequence (x_n), such that $x_1 = 2$ and

$$x_{n+1} = x_n + \sqrt{2x_{n+1} + 2x_n}, n = 1, 2, \ldots$$

Find x_{31}.

Problem 6. Given that the circle ω and the graph of function $y = x^3 - 9x^2 + 48x - 100$ intersect at points $A_i(x_i, y_i)$, $i = 1, 2, \ldots, 6$. Find the value of $x_1 + x_2 + \cdots + x_6$.

Problem 7. Given that

$$\begin{cases} x^2 + xy + y^2 = 169, \\ y^2 + yz + z^2 = 400, \\ x^2 + xz + z^2 = 441. \end{cases}$$

Find the value of $\frac{231y+41z}{x}$.

Problem 8. Let x_1, x_2, x_3 be pairwise distinct real numbers and roots of the equation

$$x^4 - 6x^3 + 11x^2 + bx + c = 0.$$

Given that

$$x_1 + 2x_2 + 3x_3 = 14.$$

Find $|b| + |c|$.

Problem 9. Let x, y, z be real numbers. Given that the smallest value of

$$\frac{(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1)}{(xyz)^2 - xyz + 1}.$$

is equal to m. Find the value of $(m+3)^2$.

3.11 Problem Set 11

Problem 1. Let (b_n) be a geometric progression. Given that

$$b_3 + \sqrt[3]{2}b_7 + \sqrt[3]{4}b_{20} = 0.$$

Find the value of the expression

$$\frac{b_3^3 + 2b_7^3 + 4b_{20}^3}{b_{10}^3}.$$

Problem 2. Solve the equation

$$\sqrt{-2x^2 + 12x - 15 - 2\sqrt{2}} + \sqrt{-3x^2 + 18x - 21 - 4\sqrt{2}} = 1.$$

Problem 3. Find the number of solutions of the equation

$$20\sqrt{[x]} - \sqrt{\{x\}} = x,$$

where we denote by $[x]$ the integer part and by $\{x\}$ the fractional part of a real number x.

Problem 4. Let $p(x)$ be n-th degree polynomial with integer coefficients, such that

$$p(1+\sqrt{2}) = 54 + 29\sqrt{2},$$

$$p(1+\sqrt{3}) = 89 + 44\sqrt{3},$$

and

$$p(1+\sqrt{5}) = 189 + 80\sqrt{5}.$$

Find the smallest possible value of n.

Problem 5. Find the value of the expression

$$\Big(\frac{y(x-z)}{xy+yz-2xz} + \frac{z(y-x)}{xz+yz-2xy} + \frac{x(z-y)}{xy+zx-2yz}\Big)\Big(\frac{xy+yz-2xz}{y(x-z)} + \\ + \frac{xz+yz-2xy}{z(y-x)} + \frac{xy+zx-2yz}{x(z-y)}\Big).$$

Problem 6. Given that x, y, z are real numbers, such that

$$x^2 - xy = y^2 - yz = z^2 - zx.$$

Find the number of possible values of the expression

$$(x+y-z)(y+z-x)(x+z-y) - xyz.$$

Problem 7. Let a, b, c be real numbers, such that the following inequality

$$axy + byz + cxz \le x^2 + y^2 + z^2,$$

holds true for any real-valued variables x, y, z. Find the greatest possible value of the expression

$$a^2 + b^2 + c^2 + abc.$$

Problem 8. Given that $a_1, a_2, \ldots, a_{101}$ are nonzero numbers, such that any of the polynomials

$$a_{i_1}x^{100} + a_{i_2}x^{99} + \cdots + a_{i_{101}},$$

has an integer root, where i_1, $i_2, \ldots,$ i_{101} is a random permutation of numbers $1, 2, \ldots, 101$. Find the number of possible values of the sum $a_1 + a_2 + \cdots + a_{101}$.

Problem 9. Find the possible smallest value of M, if given that the following inequality

$$12(abc + abd + acd + bcd) - (ab + ac + ad + bc + bd + cd)^2 \le M,$$

holds true for any real numbers a, b, c, d, where $a + b + c + d = 0$.

3.12 Problem Set 12

Problem 1. Find the number of positive solutions of the equation

$$x^5 - 80x + 128 = 0.$$

Problem 2. Let x, y be real numbers. Find the greatest possible value of the expression

$$3(xy - 2x^2 - 2y^2 - x - y + 2).$$

Problem 3. Given that

$$\frac{x}{x^2 + 3x + 1} = \frac{15 + \sqrt{130}}{19}.$$

Find the value of the following expression

$$\frac{x^2}{x^4 - 3x^2 + 1}.$$

Problem 4. Let a, b, c be positive numbers, such that the numbers

$$\frac{1}{a^2 - ab + b^2} + \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2},\ a + b + c,\ \frac{c}{a^3 + b^3} + \frac{a}{b^3 + c^3} + \frac{b}{c^3 + a^3}$$

are consequent terms of an arithmetic progression. Find the value of the following expression

$$\frac{1}{a^3 + b^3} + \frac{1}{b^3 + c^3} + \frac{1}{c^3 + a^3}.$$

Problem 5. Find the greatest even number n smaller than 30, such as one can arrange 30 numbers on a circle in such a way that the product of any consequently written n numbers is negative.

Problem 6. Find the value of the expression

$$\frac{1}{\sqrt[3]{\sqrt[3]{2} - 1}} \left(\sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \cdots - \sqrt[3]{\frac{128}{9}} + \sqrt[3]{\frac{256}{9}} \right).$$

Problem 7. Let x, y be rational numbers, such that

$$x^2 - 16xy + 19y^2 + 16x - 8y - 16 = 0.$$

Find the value of $x + y$.

Problem 8. Find the number of solutions of the equation

$$\sqrt[4]{x^4-x^2-2x+18}+\sqrt{x^4-2x^3+2x^2-2x+3}=\frac{\sqrt{2}}{\sqrt{2}-1}.$$

Problem 9. Find the value of the expression

$$\frac{\sqrt{\sqrt{2016}+\sqrt{1}}+\sqrt{\sqrt{2016}+\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}+\sqrt{2015}}}{\sqrt{\sqrt{2016}-\sqrt{1}}+\sqrt{\sqrt{2016}-\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}-\sqrt{2015}}}-$$

$$-\frac{\sqrt{\sqrt{2016}-\sqrt{1}}+\sqrt{\sqrt{2016}-\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}-\sqrt{2015}}}{\sqrt{\sqrt{2016}+\sqrt{1}}+\sqrt{\sqrt{2016}+\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}+\sqrt{2015}}}.$$

Problem 10. Find the greatest possible value of k, such that the following inequality

$$k\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq\left(\frac{1+abc}{\sqrt{abc}}\right)^3.$$

holds true for any positive numbers a,b,c.

Problem 11. Let x_1, x_2,..., x_{100} be positive integers, such that $x_1\leq x_2\leq\cdots\leq x_{100}$ and $x_1+x_2+\cdots+x_{100}=2016$. Given that the value of the expression $p(x_1)+p(x_2)+\cdots+p(x_{100})$ is the smallest possible, where $p(x)=x^3-6x^2+11x+16$. Find the value of

$$x_{61}+x_{62}+\cdots+x_{100}.$$

Problem 12. Given that a circle with the radius 10 intersects hyperbola $y=\frac{3}{x}$ at point $A_i(x_i,y_i)$, where $i=1,2,3,4$. Find the value of the following expression

$$x_1^2+x_2^2+x_3^2+x_4^2+y_1^2+y_2^2+y_3^2+y_4^2.$$

3.13 Problem Set 13

Problem 1. Find the number of real solutions of the equation

$$\sin(\pi x)=\sin^3(\pi x)+\sqrt{-x^3+3x^2-2x}.$$

Problem 2. One writes (in a random way) in the squares of 3×3 grid square the numbers $1,\ldots,9$, such that in any square is written only one number. Then, consider (eight) sums of three numbers written in the same row, column or diagonal. At most, how many sums among those eight sums can be a square of a positive integer?

Problem 3. Evaluate the expression

$$\left[\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\dots\cdot\frac{100}{99}\right],$$

where by $[x]$ we denote the integer part of a real number x.

Problem 4. Find the product of all real solutions of the equation

$$\frac{x^{2^{2016}}+x^{2^{2015}}+1}{(x^2-x+1)(x^4-x^2+1)\cdot\dots\cdot(x^{2^{2015}}-x^{2^{2014}}+1)}=20x-16.$$

Problem 5. Let the circle ω intersects the hyperbola $y=\frac{30}{x}$ at points $A_i(x_i,y_i)$, where $i=1,2,3,4$. Find $y_1y_2y_3y_4$.

Problem 6. Let the vertices of parabolas $y=ax^2+bx+c$ and $y=a_1x^2+b_1x+c_1$ be different. Given that any of those two vertices is on the parabola corresponding to the other one. Find $a+a_1$.

Problem 7. Find the integer part of the greatest solution of the equation

$$\frac{1}{[x]}+\frac{1}{\{x\}}=\frac{15}{x},$$

where by $[x]$ we denote the integer part and by $\{x\}$ the fractional part of a real number x.

Problem 8. Given that

$$\begin{cases} x^3+x^2y+xy^2+y^3=-\frac{1640}{x^4+y^4}, \\ y^3+y^2z+yz^2+z^3=\frac{255}{y^4+z^4}, \\ z^3+z^2x+zx^2+x^3=-\frac{1649}{z^4+x^4}. \end{cases}$$

Find the value of the expression

$$\frac{9x+1895y}{2z}.$$

Problem 9. At most, how many numbers can one choose among the numbers $1,\dots,15$, such that if we consider four numbers among the chosen ones, then the sum of any two is not equal to the sum of the two others?

Problem 10. Let, by one step, from the couple (m,n) is possible to obtain the couple $(m+2,n+m+1)$. Given that, by several steps, from the couple $(-2014,-1016)$ is possible to obtain the couple $(2016,p)$. Find p.

Problem 11. Let real numbers p, q be such that the following inequality

$$|x^3-3x^2-px-q|\le\frac{3\sqrt{3}+1}{2},$$

holds true for any $x\in[1,4]$. Find $p^2+(2q+17)^2$.

Problem 12. Let quadratic trinomial $p(x)$ be, such that the inequality

$$p(x^3) \geq p(3x^2 - x + 3),$$

holds true for any real value of x. Find the value of the expression

$$\frac{p(2) - p(1)}{p(26) - p(25)}.$$

3.14 Problem Set 14

Problem 1. Let the sum of the first 2016 terms of a geometric progression be equal to 2016 and the product of 1000th and 1017th terms be equal to 16. Find the sum of multiplicative inverses of the first 2016 terms.

Problem 2. Let x, y be real numbers, such that

$$x^3 - y^3 + 6x^2 + 9y^2 + 17x - 32y + 60 = 0.$$

Find $y - x$.

Problem 3. Let x_0 be the greatest real solution of the following equation

$$\frac{1}{x^4} - \frac{1}{x^3(x+1)} - \frac{1}{x^2(x+1)} - \frac{1}{x(x+1)} - \frac{1}{x+1} = 1.$$

Find $(2x_0 + 1)^2$.

Problem 4. Let the number 143 586 729 be written on the blackboard. Consider any two neighbour digits of this number. In one step, we deduce by 1 both Considered digits. If these neighbour digits are greater than 0, then we continue the steps in the same way. What is the smallest number that one can obtain after several such steps?

Problem 5. Find the sum of all integer solutions of the following inequality

$$\log_2 x \geq \frac{|x-2| + |x+2|}{4}.$$

Problem 6. Evaluate the expression

$$\left[\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2^3}\right)\cdot\dots\cdot\left(1+\frac{1}{2^{2016}}\right)\right],$$

where by $[x]$ we denote the integer part of a real number x.

Problem 7. Let $x,\ y,\ z \geq 0$ and $x+y+z=1$. Find the greatest value of the expression

$$\sqrt{4x+(y-z)^2}+\sqrt{4y+(z-x)^2}+\sqrt{4z+(x-y)^2}.$$

Problem 8. Let $p(x)$ be a polynomial with real coefficients, such that for any x it holds true

$$x^4+4x^3-8x^2-48x-47 \leq p(x) \leq 2016|x^4+4x^3-8x^2-48x-47|.$$

Find $p(4)$.

Problem 9. Let $a,\ b,\ c,\ d$ be pairwise distinct complex numbers, such that $a+b+c+d \neq 0$. Given that $|a|=|b|=|c|=|d|=\sqrt{10}$ and $|a+b+c+d|=10$. Find the value of the expression $|abc+abd+acd+bcd|$.

Problem 10. Let $p(x)$ be a polynomial of degree four with integer coefficients, such that

$$2x^4-x^3+3x^2-36x-38 < p(x) < 2x^4-x^3+3x^2-36x-36,$$

holds true on some pairwise disjoint line segments for which the sum of the lengths is equal to 8. How many real solutions does the equation $p(x)=x^4-3x^3+2x^2-36x-36$ have?

Problem 11. Let sequence (x_n) be, such that $x_1=-1062000$ and

$$x_{n+1}=x_n+\sqrt{3844-4x_n}-1,\ n=1,2,\ldots$$

Find $|x_{1000}|$.

Problem 12. Find the number of all sets M with real elements, such that

a) M has three elements.
b) If $a \in M$, then $2a^2-1 \in M$.

3.15 Problem Set 15

Problem 1. Given that

$$x+\frac{1}{x}=1.$$

Evaluate the expression

$$x^4+\frac{1}{x^4}+\frac{4}{x}-\frac{4}{x^2}.$$

Problem 2. In how many ways can one insert numbers 9, 12, 16, 45 instead of $*$ in the following expression (inserted numbers can be equal)

$$\sqrt{* + \sqrt{* + \sqrt{* + \sqrt{*}}}},$$

such that the obtained number is a rational number?

Problem 3. Let x, y, z be such numbers that

$$(x-y)(y-z)(z-x) = -1.$$

Evaluate the expression

$$\frac{1}{(x-y)^3(y-z)^3} + \frac{1}{(y-z)^3(z-x)^3} + \frac{1}{(z-x)^3(x-y)^3}.$$

Problem 4. Let $*$ be a mathematical operation defined on the set of positive integers. Given that for any positive integers a, b it holds true $a*a = 2a$, $a*b = b*a$ and

$$a*(a+b) = a*b + \frac{a^2}{(a,b)},$$

where by (a,b) we denote the greatest common divisor of a and b. Find $24*10$.

Problem 5. Let $x^2+y^2 = x+y+xy$, where $x, y \in \mathbb{R}$. Find the greatest possible value of x^2+y^2.

Problem 6. Let the distance from the point $M(7,14)$ to the graph of the function $y = \sqrt[3]{x}$ is equal to d. Find d^2.

Problem 7. Evaluate the expression

$$\left[\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\cdot\ldots\cdot\left(1+\frac{1}{100^2}\right)\right],$$

where by $[x]$ we denote the integer part of a real number x.

Problem 8. Let x_0 be the greatest real solution of the equation

$$\frac{1}{(x-1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(2x+1)^2} = \frac{9}{4}.$$

Find $(4x_0+5)^2$.

Problem 9. Let (a_n) be such sequence that $a_1 = 1+2^{2016}$ and

$$a_{n+1} = 1.25a_n - 0.75\sqrt{a_n^2 - 2^{2018}},\ n = 1, 2, \ldots, 1008.$$

Find the remainder of division a_{1009} by 100.

Problem 10. Let sequence (x_n) be such that

$$x_1 = 25\sqrt{2},$$

$$x_2 = \frac{25\sqrt{2}}{\cos\frac{7\pi}{1200} + \sqrt{3}\sin\frac{7\pi}{1200}},$$

$$\dots$$

$$x_{n+2} = \frac{x_n x_{n+1}}{2\cos\frac{7\pi}{1200}\cdot x_n - x_{n+1}},\ n = 1,2,\dots,99.$$

Find x_{101}.

Problem 11. Let a, b, c, d be positive numbers, such that $c^2+d^2-a^2-b^2 = \sqrt{3}(cd-ab)$, $b^2+c^2-a^2-d^2 = ad+bc$ and $\sqrt{b^2+d^2}+\sqrt{a^2+c^2} > \sqrt{a^2+d^2+ad} + \sqrt{b^2+c^2-bc}$. Find the value of the following expression

$$\left(\frac{ab+cd+\sqrt{3}ad}{bc}\right)^2.$$

Problem 12. Let $p(x)$, $q(x)$, $r(x)$ be quadratic trinomials with real coefficients, such that for any x it holds true

$$|p(x)|+|q(x)| = |r(x)|.$$

Given that $p(1) = q(2) = 0$ and $r(3) = 9$, $r(4) = 29$. Find $r(10)$.

Chapter 4
Calculus

Introduction. In the chapter Calculus, there are many problems related to finding a sum or a product. In order to find a sum of the form $\sum_{k=1}^{n} a_k$, we use either the representation $a_k = b_{k+1} - b_k$, $k = 1, 2, \dots, n$ or the representation $a_k + a_{n-k+1} = c$, $k = 1, 2, \dots, n$, where c is a constant. In the similar way, in order to find a product of the form $\prod_{k=1}^{n} a_k$ we use the representation $a_k = \frac{b_{k+1}}{b_k}$, $k = 1, 2, \dots, n$.
Many problems are devoted to the estimation of the terms of the given sequence and finding the required limits. In order to find the limits of the functions, we often use the following theorems and proof techniques.

Theorem 4.1 (The Monotone Convergence Theorem). *If (x_n) is a monotone sequence of real numbers, then (x_n) is convergent if and only if (x_n) is bounded.*

Theorem 4.2. *Prove that*

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Theorem 4.3. *Prove that*

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e.$$

In this chapter, there are many problems concerning to even functions and odd functions. In order to deal with these type of problems, as a proof tool we will mainly use the following theorems.

Theorem 4.4. *Let $f(x)$ be an odd function. Prove that, if $f(x)$ is continuous on $[-a, a]$, then*

$$\int_{-a}^{a} f(x)\, d(x) = 0.$$

Theorem 4.5. *Let $f(x)$ be an increasing (non-decreasing) odd function defined on $\mathbb{R}$. Prove that the unique root of the equation $f(x+1) + f(x+2) + \cdots + f(x+n) = 0$ is the number $\frac{-n-1}{2}$.*

H. Sedrakyan and N. Sedrakyan, *The Stair-Step Approach in Mathematics*,
Problem Books in Mathematics, https://doi.org/10.1007/978-3-319-70632-0_4

In this chapter, we consider many examples of functional equations and inequalities. In few of them, the notation of the derivative of a function is introduced and used.

Theorem 4.6. *Let n be a positive integer,* $0 < a_1 < a_2 < \cdots < a_n$ *and* $c_1, c_2, \ldots, c_n$ *be real nonzero numbers. Prove that the number of solutions of the equation* $c_1 a_1^x + c_2 a_2^x + \cdots + c_n a_n^x = 0$ *is not more than the number of the negative terms of the sequence* $c_1 c_2$, $c_2 c_3, \ldots$, $c_{n-1} c_n$.

4.1 Problem Set 1

Problem 1. Evaluate the expression

$$\lim_{n\to\infty} \frac{4}{n^2}\left(1+\frac{6}{1\cdot 4}\right)\left(1+\frac{8}{2\cdot 5}\right)\cdot\cdots\cdot\left(1+\frac{2n+4}{n\cdot(n+3)}\right).$$

Problem 2. Let $f(x) = (x+1)\left(\frac{1}{2}x+1\right)\left(\frac{1}{3}x+1\right)\cdot\cdots\cdot\left(\frac{1}{2014}x+1\right)$.

Denote by $f^{(n)}$ the n^{th} derivative of a function f. Find $f^{(2014)}$.

Problem 3. Let $f(x) = (x-7)^{2014}$. Find the solution of the equation

$$f'(x) + f'''(x-2014) + f'''(x+2014) = 0.$$

Problem 4. Evaluate the expression

$$\int_0^2 \log_2(x + \sqrt{x^2-2x+2} - 1)dx.$$

Problem 5. Let the sequence (x_n) be such that $\lim\limits_{n\to\infty}(x_n - x_{n-1}) = 0$. Find $\lim\limits_{n\to\infty} \frac{x_n}{n}$.

Problem 6. Find the minimum value of a natural number a, such that the equation $x^3 + (3-a)x^2 + 3x + 1 = 0$ has three solutions in the set of real numbers.

Problem 7. Let the sequence (x_n) be such that $0 < x_1 < 1$ and $x_{n+1} = x_n(1-x_n)$. Find $\lim\limits_{n\to\infty}(nx_n)$.

Problem 8. Find the product of the solutions of the following equation $3^{\frac{x}{2}} - 2^{x-1} = 1$.

Problem 9. Evaluate the expression

$$\lim_{n\to\infty}\left(6\sin\left(\frac{\pi}{3}+\frac{1^2}{n^3}\right)+6\sin\left(\frac{\pi}{3}+\frac{2^2}{n^3}\right)+\cdots+6\sin\left(\frac{\pi}{3}+\frac{n^2}{n^3}\right)-3\sqrt{3}n\right).$$

4.2 Problem Set 2

Problem 1. How many integer values does the function $y = 8tgx\cos x$ have?

Problem 2. Evaluate the expression

$$\lim_{x\to 0}\frac{3^{tg50x}-1}{3^x-1}.$$

Problem 3. Find the greatest value of C in the set of real numbers, such that the inequality $|tgx - tgy| \geq C|x-y|$ holds true for any $x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Problem 4. Find the number of the solutions of the equation $8\cdot 2^x - 6\cdot 3^x + 5^x - 3 = 0$.

Problem 5. Evaluate the expression

$$\lim_{x\to\pi}\frac{40x\cos\frac{x}{2}}{\pi^2-x^2}.$$

Problem 6. Let $a_1,\ldots,a_n$ be the sides of a convex n-gon, which is in a unit square. Given that there exists $\alpha < 0$ such that $a_1^\alpha + \cdots + a_n^\alpha < 4$. Find all possible values of n.

Problem 7. Let (u_n) be a sequence of real numbers, such that $u_1 = 1, u_{n+1} = u_n + \frac{1}{u_n}, n = 1,2,\ldots$ Find $[50u_{100}]$, where by $[x]$ we denote the integer part of a real number x.

Problem 8. We call two infinite sequences (a_n) and (b_n) "similar", if $a_n \neq b_n$ only for finite number of values of n (any sequence is considered similar to itself). We call any convergent sequence (x_n) of real numbers "nice", if $x_{n+1} = x_n^2 - 2$, for any $n = 1,2,\ldots$ (where x_1 is any number). We have a list of m sequences, such that any "nice" sequence is "similar" to one of the sequences in the list. Find the minimum possible value of m.

Problem 9. Let $f(x)$ be a continuous function on $[0,1]$, such that

$$\int_0^1 f^2(x)dx = \frac{e^2}{2} + \frac{11}{6},$$

$$\int_0^1 f(x)e^x dx = \frac{e^2}{2} + \frac{1}{2}$$

$$\int_0^1 f(x)xdx = \frac{4}{3}.$$

Find $f(0)$.

4.3 Problem Set 3

Problem 1. Consider the function $y = \{x\} \cdot [x]$, where by $\{x\}$ and $[x]$ we denote the rational and integer part of a real number x, respectively. Find the number of real numbers, which do not belong to the domain of the function y.

Problem 2. Evaluate the expression

$$\lim_{x\to 0} \frac{4(\sqrt[20]{1+x}-1)}{\sqrt[15]{1+x}-1}.$$

Problem 3. Find the greatest value of C in the set of real numbers, such that the inequality $|lnx - lny| \geq C|x-y|$ holds true for any $x, y \in (0,1]$.

Problem 4. Find the greatest integer value of a, such that the equation $\sqrt{900-x^2} = \sqrt{x-a} + 7\sqrt{10}$ has a solution.

Problem 5. Evaluate the expression

$$\lim_{x\to 0} \frac{\ln(\sin 3x + \cos 3x)}{\ln(\sin x + \cos x)}.$$

Problem 6. Let f be a non-deceasing and continuous function on $[0,1]$, such that

$$\int_0^1 f(x)dx = 2\int_0^1 xf(x)dx.$$

Given that $f(1) = 10.5$. Find the value of $f(0) + f(0.5)$.

Problem 7. Evaluate the expression

$$\frac{3^{\frac{1}{103}}}{3^{\frac{1}{103}}+\sqrt{3}} + \frac{3^{\frac{2}{103}}}{3^{\frac{2}{103}}+\sqrt{3}} + \cdots + \frac{3^{\frac{102}{103}}}{3^{\frac{102}{103}}+\sqrt{3}}.$$

Problem 8. Let (u_n) be a sequence of real numbers, such that $u_1 = 10^9, u_{n+1} = \frac{u_n^2+2}{2u_n}$, $n = 1, 2, \ldots$ Find $[10^{13}(u_{36} - \sqrt{2})]$, where by $[x]$ we denote the integer part of a real number x.

Problem 9. Let $f : \mathbb{N} \to \mathbb{Z}$ and $f(1) = 5$. For any positive integers a, b, c given that $a+b+c \mid f(a)+f(b)+f(c)-3abc$. Find $f(9)$.

4.4 Problem Set 4

Problem 1. Let M and m be the greatest and smallest values of function $f(x) = \sqrt{x-5} + \sqrt{28-2x}$, respectively. Find the value of $\frac{M^{10}}{m^9}$.

Problem 2. Evaluate the expression

$$\lim_{x\to 0}(\cos x)^{ctgx}.$$

Problem 3. Let $f(x) = x + \sin x$ and $g(x) = e^{-x} - x^3$. Find the greatest integer solution of the inequality $f(g(f(x))) > f(g(f(\sqrt{2015})))$.

Problem 4. Evaluate the expression

$$\int_{2-\sqrt{3}}^{2+\sqrt{3}} \frac{3^{3-x} - 3^{x-1}}{2^{x-1} + 2^{3-x}} dx.$$

Problem 5. Find the solution of the following equation in the set of real numbers

$$\left(\frac{19x - 157}{6x^2 - 31x - 17}\right)'' = 0.$$

Problem 6. Evaluate the expression

$$\lim_{n\to\infty} \frac{16}{\pi}\left(arctan\frac{3}{1^2 + 3\cdot 1 + 1} + \cdots + arctan\frac{3}{n^2 + 3\cdot n + 1}\right).$$

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Given that for any x it holds true $f'(x)\cos x + (\cos x - \sin x)f(x) = 0$ and $f(0) = e^{\frac{\pi}{3}}$. Find $f\left(\frac{\pi}{3}\right)$.

Problem 8. Let (u_n) be a sequence of real numbers, such that $u_1 = 1, u_{n+1} = u_n + \frac{1}{u_n^2}$, $n = 1, 2, \ldots$ Find $[10u_{9000}]$, where by $[x]$ we denote the integer part of a real number x.

Problem 9. Let $f : (0, +\infty) \to (0, 1]$, $g : (0, +\infty) \to (0, 1]$ and g be a non-decreasing function. Given that for any positive x, y it holds true $f(x)f(y) = g(x)f(yf(x))$. Find the number of the solutions of the following inequality $f(x) > g(x)$.

4.5 Problem Set 5

Problem 1. Let a be a real number. It is called a "special" number, if the function $y = [x]\cdot\{x\}$ accepts the value equal to a at finitely many points. We denote by $\{x\}$ the fractional part of a real number x and by $[x]$ the integer part. Find the number of all "special" numbers.

Problem 2. Let us denote by $\{x\}$ the fractional part of a real number x. Evaluate the expression

$$\lim_{n\to\infty}\{(2+\sqrt{3})^n\}.$$

Problem 3. Find the smallest positive integer a, such that any $x \in \left[\frac{\pi}{2}, \pi\right]$ is a solution of the following inequality

$$\frac{a+5-(a^2-4)\cos x}{a^2+2a+2.5-0.5\cos 2x}<1.$$

Problem 4. Evaluate the expression

$$\lim_{x\to 0}\frac{24\arcsin(\sqrt{1+x}-1)}{\arcsin(\sqrt[3]{1+x}-1)}.$$

Problem 5. Find the sum of all such numbers a that for any of them the equation

$$|||||x-5|-1|-1|-1|-1|=x-a$$

has infinitely many solutions.

Problem 6. Evaluate the expression

$$\int_{0.5-\sqrt{3}}^{0.5+\sqrt{3}}\frac{\sqrt{3}\cdot 3^x}{3^x+\sqrt{3}}dx.$$

Problem 7. Given that a is such number that the following inequality

$$\log_a(2+\sqrt{x^2+ax+3})\log_3(x^2+ax+4)-\log_a 2\le 0$$

has a unique solution. Find a^2.

Problem 8. Let $u_0=0.001$, $u_{n+1}=u_n(1-u_n)$, $n=0,1,\ldots$. We denote by $[x]$ the integer part of a real number x. Find $[2000u_{1000}]$.

Problem 9. Let $f:(0,+\infty)\to(0,1]$, $g:(0,+\infty)\to(0,1]$. Given that f is non-increasing function, g is non-decreasing and $f(x)f(y)=g(x)f(yf(x))$ holds true for any $x,y>0$. Find the number of possible values of $\frac{f(2015)}{g(2015)}$.

4.6 Problem Set 6

Problem 1. Given that

$$f(x)=\frac{1}{x(x+1)}+\frac{2}{(x+1)(x+2)}+\cdots+\frac{999}{(x+999)(x+1000)}$$

and

$$g(x)=\frac{1}{x}+\frac{1}{x+1}+\cdots+\frac{1}{x+999}.$$

Evaluate the following expression $f(-1001)-g(-1001)$.

Problem 2. Evaluate the expression

$$\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt[3]{1+3x}}{1-\cos x}.$$

Problem 3. Given that

$$f(x) = \frac{x^2}{x^2-100x+5000}.$$

Evaluate the expression

$$f(1)+f(2)+\cdots+f(100).$$

Problem 4. Let

$$A = \pi^2 \int_0^1 \frac{\sin(\pi x)}{1+\sin(\pi x)}dx,$$

and

$$B = \int_0^{\pi} \frac{x\sin x}{1+\sin x}dx.$$

Find the value of the expression $\frac{A}{B}$.

Problem 5. Given that a is such a number that any number belonging to $[-1,3]$ is a value of the following function

$$f(x) = \frac{a+2\sin x+1}{\cos^2 x+a^2+|a|+1}.$$

Find a.

Problem 6. Given that the line, defined by the following equation $y = kx+b$, is tangent at two points to the graph of the following function

$$f(x) = x^4-6x^3+13x^2-6x+1.$$

Find k.

Problem 7. Let a be a real number, such that the equation

$$|2x+a|-|x-a|+|x-2a| = -x^2-ax-1.25a^2+5a-4$$

has only one real solution. Find a.

Problem 8. Let (u_n) be an increasing sequence of positive integers, such that

$$u_1^3+\cdots+u_n^3 = (u_1+\cdots+u_n)^2,\ n = 1,2,\ldots$$

Find the number of possible values of u_{2015}.

Problem 9. Let $p(x)$ and $q(x)$ be polynomials with real coefficients. Given that $q(x)$ is increasing polynomial function, $p(x)$ is a polynomial of even degree and the equation $p(x)-q(2x)=0$ has two real roots. Let $f:\mathbb{R}\to\mathbb{R}$ and $p(f(x+y))=q(f(x)+f(y))$, for any real numbers x,y. Find the number of all such functions f.

4.7 Problem Set 7

Problem 1. Given that

$$\lim_{n\to\infty}\left(\frac{(n+1)^4}{n^4+1}\right)^n=e^a.$$

Find a.

Problem 2. Find the number of elements of the range of the following function

$$f(x)=\arctan x+\arctan\frac{1-x}{1+x}.$$

Problem 3. Let $y=3x-5$ be the equation of the tangent line at point x_0 of function $f(x)$. Find the value of the first derivative of the function $\frac{f(x)}{x}+6f(x)+\frac{5}{x}-2x+7$ at point x_0.

Problem 4. Let f be a continuous function defined on $[0,1]$. Given that the following numbers $\int_0^1(f(x))^{2014}dx$, $\int_0^1(f(x))^{2015}dx$, $\int_0^1(f(x))^{2016}dx$ make an arithmetic progression. Find the value of the following expression

$$\int_0^1(f(x))^2+(1-f(x))^2dx.$$

Problem 5. Let

$$f(x)=\frac{a+\sin x-\cos x}{a+\sin x+\cos x}.$$

How many values of a are there, such that $f(-x)=-f(x)$, for any $x\in[-\frac{\pi}{4},\frac{\pi}{4}]$?

Problem 6. Let

$$f(x)=\frac{6x+21+28\sqrt{3x+2}}{9x+18-12\sqrt{3x+2}}.$$

Find the value of the following expression $f(f(1))+f(f(2))+\cdots+f(f(40))$.

Problem 7. Find the value of the following expression

$$\lim_{n\to\infty}\left(16n-\frac{64}{\pi}\cdot\arctan\frac{1^2+3\cdot1-2}{1^2+3\cdot1+4}-\cdots-\frac{64}{\pi}\cdot\arctan\frac{n^2+3\cdot n-2}{n^2+3\cdot n+4}\right).$$

Problem 8. Find the sum of all the possible integer values of a, such that for any of them the following equation

$$a^2+10|x-1|+\sqrt{3x^2-6x+4}=17a-3|2x-3a|-20$$

has at least one solution.

Problem 9. Find the number of all polynomials $p(x)$ with integer coefficients, such that the inequality $x^2 \le p(x) \le x^4+1$ holds true for any x.

4.8 Problem Set 8

Problem 1. Evaluate the expression

$$\lim_{x\to 0}\frac{27\pi(\sqrt[3]{10x+27}-3)}{\sin(\pi x)}.$$

Problem 2. At how many points does the function

$$f(x)=\cos x+\cos(\sqrt{2}x)$$

accept the value $f(0)$?

Problem 3. Evaluate the expression

$$\lim_{x\to 0}\frac{\ln(1+\tan(50x))}{\sin x}.$$

Problem 4. Find the sum of all integer numbers belonging to the range of the function

$$f(x)=\sqrt{16-x}+\sqrt{9+x}.$$

Problem 5. Evaluate the expression

$$\lim_{x\to 0}\frac{\sin(2\sin(3\sin 4x))}{\arcsin(\arcsin(\arcsin x))}.$$

Problem 6. Find the value of the following sum

$$\int_0^1 (x^3+1)^5\,\mathrm{d}x+\int_1^{32}\sqrt[3]{\sqrt[5]{x}-1}\,\mathrm{d}x.$$

Problem 7. Let f be a continuous function defined on $[0,1]$. Given that the numbers $\int_0^1 f^{2014}(x)\,\mathrm{d}x$, $\int_0^1 f^{2015}(x)\,\mathrm{d}x$, $\int_0^1 f^{2016}(x)\,\mathrm{d}x$ make a geometric progression. Find the value of the expression

$$\frac{f(0)+100f(0.5)+200f(1)}{f(0.25)}.$$

Problem 8. Find the smallest value of the function

$$f(x)=\frac{3-2x+3x^2}{\sqrt[3]{(x^3-9x^2+3x-3)^2}}.$$

Problem 9. Let $f:\mathbb{R}\to\mathbb{R}$. Given that

$$\lim_{x\to 0}\left(3\frac{f(4x)}{x}-5\frac{f(2x)}{x}+2\frac{f(x)}{x}\right)=4,$$

and

$$\lim_{x\to 0} f(x)=0.$$

Find the value of

$$\lim_{x\to 0}\frac{f(x)}{x}.$$

4.9 Problem Set 9

Problem 1. Evaluate the expression

$$\lim_{x\to \frac{\pi}{2}}(\sin(5x))^{\tan x}.$$

Problem 2. Find the smallest solution of the inequality

$$\frac{2^x+3^x+4^x}{5^x+6^x}\leq\frac{29}{61}.$$

Problem 3. Given that

$$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{2}{n}\right)^{n+2}\cdot\cdots\cdot\left(1+\frac{10}{n}\right)^{n+10}\right)=e^a.$$

Find a.

Problem 4. Find the number of all possible values of a, such that for any a the function

$$f(x)=\ln(\sqrt{x^2+1}+ax),$$

is an odd function.

Problem 5. Find the values of the function

$$f(x) = \cos x + \cos(\sqrt{2}x),$$

such that the function accepts any of those values at finite number of points.

Problem 6. Let function $f(x)$ be defined on $\mathbb{R}$ and be non-decreasing. Given that for any x it holds true

$$f(x^2 - 23x + 144) \geq f^2(x) - 23f(x) + 144.$$

Find $f(2015)$.

Problem 7. Find all possible positive values of a, such that the equation

$$x^2|x+a| = 3a + 14,$$

has three solutions.

Problem 8. Let function $f(x)$ be defined on $\mathbb{R}$ and be infinitely differentiable. Given that $f(0) = f(101) \neq f(1)$ and for any x it holds true

$$f'''(x) + 3f''(x)f'(x) + (f'(x))^3 = 0.$$

Find the value of the following expression

$$\frac{|f(1)| + |f(2)| + \cdots + |f(100)|}{|f(1)| + |f(2)| + \cdots + |f(50)|}.$$

Problem 9. Let $f : (0, +\infty) \to \mathbb{R}$ and $f(2) \neq 0$. Given that for any positive numbers x, y it holds true

$$xf(x) - yf(y) = xy^2 f\left(\frac{x}{y}\right).$$

Find $\frac{4f(8)}{f(2)}$.

4.10 Problem Set 10

Problem 1. Let

$$x_n = \left(\frac{[n + \sin n]}{n + \sin n}\right)^{\frac{n+\sin n}{\{n+\sin n\}}}.$$

Evaluate the following expression

$$e \lim_{n\to\infty} x_n,$$

where $[x]$ is the integer part and $\{x\}$ is the fractional part of a real number x.

Problem 2. Find all possible values of a, such that for any of them the function

$$f(x) = \ln(\sqrt{x^2+1}+ax)$$

is even.

Problem 3. Evaluate the expression

$$\lim_{x\to+\infty}(\sin(\sin(\sin\sqrt{x^2+1})) - \sin(\sin(\sin x))).$$

Problem 4. Given that point $M(x_0, y_0)$ is a centre of symmetry of the graph of function

$$y = \sqrt[3]{x+1} + \sqrt[3]{x+2} + \sqrt[3]{x+3} + 100.$$

Find $x_0 + y_0$.

Problem 5. Evaluate the expression

$$\int_0^{\frac{\pi}{2}} \frac{2\sin x - \cos x - \cos 3x}{\sin x + \cos x}\,dx.$$

Problem 6. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$. Given that function g is monotonic on $\mathbb{R}$ and for any x, y it holds true

$$f(x) - f(y) \le |x-y||g(x)-g(y)|.$$

Find the number of possible values of $f(2015)$, if $f(1) = 1$.

Problem 7. Given that the domain of the function

$$f(x) = \sqrt{x^3 - ax^2 + (a+b)x - 1 - b} + \sqrt[4]{-x^3 + (c+15)x^2 - (15c+26)x + 26c}$$

consists of three points. Find the possible smallest value of $a+b+c$.

Problem 8. Find the smallest value of function $f(x) = 4^{\sin x} + 4^{\cos x}$ on $[0, \frac{\pi}{2}]$.

Problem 9. Find the sum of the cubes of the roots of the following equation

$$2^{x+3} - 2\cdot 3^{x+1} + 5^x = 3.$$

4.11 Problem Set 11

Problem 1. Given that

$$\lim_{x\to\frac{\pi}{2}}(1+\cos(13x))^{\tan x} = e^a.$$

Find a.

Problem 2. Find the smallest value of a, such that the function

$$f(x) = \ln(x^{10}+1) - a\ln(|x|+1)$$

is upper bounded.

Problem 3. Find the smallest value of the function

$$f(x) = \Big(\frac{1}{\sqrt{1-x^2}}\Big)^3 + \Big(\frac{1}{1-\sqrt{1-x^2}}\Big)^3.$$

Problem 4. Find the positive value of a, such that the equation

$$x^2 - 3ax + 4a|x-3a| + 12.5a^2 - 1250 = 0.$$

has a unique solution.

Problem 5. Solve the equation

$$\sqrt[6]{x} + \sqrt[3]{4\sqrt{x}-5} = \sqrt{6\sqrt[3]{x}+1}.$$

Problem 6. Evaluate the expression

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos 3x}{\sin x + \cos x}dx.$$

Problem 7. Find the greatest value of a, such that the sequence

$$x_1 = a, \;\; x_{n+1} = 24x_n - x_n^3, n = 1,2,\dots$$

is bounded.

Problem 8. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$. Given that $f(1)=1$, function g is monotonic on $\mathbb{R}$ and for any numbers x, y it holds true the following inequality

$$f(x) - f(y) \le |\arctan x - \arctan y||g(x) - g(y)|.$$

Find the value of $f(1) + 2f(2) + \cdots + 40f(40)$.

Problem 9. Let $p(x)$, $q(x)$ and $r(x)$ be polynomials with real coefficients, such that

$$p(q(x)) + p(r(x)) = x^2,$$

for any x. Given that $q(1)+r(1)=1$. Find the greatest possible value of the degree of polynomial $q(x)+r(x)$.

4.12 Problem Set 12

Problem 1. Evaluate the expression

$$\lim_{x\to 0}\frac{|x|^{x+2}}{\sqrt{1+x^2}-1}.$$

Problem 2. Find the number of real numbers a, such that the equation

$$|\ln(x+\sqrt{x^2+1})|=a^2-1,$$

has a unique solution.

Problem 3. Evaluate the expression

$$\lim_{x\to 0}\frac{\sin\tan x-\tan\sin x}{x^3}.$$

Problem 4. Find the number of roots of the equation

$$\sin x=\frac{x^3-3x^2+5x}{4-3x}.$$

Problem 5. Find the number of functions $f:\mathbb{R}\to\mathbb{R}$, such that (for each of them) holds true $f(x)-f(y)>\sqrt{x-y}$ for any x, y satisfying the condition $x>y$.

Problem 6. Find the number of real numbers a, such that the equation

$$x^2-2ax+4a|x-a|+2a^2-16=0,$$

has three solutions.

Problem 7. Let sequence (x_n) be such that $x_0=x_{k+1}=0,\ \ x_i>0$ and

$$x_{i+1}>\frac{\sqrt{6}+\sqrt{2}}{2}x_i-x_{i-1},\ \ i=1,2,\dots,k.$$

Find the possible smallest value of k.

Problem 8. Let real number a, b be, such that the inequality

$$|x^2-ax-b|\le\frac{1}{8},$$

holds true for any $x\in[1,2]$. Find the value of $20a+16b$.

Problem 9. Let function $f:(0,\pi)\to\mathbb{R}$ be increasing on $(0,\pi)$, has a second-order derivative and the equation

$$(f(x))^2+(f'(x))^2=1$$

holds true for any $x\in(0,\pi)$. Find the number of all such functions f.

Problem 10. Let function $f:[0,1]\to\mathbb{R}$ be non-decreasing on $[0,1]$ and continuous. Given that

$$\int_0^1 \arcsin x(f(x)-f(\sin x))(f(x)-f(\sin(\sin x)))dx+$$

$$+\int_0^1 \sin x(f(\sin(\sin x))-f(x))(f(\sin(\sin x))-f(\sin x))dx=$$

$$=\int_0^1 x(f(x)-f(\sin x))(f(\sin x)-f(\sin(\sin x)))dx.$$

Find the number of possible values of $f(1)-f(0)$.

Problem 11. Let $x_1=a$, $x_{n+1}=2016x_n^3-2015x_n$, $n=1,2,\dots$ Find the number of real numbers a belonging to $[-1,0)$, such that for any a the elements of sequence (x_n) are negative.

Problem 12. We call a number c “sympathetic”, if there exists a function $f:(0,+\infty)\to(0,+\infty)$, such that it is continuous on $[2016,+\infty)$ and the inequality

$$\frac{f(x+f(x))}{f(x)}\le c,$$

holds true for any positive x. Find the smallest “sympathetic” number.

4.13 Problem Set 13

Problem 1. Given that

$$\lim_{x\to\infty}\left(\frac{x^3-1}{x+5}-ax^2-bx-c\right)=0.$$

Find $a+b+c$.

Problem 2. Find the product of all numbers belonging to the domain of function

$$f(x)=\sqrt{-x^2+11x-24}+\sqrt{\log_2(\cos\pi x)}.$$

Problem 3. Evaluate the expression

$$\int_0^{\pi}\ln\frac{2-\cos x}{2+\cos x}dx.$$

Problem 4. Evaluate the expression

$$\lim_{x\to 0}\frac{\sin(\tan x)-\ln(1+x)}{\sin^2\frac{x}{10}}.$$

Problem 5. Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable function. Given that

$$\int_0^1 f(x)dx = f(1)-\frac{2\sqrt{3}}{3}.$$

Find the smallest possible value of the expression

$$\int_0^1 (f'(x))^2 dx.$$

Problem 6. Given that point $M(x_0,y_0)$ is the centre of symmetry of the graph of function

$$f(x)=x+301+\log_2\frac{x^3-13x^2+54x-72}{x^3+x^2-2x}.$$

Find x_0+y_0.

Problem 7. Let x_1, x_2 be, respectively, the smallest and greatest solutions of the equation

$$\left(\frac{4}{3}\right)^{\cos x}=\sin x,$$

in the interval $[0,2\pi]$. Find $\frac{3x_2}{x_1}$.

Problem 8. Evaluate the expression

$$\frac{100}{\pi}\cdot\lim_{n\to\infty}\left(\arctan\frac{1}{2}+\arctan\frac{2}{11}+\cdots+\arctan\frac{2n+2}{n^4+4n^3+7n^2+6n+4}\right).$$

Problem 9. Given that $\frac{2\pi}{5}$ is the period of function

$$f(x)=a\sin^5 x+10\sin^3 x+b\sin x.$$

Find ab.

Problem 10. Find the difference of the greatest and smallest values of the function

$$f(x)=\frac{20}{\pi}\left(\arcsin x+\arcsin\left(x\sqrt{1-x^2}-\sqrt{3}x^2+\frac{\sqrt{3}}{2}\right)\right).$$

Problem 11. Let function $f:(0,\pi)\to\mathbb{R}$ and the inequality

$$f(x)\sin y - f(y)\sin x \le \sqrt[3]{(x-y)^4},$$

holds true for any numbers x, y. Given that $f\left(\frac{\pi}{2}\right)=100\sqrt{3}$. Find $f\left(\frac{2\pi}{3}\right)$.

Problem 12. Let C be the smallest real number, such that the inequality

$$a^{12}+(ab)^6+(abc)^4+(abcd)^3 \leq C(a^{12}+b^{12}+c^{12}+d^{12}),$$

holds true for any real numbers a, b, c, d. Find $[100C]$, where we denote by $[x]$ the integer part of a real number x.

4.14 Problem Set 14

Problem 1. Let

$$f(x)=\log_2 \frac{2^x+1}{2^x-1}.$$

Evaluate the expression

$$f(f(1))+f(f(2))+\cdots+f(f(40)).$$

Problem 2. Let a, b, c, d be real numbers. Given that sequence

$$x_n=\sqrt[3]{an^3+bn^2+cn+d}-3n-1,$$

is convergent. Find a.

Problem 3. Evaluate the expression

$$\int_{-4}^{4} \frac{\sqrt{1+x^2+x^4}}{1+x+x^2+\sqrt{1+x^2+x^4}}dx.$$

Problem 4. Find the sum of all positive integers a less than 30, such that the domains of functions

$$y=\frac{\sin x}{9\cos^2 x-1},$$

and

$$y=\frac{1}{9\cos^2 x-1}+\frac{1}{9\cos 2x+a},$$

coincide.

Problem 5. Given that m, n, k are such real numbers, that

$$\int \sin x \ln(1+\sin x)dx = mx+n\cos x+k\cos x\ln(1+\sin x)+C.$$

Find $m^2+n^2+k^2$.

Problem 6. Evaluate the expression

$$\lim_{x\to 0}\frac{\tan(e^x-1)-\ln(1+\sin x)}{\sqrt[3]{1+x^2}-1}.$$

Problem 7. Let non-decreasing, continuous function $f(x)$ be defined on $\left[0,\frac{\pi}{2}\right]$. Given that

$$\int_0^{\frac{\pi}{2}} f(x)dx = 10,$$

and

$$\int_0^{\frac{\pi}{2}} f(x)\sin^2 x dx = 5.$$

Find

$$\int_0^{\frac{\pi}{4}} f(x)dx.$$

Problem 8. Let (x_n) be a sequence of positive real numbers, such that

$$\lim_{n\to\infty}\frac{\sqrt{1+x_{n+1}}-31}{\sqrt{1+x_n}-31}=0.1.$$

Find the limit of the sequence (x_n).

Problem 9. Solve the equation

$$\log_{25}(625x)\log_{25}(\log_{25}x)=\log_{25}x\log_{25}(\log_{25}(625x)).$$

Problem 10. Evaluate the expression

$$\frac{2016}{\pi}\sum_{n=1}^{\infty}\arccos\frac{n^2+n+3}{\sqrt{n^2+3}\cdot\sqrt{n^2+2n+4}}.$$

Problem 11. Let $x_1, x_2, \ldots, x_{2016}\geq 0$ and $x_1+x_2+\cdots+x_{2016}=1$. Given that the greatest value of the expression

$$x_1^5(1-x_1)+x_2^5(1-x_2)+\cdots+x_{2016}^5(1-x_{2016}),$$

is equal to M. Find $(27M+14)^2$.

Problem 12. Let continuous function $f(x)$ be defined on $[1,17]$. Given that the equation

$$\frac{f^2(x)-24f(x)+x^2-18x+125}{\sqrt{f^2(x)-36f(x)+x^2-2x+325}\cdot\sqrt{f^2(x)-12f(x)+x^2-34x+325}}=$$

$$= \frac{f^2(y) - 24f(y) + y^2 - 18y + 125}{\sqrt{f^2(y) - 36f(y) + y^2 - 2y + 325} \cdot \sqrt{f^2(y) - 12f(y) + y^2 - 34y + 325}}.$$

holds true for any numbers x, y belonging to $(1, 17)$. Given also that $f(10) = 15$. Find $2|f'(15)|$.

4.15 Problem Set 15

Problem 1. Let $f(x) = x^{\sqrt[3]{2}}$. Evaluate the expression

$$f(f(f(1))) + f(f(f(2))) + \cdots + f(f(f(13))).$$

Problem 2. Evaluate the expression

$$\lim_{n\to\infty} \left(\frac{1^2 - 2\cdot 1 + 3}{1^2 + 2\cdot 1 + 3} \cdot \frac{2^2 - 2\cdot 2 + 3}{2^2 + 2\cdot 2 + 3} \cdot \cdots \cdot \frac{n^2 - 2\cdot n + 3}{n^2 + 2\cdot n + 3} \cdot n^4 \right).$$

Problem 3. Let $f(x) = \frac{x}{\sqrt{1-x^2}}$. Find $\underbrace{f(f(\ldots(f}_{2016}\left(\frac{1}{\sqrt{2017}}\right))\ldots))$.

Problem 4. Find the sum of all integer numbers belonging to the range of the function

$$f(x) = \sqrt{x-1} + \sqrt{11-x} + \cos \pi x.$$

Problem 5. Given that

$$f(x) = (x^2 + x + 1)^{20}(x^2 - x + 1)^{10}.$$

Evaluate the expression

$$\frac{6f^{(59)}(1)}{f^{(60)}(1)}.$$

Problem 6. Find the number of all couples (a, b), where $a, b \in \mathbb{R}$, such that

$$f(x) = \ln \frac{3 + a\sin x}{b + 5\sin x},$$

is an odd function.

Problem 7. Evaluate the expression

$$\int_{-1}^{5} \frac{1}{1 + 2^{\sqrt[3]{x-2}}}\, dx.$$

Problem 8. Evaluate the expression

$$\frac{100}{\pi}\lim_{n\to\infty}\left(\arccos\frac{1^4+2\cdot 1^3+3\cdot 1^2+2\cdot 1}{1^4+2\cdot 1^3+3\cdot 1^2+2\cdot 1+2}+\cdots+\arccos\frac{n^4+2n^3+3n^2+2n}{n^4+2n^3+3n^2+2n+2}\right).$$

Problem 9. Find the smallest positive integer number a, such that there exists a function $f:\mathbb{R}\to\mathbb{R}$ satisfying the following assumptions

a) function $f(x)$ is twice differentiable at any point.

b) $f(-a)=f(a)=0$.

c) $f(x)>0$, if $x\in(-a,a)$.

d) $f(x)+7f''(x)\geq 0$, if $x\in(-a,a)$.

Problem 10. Let $f:\mathbb{R}\to\mathbb{R}$ and for any real numbers x, y the following inequality holds true

$$f(y)(1-\ln(1+(x-y)^2))\leq f(x).$$

Given that $f(3)=14$. Find $f(14)$.

Problem 11. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable and $f'(x)$ be continuous in $\mathbb{R}$. Given that $f(4)-f(0)=20$ and

$$\int_0^4 (f'(x))^2dx=100.$$

Find $f(3)-f(1)$.

Problem 12. Let $f:\mathbb{R}\to\mathbb{R}$ and for any real numbers x, y, z holds true the following inequality

$$|f(x)-f(y)||x+y|\leq|f(x)-f(z)||x+z|+|f(z)-f(y)||z+y|.$$

Given that $f(0)=-5$ and $f(20)=16$. Find $f(40)$.

Chapter 5
Combinatorics

Introduction. In the chapter Combinatorics, the majority of the problems can be treated by the basic rules of counting.
The rule of sum/Addition principle. If there are n choices for one action, and m choices for another action and the two actions cannot be done at the same time, then there are $n+m$ ways to choose one of these actions.
The rule of product/Multiplication principle. If there are n choices for one action, and m choices for another action after that, then there are $n \cdot m$ ways to perform both of these actions.
Let us now list some basic terms and definitions used in this chapter.
A *permutation* of a set of objects is an ordering (rearrangement) of those objects. There are two types of *permutations with repetition* and *permutations without repetition*.
A *permutation with repetition* (or r-tuple) is an ordered selection of r elements from a set of n elements in which repetition is allowed.

Theorem 5.1. *The number of permutations with repetition for selecting r elements from a set of n elements in which repetition is allowed is equal to n^r.*

The number of permutations of obtaining an ordered subset of r elements from a set of n elements is denoted by nP_r.

Theorem 5.2. *Prove that $nP_r = \frac{n!}{(n-r)!} = n(n-1)\cdot\dots\cdot(n-(r-1))$, where $n! = 1\cdot 2\cdot\dots\cdot n$.*

The number of permutations of obtaining an ordered subset of n elements from a set of n is denoted by P_n.

Theorem 5.3. *Prove that $P_n = n!$.*

The number of ways of obtaining an unordered subset of r elements from a set of n is called *combination* and is usually denoted by nC_r or $\binom{n}{r}$ or C_n^r.
Hence, one could say that a permutation is an ordered combination.

H. Sedrakyan and N. Sedrakyan, *The Stair-Step Approach in Mathematics*,
Problem Books in Mathematics, https://doi.org/10.1007/978-3-319-70632-0_5

Theorem 5.4. *Prove that* $nC_r = \frac{n!}{r!(n-r)!}$.

5.1 Problem Set 1

Problem 1. How many three-digit numbers (with distinct digits) are divisible by 11?

Problem 2. There are three boys and seven girls. In how many ways is it possible to divide them into three different groups, such that each group consists of a boy, two groups consist of three people and the third group consists of four people?

Problem 3. In how many ways is it possible to put the book series of two different authors consisting of three and four books in the bookshelf, such that three books of the first author are in the correct consecutive order?

Problem 4. Unknown size chess board is divided into n parts, each part has the following form . Given that n is an odd number. Find the possible minimum value of n.

Problem 5. There are 10 knights and n rooks on a chess board, such that none of them is under attack. Find the greatest possible value of n.

Problem 6. The sides of some squares of a chess board are painted in red. How many sides at least one needs to paint, so that each square has at least two red sides?

Problem 7. Given 999×1000 chess board. It is necessary to paint some of squares (but not all of them), such that each unpainted square has exactly one painted *neighboring* square (we call two squares neighbouring squares, if they have a common side). How many unpainted squares at least can have this board?

Problem 8. Given 4×4 chess board. Find the minimum number of sides (corresponding to the squares of the board), which need to be removed from the board, so that the rest of the figure could be drawn using a pencil and without lifting the pencil from the board (it is forbidden to pass over the same side twice).

Problem 9. Given seven points on the plane, the distance between them is expressed by numbers $a_1, a_2, \ldots, a_{21}$. What is the maximal number of times that we may have the same number among those 21 distances?

5.2 Problem Set 2

Problem 1. There are five teachers of mathematics, three teachers of physics and two teachers of chemistry. We choose some of them, such that there is a teacher chosen corresponding to each subject. In how many ways can we make such a choice?

Problem 2. Consider the points $A_i(i,1)$, $i = 1,2,\ldots,15$ and $A_i(i-15,4)$, $i = 16,17,\ldots,30$. Find the number of all isosceles triangles, with all vertices in a given set of points $A_1, A_2, \ldots, A_{30}$.

Problem 3. How many rooks at maximum can be placed on a chess board so that each rook is not under attack of more than one rook?

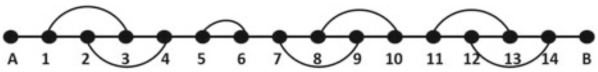
Problem 4. In how many ways can one divide the set of natural numbers into two disjoint subsets N_1 and N_2 satisfying the following condition: the difference of any two numbers from the same subset is not a prime number greater than 100?

Problem 5. Let some of the sides of the squares of a chess board be painted in red. What is the smallest number of sides, which need to be painted, so that each square has at least three red sides.

Problem 6. Consider 27 points on the plane, such that no three of them lie on a straight line. Some of these points are connected by segments, such that their number is more than 325. Given that for any four points A, B, C, D the following condition holds true: if A and B are connected, B and C are connected and C and D are connected, then A and D are connected as well. Find the number of all segments.

Problem 7. Let the entries of 16×16 table be real numbers. Given that the sum of the numbers of each 4×4 square is not negative and the sum of the numbers of each 5×5 square is not positive. Denote the sum of all the entries of 16×16 table by D. Find the number of all the possible values of D.

Problem 8. In how many ways can one go from the city A to the city B, if it is possible to pass through each of the places $1, 2, \ldots, 14$ not more than once.



Problem 9. A ten-digit number $\overline{a_1 a_2 \ldots a_{10}}$ is called "interesting", if its all digits are nonzero and each of $\overline{a_1 a_2}, \overline{a_2 a_3}, \overline{a_3 a_4}, \ldots, \overline{a_9 a_{10}}$ is divisible by one of the numbers 13, 17, 23, 37. Denote by n the number of all "interesting" ten-digit numbers. Find $\frac{n}{6}$.

5.3 Problem Set 3

Problem 1. Find the number of all six-digit positive integers with the distinct digits which are divisible by 11 and consist of the digits $1, 2, 3, 4, 5, 8$.

Problem 2. Find the number of all three-digit positive integers with distinct digits, such as for any of them the biggest digit is the middle one.

Problem 3. Find the number of all four-digit positive integers $\overline{abcd}$ with the digits $1, 2, 3, 4, 5$, such as $\overline{ab} \neq 23$ and $\overline{bc} \neq 23$ and $\overline{cd} \neq 23$.

Problem 4. Find the number of all positive divisors of 2015^8, such as any of them is not a square of a positive integer.

Problem 5. Find the number of all seven-digit numbers, which start with the digit 1 and end with the digit 9, such as the difference of any two neighbour digits is 1 or 2.

Problem 6. Let n be a positive integer. Given that after placing (in a random way) n dominos on a chess board there exists 2×2 square which does not have a square

covered by any domino. Each domino covers exactly two squares of a chess board. Find the possible greatest value of n.

Problem 7. Given that for any n black figures on a chess board there exists a square, such that the white knight placed on that square does not attack any of these black figures. Find the possible greatest value of n.

Problem 8. Consider 4×9 rectangular grid. In how many ways can it be covered by L-shape trominos, such that each tromino covers exactly three squares and each square is covered by exactly one tromino?

Problem 9. Given that from any n squares of a chess board one can choose two squares, such that a knight needs at least three steps in order to go from the one square to the other one. Find the possible smallest value of n.

5.4 Problem Set 4

Problem 1. At least, with how many L-shape trominos one can cover a chess board, such that each tromino covers exactly three squares and each square is covered by the same number of trominos?

Problem 2. Find the number of all three-digit numbers, such that for each of them its digits are side lengths of some triangle.

Problem 3. Consider 12 cards, such that on three of them is written the letter A, on three of them the letter B, on three of them the letter C and on the last ones the letter D. We choose 4 cards among them and put in some order. How many distinct combinations of letters can we have after such a choice?

Problem 4. A teacher invites guests to celebrate his birthday. Given that of any five guests there exist two guests who have met each other at the teacher's house. Given also that every guest visits teacher's birthday only once. At least with how many photos can he have the pictures of all guests?

Problem 5. Consider 10×11 size rectangle drown on paper. At most, how many 1×7 size rectangles can one cut from 10×11 rectangle, such as the sides of any of them are parallel to the sides of 10×11 rectangle?

Problem 6. At most, how many numbers can one choose among the numbers $1, 2, \ldots, 2012$, such that the difference of any two chosen numbers is not a prime number?

Problem 7. Consider a convex dodecagon (12 sided polygon) and its diagonals. Given that one cannot choose three diagonals, such that they have the same intersection point (located in the interior part of the dodecagon). Find the number of all triangles, such that for any of them all the sides are on the diagonals and any vertex is not one of the vertices of the dodecagon.

Problem 8. Without lifting a pencil from the paper how many line segments one should draw, such that they include all 144 vertices of some 11×11 square grid?

Problem 9. Find the minimum value of n, such that for any positive integers a, b, ($a > 6$, $b > 6$) one can cut not more than n squares from $a \times b$ paper rectangle grid, in a way that the rest of the figure one can cut into 1×7 rectangles.

5.5 Problem Set 5

Problem 1. Find the number of all three-digit numbers $\overline{abc}$, such that a quadratic expression ax^2+bx+c has a integer root.

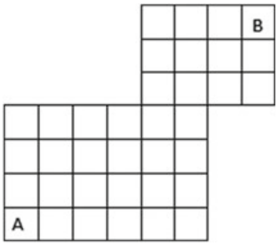
Problem 2. Find the number of all six-digit numbers, with distinct digits and belonging to the set $\{1,2,3,4,5,6\}$, such that for each of those six-digit numbers digits 1 and 2, 5 and 6 are not consecutive digits.

Problem 3. In how many ways can one put six identical books of mathematics and four different books of physics in a bookshelf, such that on one side of each book of physics the neighbouring-book is a book of mathematics and on the other side the neighbouring-book is a book of physics?

Problem 4. Two friends agreed to go to a cafe and meet there at some point in time starting from 6 PM till 8 PM. If any of them comes not-later than half past seven, then he needs to wait 30 minutes. If any of them comes later than half past seven, then he needs to wait till 8 PM. Let p be the probability of their meeting in the cafe. Find $16p$.

Problem 5. Let the entries of 4×4 grid square be numbers 0 or 1, such that every column sum is equal to 2 and every row sum is equal to 2. How many such 4×4 grid squares are there?

Problem 6. Two squares are called neighbouring if they have a common side. In how many ways can one go from the square A to the square B? (see the figure below). From each square, one can move either to the up or right neighbouring square.



Problem 7. A set is called a "good" set, if its elements belong to $\{1,2,\ldots,10\}$ and their sum is divisible by 4. Find the number of all "good" subsets of the set $\{1,2,\ldots,10\}$.

Problem 8. At least, how many 1×1 squares are required in order to cover any $n\times n$ square by 1×1, 2×2, 3×3 squares?

Problem 9. Consider 11×11 paper grid square. One cuts from it several figures of this form . Given that all the cuts are done along the sides of the squares. At least, how many squares could be left (after the cut) in the initial paper grid square?

5.6 Problem Set 6

Problem 1. In how many different ways can one put into a bookshelf 7 numbered books of the same author, such that book number 2 is between books number 1 and 3?

Problem 2. In how many different ways can one go from the city A to the city B and come back (see the figure below), if it is known that the ways for going and coming back do not include parts with the same numbers (e.g., the ways $1, 4, 8, 10, 5, 2$).

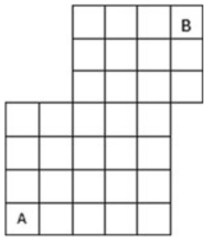
Problem 3. In how many different ways can one read the word "alternation"?

ALTERNATION
LTERNATION
TERNATION
ERNATION.

We start to read the first letter, then we read the letter on the right side or down of that letter.

Problem 4. Let the entries of 3×3 grid square be numbers 0 or 1, such that the total sum is an even number. How many such squares are there?

Problem 5. In how many ways can one go from the square A to the square B (see the figure below), if it is allowed to move either to the right or up "neighbouring" squares? Two squares are called "neighbouring" squares, if they have a common side.



Problem 6. At most, how many trominos can one put on 8×14 grid rectangle, such that each tromino covers exactly three squares of the grid rectangle and any two trominos do not have a common point.

Problem 7. At least, how many squares one needs to remove from 2011×2015 grid rectangle, such that the rest of the figure is possible to cut into 2×2, 3×3 squares?

Problem 8. On some squares of 150×150 grid square, one puts some playing cards (one playing card per square). Given that, for any playing card, the row or the column including that playing card has no more than three playing cards. At most, how many playing cards can one put on 150×150 grid square?

Problem 9. A figure on the chess board is called a "fast" knight, if it makes simultaneously two steps of a regular knight. At most, how many fast knights can one put on the chess board, such that none of them is under attack?

5.7 Problem Set 7

Problem 1. In how many different ways can one put into a bookshelf 6 numbered books of the same author, such that the books number 1 and 3 are not (simultaneously) the neighbours of the book number 2?

Problem 2. At most, how many numbers can one choose from the numbers $1, 2, \ldots, 14$, such that pairwise subtractions of all the chosen numbers are distinct positive numbers?

Problem 3. On each edge of a triangular pyramid are given four points. Let us consider these 24 points and four vertices of the pyramid. Find the number of all lines passing through any two points from those 28 points.

Problem 4. During the hockey championship, any two teams of 10 participant teams has played with each other exactly once. Given that all the teams have obtained different final scores. At most, how many games could win the team on the last place? Note that, in hockey, the victory is 2 scores, draw is 1 score, defeat is 0 score.

Problem 5. The entries of 8×8 grid square are positive integers. Given that in any two squares having a common side are written numbers, such that the ratio of the greater one to the smaller one is equal to 2. At most, how many pairwise distinct numbers can be written in the given 8×8 grid square?

Problem 6. Consider n sets, such that the following properties hold true:

a) any set has exactly three elements,

b) there exists a set that does not intersect at most with two sets,

c) any element can belong to at most three sets.

Find the greatest possible value of n.

Problem 7. Find the number of all 8-digit numbers with digits 1, 2 or 3, such that any of them either does not have the digit 2 or for any digit 2 one of the neighbours is digit 1 and the other neighbour is digit 3.

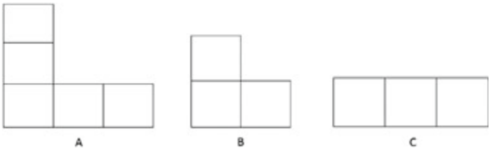
Problem 8. Let n be a positive integer. Given that after placing (in a random way) n dominos on 9×10 rectangular grid there exists 2×2 square which does not have a square covered by any domino. Each domino covers exactly two squares of a chess board. Find the possible greatest value of n.

Problem 9. The entries of 4×4 grid square are the numbers 0, 1 or 2, such that any row and column sum of the grid square is equal to 2. Find the number of all such grid squares (different from each other).

5.8 Problem Set 8

Problem 1. On some squares of 8×8 grid square are placed candies. Given that the number of candies on each row and on each column is not more than five and it is odd number. At most, how many candies can be placed on the squares of the given grid square?

Problem 2. Assume that a grid rectangle is divided into n parts of the forms A, B and C (such that there are parts of all these forms). Find the smallest possible value of n.



Problem 3. A man puts n identical coins into eight begs, such that the number of coins in difference begs is different. The begs are for his wife and seven children. Given that no matter which beg chooses the wife, she can distribute her all coins to children, such that all children have equal number of coins. Find the smallest possible value of n.

Problem 4. Find the number of all finite sequences, such that for any of them hold true the following properties:

a) the first term is equal to 0.

b) every term is greater than the previous term either by 1 or by 10.

c) any term does not have a digit greater than 5, and at least one of the digits of the last term is equal to 5.

Problem 5. Find the number of all 14-digit numbers with digits $1, 2, 3$, such that any of them either does not have a digit 2 or digit 2 is in between digits 1 and 3. Moreover, any two neighbouring digits are not equal.

Problem 6. In how many ways can one cover 3×10 grid rectangle by dominos, such that any domino covers exactly two squares and that every square is covered exactly by one domino?

Problem 7. We have chosen arbitrarily five numbers from numbers $1, 2, \ldots, n$. Given that from those five numbers, one can choose some numbers, such that the chosen numbers can be divided into two groups with equal sums (a group can consist of only one term). Find the greatest possible value of n.

Problem 8. In how many ways can one cover 3×11 grid rectangle by dominos after deleting the bottom left corner square, such that any domino covers exactly two squares and that every square is covered exactly by one domino?

Problem 9. Arrangement of 12 dominos on 9×10 grid rectangle is called "convenient", if the following two properties hold true:

a) any domino covers exactly two squares of 9×10 grid rectangle

b) any 2×2 square that covers exactly four squares of 9×10 grid rectangle has a common square at least with one domino.

Find the number of all "convenient" arrangements.

5.9 Problem Set 9

Problem 1. Given that n mathematicians take part in a mathematical conference. Let the following properties hold true:

a) Any two participants have a common acquaintance participant.

b) Any participant is acquainted with not more than three participants.

Find the possible greatest value of n.

Problem 2. In how many ways can one cover 3×18 grid rectangle with L-shape trominos, such that each L-shape tromino covers exactly three squares and each square is covered only by one tromino?

Problem 3. Find the number of all sequences $x_1, x_2, \dots, x_8$, such that for any of them it holds true:

a) $x_i \neq x_j$, if $i \neq j$.

b) $x_i \in \{1, 2, \dots, 8\}$, where $i = 1, 2, \dots, 8$.

c) $3 \mid x_i + x_{i+1} + x_{i+2}$, where $i = 1, 2, \dots, 6$.

Problem 4. Let $ABCDEF$ be a convex hexagon. Given that its diagonals AD, BE and CF do not intersect at one point. At least, how many points does one need to choose inside of $ABCDEF$, such that inside of twenty triangles created by vertexes A, B, C, D, E, F there is at least one point from the chosen points?

Problem 5. In a king's wine cellar, there are 120 barrels of wine that are numbered by $1, 2, \dots, 120$ numbers. The king knows that one of his servants has poisoned one of the barrels, but he does not know the number of the poisoned barrel. Given that after drinking any portion of the poisoned wine the person dies the next day. Given also that the king can make his 10 servants to taste the wine from any barrel he chooses. At least, how many servants does the king need to send to the wine cellar to taste the wines in order to know (the next day) the number of the poisoned barrel?

Problem 6. Find the number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_7 = 15,$$

such that it holds true $1 \leq x_i \leq 3$, $i = 1, 2, \dots, 7$.

Problem 7. Given that if any n squares of 19×61 grid rectangle are painted in black, then there is a L-shape tromino consisting of three black squares. Find the possible smallest value of n.

Problem 8. At most, how many diagonals can one consider in a convex 101-gon, such that any of them has a common (inner) point with not more than one diagonal (from the considered diagonals)?

Problem 9. A 10×10 table consists of 100 unit cells. A *block* is a 2×2 square consisting of 4 unit cells of the table. A set C of n blocks covers the table (i.e., each cell of the table is covered by some block of C) but no $n-1$ blocks of C cover the table. Find the greatest possible value of n.

5.10 Problem Set 10

Problem 1. In how many ways can one put six ordered books in the bookshelf, such that the neighbouring-books of the book number four are not the books number one, two and three?

Problem 2. Find the number of all solutions (x,y,z) of equation $x+y+z=64$, such that x, y, z are positive integers and y is an even number.

Problem 3. Find the number of all positive integers smaller than 10^4, such that any of them is divisible by 11 and the sum of its digits is equal to 19.

Problem 4. Find the number of three-digit numbers, such that any of them has exactly eight divisors.

Problem 5. Given that 64 mathematicians take part in a mathematical conference, such that among any three, at least two participants are acquaintances. Find the smallest possible number of all couples of acquaintances.

Problem 6. A grasshopper is in the centre of the rightmost square of 1×14 grid rectangle. It can jump one or two squares either to the right or to the left. A "successful journey" is a journey consisting of 13 jumps, such that the grasshopper has managed to be on all squares of 1×14 grid rectangle. Find the number of all successful journeys.

Problem 7. Consider any 301 rectangles with positive integer sides, such that all sides are less than or equal to 100. Among those rectangles are chosen the maximum number of rectangles $A_1, A_2, \dots, A_k$, such that any of them, except the last one, is possible to cover by the next one. Find the smallest possible value of k.

Problem 8. Consider all 10×10 grid squares, such that the entries of any of them are numbers from 1 to 100 (written in a random way and in any square of any 10×10 grid square is written only one number). For any 10×10 grid square, consider all positive differences of numbers written in any two squares that have at least one common vertex. Let M be the greatest among those differences. Find the possible smallest value of M.

Problem 9. Find the number of all sequences $x_1, x_2, \dots, x_{13}$, such that for any of them it holds true

a) $x_i \neq x_j$, if $i \neq j$, where $i,j \in \{1,2,\dots,13\}$.

b) $x_i \in \{1,2,\dots,13\}$, where $i=1,2,\dots,13$.

c) $|x_i - x_{i+1}| \in \{1,2\}$, where $i=1,2,\dots,12$.

5.11 Problem Set 11

Problem 1. Find the number of all three-digit numbers, such that for any of them the sum of all digits is a square number.

Problem 2. Find the number of all solutions (x,y,z) of the equation $x+y+z=100$, such that x, y, z are positive integers and $13 \mid x+z$.

Problem 3. The squares of 3×3 grid square are painted either in red or blue or yellow. Given that any two squares with a common side are painted in different colours. In how many such ways can one paint 3×3 grid square?

Problem 4. One covers 2016×2016 grid square by dominos, such that every domino covers exactly two squares and every square is covered by only one domino. Let k be the number of 2×2 grid squares, such that any of them is covered by exactly two dominos. Find the smallest possible value of k.

Problem 5. Find the number of all sequences $x_1, x_2, \ldots, x_{100}$, such that for any of them it holds true

a) $x_i \neq x_j$, if $i \neq j$, where $i,j \in \{1,2,\ldots,100\}$.

b) $x_i \in \{1,2,\ldots,100\}$, where $i = 1,2,\ldots,100$.

c) $|x_i - x_{i+1}| = 1$, where $i = 1,2,\ldots,94$.

Problem 6. Fourteen participants took part in a chess championship, such that any two participants played together only once. Given that the sum of the final points of the participants in the first three places is six times more than the sum of the final points of the participants in the last four places. Find the sum of the final points of the other seven participants, if a win is worth one point to the victor and none to the loser, a draw is worth a half point to each player.

Problem 7. Find the smallest number n, such that among any n numbers chosen from the numbers $1,2,\ldots,1000$ there are two numbers, such that their ratio is equal to 3.

Problem 8. At least, how many squares of 8×8 grid square does one need to paint in black, such that at least one of the squares of any 2×3 grid rectangle (consisting of the six squares of considered 8×8 grid square) is black?

Problem 9. At most, how many diagonals can one choose in a convex 500-gon, such that any of them has a common inner point with no more than two chosen diagonals?

5.12 Problem Set 12

Problem 1. Find the number of all two-digit numbers, such that each of them is divisible by all its digits.

Problem 2. Let every square of 4×5 grid rectangle be painted in one of the following colours: red, blue, yellow or green. Given that any two squares with a common vertex are painted in different colours. In how many distinct ways can the given grid rectangle be painted in such a way?

Problem 3. In how many distinct ways can one choose three numbers among the numbers $1,2,\ldots,20$, such that their sum is divisible by 3?

Problem 4. In how many distinct ways can one arrange the numbers $1,2,\ldots,9$ on a circle, such that the sum of any three consequently written numbers is divisible by 3?

Problem 5. In how many distinct ways can one divide the set of natural numbers into two subsets, such that in any subset the ratio (quotient) of any two elements is not a prime number?

Problem 6. Find the number of all sequences $x_1, x_2, \dots, x_{101}$, such that for any of them it holds true:

a) $x_i \in \{1,2,\dots,101\}$, where $i = 1,2,\dots,101$.

b) $|x_i - x_j| \geq |i-j|$, where $i,j \in \{1,2,\dots,101\}$.

Problem 7. One writes the numbers 1, 2,..., 12 in the squares of 3×4 grid rectangle, such that in each square is written only one number. Afterwards, one considers the product of three numbers written in each column. Find the smallest possible value of the greatest product among those four products.

Problem 8. Let the circle be divided into 13 equal arcs. At most, in how many colours does one need to paint those 13 separation points, such that for any colouring there are three points of the same colour that are the vertices of an isosceles triangle?

Problem 9. Let each square of $n \times n$ grid square be painted in one of the given three colours, such that any square has at least two neighbour (having a common side) squares painted in different colours. Moreover, for any square, the number of neighbour squares painted in different colours are equal. Find the possible greatest value of n.

Problem 10. A 6-digit number is called "interesting", if it has distinct digits and is divisible by 999. Find the number of all "interesting" numbers.

Problem 11. There are 2016 excellent pupils in the city and they all take part in the annual meeting of the excellent pupils. Given that any participant has the same number of acquaintances (excellent pupils). Moreover, for any boy, the number of female acquaintances is bigger than the number of male acquaintances, and for any girl, the number of female acquaintances is not less than the number of male acquaintances. Given also that the number of female excellent pupils is not more than 1120. Find the smallest possible value of the number of the acquaintances of each pupil.

Problem 12. Given that 20 teams took part in the volleyball championship, such that any two teams have played together only once. Team A is called "stronger" than team B, if either A won B or there is a team C, such that A won C and C won B. A team is called a "champion", if it is "stronger" than any other team. Given that there is no team that won all the others. Find the smallest possible value of the number of "champions".

5.13 Problem Set 13

Problem 1. In how many ways can one split the set $\{1,2,3,4,5,6,7,8\}$ into two disjoint subsets, such that in each subset the sum of all elements is a prime number?

Problem 2. In how many ways can one put six different fruits in three plates, such that at least in two plates the number of fruits is equal?

Problem 3. Let 14 teams take part in a volleyball championship. Given that any two teams have played with each other only once. A group consisting of several teams is called the strongest, if any team (except the teams belonging to that group) has lost against at least one of the teams belonging to that group. Given that in any championship there exists a strongest group consisting of n teams. Find the smallest possible value of n.

Problem 4. Find the smallest possible value of the number of diagonals of a convex hexagon, such that none of those diagonals is parallel to any side of the hexagon.

Problem 5. In how many ways can one take 2×2 grid square away from 8×8 grid square, such that the obtained figure is possible to divide into twenty L-shape trominos?

Problem 6. Given six boxes, such that for any two boxes one of them is possible to put inside of the other one. In how many ways can one arrange (put inside of each other) these boxes? (e.g., one of the possible arrangements is $\{(6,4,1),(5,3),2\}$; in this example, the boxes are enumerated by numbers $1,2,\ldots,6$).

Problem 7. Find the number of all non-empty subsets of set $\{1,2,\ldots,11\}$, such that for any of them the sum of all elements is divisible by 3.

Problem 8. Let each square of $n \times n$, $(n > 1)$ grid square be painted in one of the following colours: red, blue, yellow or green. Given that any two squares with at least one common vertex are painted in different colours. Denote by A_n, the number of all possible $n \times n$ grid squares painted in such a way. Find $\frac{A_{19}}{A_{10}}$.

Problem 9. At most, how many vertices of a regular 26-gon can one choose, such that any three vertices among the chosen ones are the vertices of a scalene triangle?

Problem 10. Let in some squares of 11×11 grid square are written asterisks. Two asterisks are called neighbours, if the squares corresponding to them have at least one common vertex. Given that any asterisk has at most one neighbour asterisk. Find the possible greatest number of all asterisks.

Problem 11. Consider 64 balls having eight different colours, such that there are eight balls of each colour. At least, how many balls does one need to take from those balls in order to be able to put them in the line, such that for any two different colours there is a ball with two neighbour balls of these colours.

Problem 12. Find the smallest positive integer n, such that the following statement holds true: if 7×7 grid square is covered (in a random way) by n rectangles, such that any rectangle covers exactly two squares of 7×7 grid square, then one can take away one of those n rectangles, in such a way that the rest of the rectangles again cover the grid square.

5.14 Problem Set 14

Problem 1. In how many ways is it possible to put five different candies in three different plates, if in the first plate is allowed to put not more than one, in the second plate not more than two and in the third plate not more than three candies?

Problem 2. Let each square of 3×3 grid square be painted in one of the following colours: red, blue or yellow. Given that any two squares having a common side are painted in different colours. Given also that in each colour are painted exactly three squares. In how many ways is it possible to paint (as it was described) this grid square?

Problem 3. In how many ways is it possible to put ten different fruits in two different plates, such that in any plate there are at least three fruits?

Problem 4. Find the smallest positive integer n, such as any square of 2016×2016 grid square is possible to paint in one of the given n colours, such that any  figure (consisting of five squares) does not have squares painted in the same colour.

Problem 5. Seven boys and some girls take part in an event. Given that any two boys have danced with distinct number of girls and any two girls have danced with equal number of boys. At least, how many people took part in that event?

Problem 6. Find the number of words consisting of two letters A, three letters O and seven letters B, such that any of them does not include consequent (neighbour) vowel letters. For example *BABABBOBOBOB*.

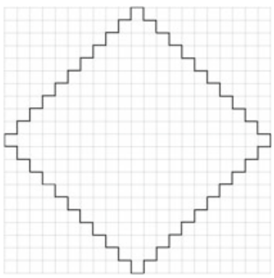
Problem 7. Let n participants take part in a mathematical conference. Given that any participant is acquainted with at most three other participants. Given also that any two participants are either acquaintances or have a mutual acquaintance. Find the greatest possible value of n.

Problem 8. Consider a set consisting of six positive integers. For any non-empty subset (of this set), consider the arithmetic mean of all its elements. At most, how many times can the same number be a term of the sequence consisting of those arithmetic means?

Problem 9. Consider 64 balls of eight different colours. Given that there are eight balls of each colour. At least, how many balls is possible to put in a row, such that for any two colours (not obliged to be different) there is a ball having neighbour balls of these colours?

Problem 10. Let in some squares of 40×40 grid square be written asterisks (only one asterisk in each square). Given that the total number of asterisks is equal to n. Find the greatest possible value of n, such that for any placement of asterisks there exists 3×3 grid square having less than three asterisks.

Problem 11. Find the greatest possible number of dominos that is possible to place inside the following figure, such that each domino covers two squares and each square is covered by only one domino.



Problem 12. Given that some cities among n cities located on the island are pairwise-connected by airways, such that each city is connected at most with three other cities and if there is no connection between any two cities, then it is possible to go from one city to the another one at least in two different ways, every time passing through only one transit city. Find the greatest possible value of n.

5.15 Problem Set 15

Problem 1. Find the number of all permutations $x_1, x_2, \dots, x_8$ of numbers $1, 2, \dots, 8$, such that $x_1 - x_2 + x_3 - x_4 + x_5 - x_6 + x_7 - x_8 = 16$.

Problem 2. In how many ways can one transport three girls and seven boys by three boats, such that in every boat there are one girl and at least two boys?

Problem 3. Let every cell of 10×10 square grid be painted in one of the following colours: red, blue or yellow. Given that the cells of any rectangle that consists of three cells of the given square grid are painted in different colours. In how many ways is possible to paint, as described above, the given square grid?

Problem 4. Find the number of words consisting of two letters A, three letters O and five letters B, such that for any of them the number of couples, consisting of two consequent vowels, is equal to 1. For example, *BAABOBBOBO*.

Problem 5. Given 8×8 square grid. Consider among 2×3 rectangular grids only those consisting of the cells of the given square grid. Find the smallest positive integer n, such that the following conditions holds true:

a) There exist n such 2×3 rectangular grids that any 2×3 rectangular grid has at least with one of them a common cell.

b) For any $n - 1$ rectangular grids size of 2×3, there exists at least one other 2×3 rectangular grid having no common cell with any of these $n - 1$ rectangular grids.

Problem 6. Find the number of all permutations $x_1, x_2, \dots, x_6$ of numbers $1, 2, \dots, 6$, such that $3 \nmid x_{i+1} - x_i$, where $i = 1, 2, \dots, 5$.

Problem 7. Let on every square of 4×4 chessboard be placed a chess-knight, such that any black knight attacks only one white knight and any write knight attacks only one black knight. Find the number of all such placements.

Problem 8. At least, at how many points can intersect the diagonals of a convex heptagon? The common endpoint of two or more diagonals is not considered as an intersection point.

Problem 9. Let the entries of 100×100 square grid be integer numbers, such that in each cell is written only one integer. Given that if integers a, b are written in any two cells having a common side, then $|a - b| \leq 1$. At least, how many times can the number, written maximum number of times in the given square grid, be written?

Problem 10. Let in some cells of 8×8 square grid be placed asterisks, only one asterisk per cell. Given that the total number of asterisks is equal to n. Find the greatest possible value of n, such that for any placement of asterisks there exists 3×3 square grid that has at most two asterisks.

Problem 11. A triangle is called "beautiful", if its all angles are less than or equal to $120°$. Given that a regular hexagon is divided into triangles. At least, how many of these triangles can be "beautiful"?

Problem 12. Let the entries of 100×100 square grid be integer numbers, such that in each cell is written only one integer. Given that if integers a, b are written in any two cells having a common side, then $|a-b| \leq 1$. Let integers c, d be written maximum number of times in the given square grid. Moreover, c is written n times and d is written k times. Find the smallest possible value of $n+k$.

Chapter 6
Hints

4.1 Problem Set 1

Problem 7. Note that

$$\frac{1}{x_{n+1}} = \frac{1}{x_n} + \frac{1}{1-x_n}, n = 1,2,\ldots$$

Problem 8. Prove that the given equation has at most two roots.
Problem 9. Prove that

$$\lim_{x\to 0} \frac{\sin(\frac{\pi}{3}+x) - \sin\frac{\pi}{3}}{x} = \frac{1}{2}.$$

4.2 Problem Set 2

Problem 7. Note that

$$u_{n+1}^2 = u_n^2 + 2 + \frac{1}{u_n^2}, n = 1,2,\ldots,99.$$

Problem 8. Prove that, if $x_n \neq 2$, $n = 1,2,\ldots$, then $\lim_{n\to\infty} x_n \neq 2$.
Problem 9. Prove that

$$\int_0^1 (f(x) - x - e^x)^2 dx = 0.$$

4.3 Problem Set 3

Problem 7. Let $f(x) = \frac{3^x}{3^x+\sqrt{3}}$. Prove that $f(x) + f(1-x) = 1$.
Problem 8. Let $x_n = u_n - \sqrt{2}$. Prove that $x_{n+1} < \frac{x_n}{2}$ and $x_{n+1} < \frac{{x_n}^2}{2}$, $n = 1,2,\ldots$

H. Sedrakyan and N. Sedrakyan, *The Stair-Step Approach in Mathematics*,
Problem Books in Mathematics, https://doi.org/10.1007/978-3-319-70632-0_6

Problem 9. Prove that

$$f(b+1)-f(b)+f(1)-f(2)-3b^2-3b+6=0,$$

where $b\in\mathbb{N}$.

4.4 Problem Set 4

Problem 7. Note that

$$(f(x)e^x\cos x)'=0.$$

Problem 8. Note that

$$u_{n+1}^3=u_n^3+3+\frac{3}{u_n^3}+\frac{1}{u_n^6},$$

where $n=1,2,\ldots$

Problem 9. Prove that $f(x)\le g(x)$, where $x\in(0,+\infty)$.

4.5 Problem Set 5

Problem 7. Prove that

$$\log_a\left(2+\sqrt{3-\frac{a^2}{4}}\right)\log_3\left(4-\frac{a^2}{4}\right)-\log_a 2=0.$$

Problem 8. Prove that

$$\frac{1}{u_{1000}}=\frac{1}{u_0}+\frac{1}{1-u_0}+\cdots+\frac{1}{1-u_{999}}.$$

Problem 9. Prove that $f(x)\le g(x)$, where $x\in(0,+\infty)$.

4.6 Problem Set 6

Problem 7.

$$|2x+a|-|x-a|+|x-2a|=\begin{cases} kx+b,\ if\ x\ge -\frac{a}{2}, where\ k\ge 0,\\ lx+m,\ if\ x<-\frac{a}{2}, where\ l\le 0.\end{cases}$$

Problem 8. Note that

$$u_1^2(u_1-1)+2u_2\left(\frac{u_2(u_2-1)}{2}-u_1\right)+\cdots+2u_n\left(\frac{u_n(u_n-1)}{2}-(u_1+\cdots+u_{n-1})\right)=0,$$

where $n=2,3,\ldots$

Problem 9. Prove that $E(f)$ is a finite set.

4.7 Problem Set 7

Problem 7. Prove that, if $k \in \mathbb{N}$, then

$$\arctan \frac{3}{k^2+3k+1} = \frac{\pi}{4} - \arctan \frac{k^2+3k-2}{k^2+3k+4}.$$

Problem 8. Note that the function

$$f(x) = 10|x-1| + \sqrt{3x^2-6x+4} + 3|2x-3a|$$

is increasing on $[1, +\infty)$ and decreasing on $(-\infty, 1]$.
Problem 9. Prove that $degp(x) \leq 4$.

4.8 Problem Set 8

Problem 7. Note that

$$\int_0^1 (f^{1007}(x) \cdot \frac{\int_0^1 f^{2015}(x)dx}{\int_0^1 f^{2014}(x)dx} - f^{1008}(x))^2 dx = 0.$$

Problem 8. Note that

$$\left(\frac{1-x}{\sqrt[3]{-x^3+9x^2-3x+3}}\right)^3 + \left(\frac{1-x}{\sqrt[3]{-x^3+9x^2-3x+3}}\right)^3 + \ldots +$$

$$+\left(\frac{1+x}{\sqrt[3]{-x^3+9x^2-3x+3}}\right)^3 = 1.$$

Problem 9. Let $g(x) = \frac{f(x)}{x} - 1$ and $\varphi(x) = 3g(2x) - g(x)$. Prove that $\lim_{x\to 0} \varphi(x) = 0$.

4.9 Problem Set 9

Problem 7. Prove that

$$\frac{8a^3}{27} - 28 - 6a = 0.$$

Problem 8. Let $g(x) = e^{f(x)}$. Prove that $g''(x) = 0$.
Problem 9. Prove that

$$\frac{f(x)}{x^2} - \frac{f(y)}{y^2} = \frac{1}{x^2 y} f\left(\frac{x}{y}\right),$$

where $x > 0$, $y > 0$.

4.10 Problem Set 10

Problem 7. Let $D(f) = \{x_1, x_2, x_3\}$. Prove that $p(x_i)q(x_i) = 0$, where $i = 1, 2, 3$.
Problem 8. Prove that

$$4^{\sin x} + 4^{\cos x} \geq 5,$$

where $x \in \left[0, \frac{\pi}{4}\right]$.
Problem 9. Prove that the numbers 0, 1, 2 are the roots of the given equation.

4.11 Problem Set 11

Problem 7. If $a = 5 + h$, where $h > 0$. Prove that $x_{2n-1} \geq 5 + (2n-1)h$, where $n = 1, 2, \ldots$
Problem 8. Prove that, if $n \in \mathbb{N}$, then

$$|f(y) - f(x)| \leq \frac{1}{n} |\arctan y - \arctan x||g(y) - g(x)|.$$

Problem 9. Prove that if $degp(x) > 1$ and $degq(x) = degr(x) \geq 1$, then for any value of x it holds true $q(x) + r(x) = 1$.

4.12 Problem Set 12

Problem 9. Note that

$$((f(x))^2 + (f'(x))^2)' = 0.$$

Problem 10. Note that

$$f(\sin(\sin x)) \leq f(\sin x) \leq f(x),$$

where $0 \leq x \leq 1$.
Problem 11. Prove that, if $a \in (-1, 0)$, then $-1 < x_1 < x_2 < \cdots < x_n < \ldots$
Problem 12. Prove that $c \geq 1$.

4.13 Problem Set 13

Problem 9. Prove that $\frac{2\pi}{5}$ is the period of the function $f'(x)$.
Problem 10. Prove that

$$x\sqrt{1-x^2} - \sqrt{3}x^2 + \frac{\sqrt{3}}{2} = \arcsin\left(2\arcsin x + \frac{\pi}{3}\right).$$

Problem 11. Prove that, if $x, y \in \left[\frac{\pi}{2}, \frac{22\pi}{3}\right]$, then

$$\left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| \leq \frac{4}{3}\sqrt[3]{(x-y)^4}.$$

Problem 12. Prove that $c < 1,43$.

4.14 Problem Set 14

Problem 9. Prove that the function $f(x) = \frac{\ln x}{x}$ is increasing on $(0,e]$ and is decreasing on $[e,+\infty)$.
Problem 10. Consider the points $A_n\left(n, \frac{n^2}{\sqrt{3}}\right)$, where $n = 1,2,\ldots$
Problem 11. Prove that, if $x \geq 0$, $y \geq 0$ and $x+y \leq \frac{2}{3}$, then

$$x^5(1-x)+y^5(1-y) \leq (x+y)^5(1-x-y).$$

Problem 12. Consider the points $A(1,18)$, $B(17,6)$, $X(x,f(x))$, $Y(y,f(y))$.

4.15 Problem Set 15

Problem 9. Consider the function

$$F(x) = \frac{\pi}{2a}f(x)\sin\left(\frac{\pi}{2a}x\right)+f'(x)\cos\left(\frac{\pi}{2a}x\right).$$

Problem 10. Prove that $f'(x) = 0$.
Problem 11. Note that

$$\int_0^4 (f'(x)-5)^2 dx = 0.$$

Problem 12. Consider the function $g(x) = f(x)+5$. Prove that for any numbers x,y it holds true

$$|g(x)-g(y)||x+y| = |g(-x)-g(y)||-x+y|.$$

Chapter 7
Solutions

7.1 Geometry and Trigonometry

7.1.1 Problem Set 1

Problem 1. Let M be a given point in rectangle $ABCD$, such that $\angle AMD = 90^\circ$, $BC = 2MD$, $CD = AM$. Find $16\cos^2\angle CMD$.

Solution. *We have that $\angle AMD = 90^\circ$ and $AD = BC = 2MD$, therefore $\angle MAD = 30^\circ$. Thus $\angle MAB = 60^\circ$ and $AB = CD = AM$. Hence, ABM is an equilateral triangle. We deduce that $\triangle BMC = \triangle AMD$, thus $MC = MD$ and $\angle CMD = 120^\circ$. Hence $16\cos^2\angle CMD = 16\cos^2 120^\circ = 4$.*

Problem 2. Let ABC be a given triangle, such that $AD = 3$, $DE = 5$, $EC = 24$ and $\angle ABE = 90^\circ$, $\angle DBC = 90^\circ$, where D and E are some points on AC. Find $3AB$.

Solution. *Let BH is the altitude of ABC triangle. Triangles ABC and DBC are right-angled, thus $AH \cdot HE = BH^2 = DH \cdot HC$. Denote by $DH = x$, hence $(3+x)(5-x) = x(29-x)$, thus $x = \frac{5}{9}$. We have that $AB^2 = AH \cdot AE$, therefore $AB = \sqrt{\frac{32}{9}\cdot 8} = \frac{16}{3}$.*

Problem 3. Let $ABCD$ be a convex quadrilateral, such that $AB = BC$ and $\angle ADC = \angle BAD + \angle BCD$. Find $\dfrac{BD}{BC}$.

Solution. *Let us prove that $\frac{BD}{BC} = 1$. We proceed by a contradiction argument. Assume that $BD > BC$, then from the triangles ABD and DBC we obtain that $\angle BAD > \angle BDA$ and $\angle BCD > \angle CDB$, thus $\angle ADC = \angle BDA + \angle CDB < \angle BAD + \angle BCD$, we obtain a contradiction. In a similar way, we obtain a contradiction for the case when $BD < BC$. Therefore $BD = BC$.*

H. Sedrakyan and N. Sedrakyan, *The Stair-Step Approach in Mathematics*,
Problem Books in Mathematics, https://doi.org/10.1007/978-3-319-70632-0_7

Problem 4. Given that

$$\cos\sqrt{(\sin x+\cos x)(1-\sin x\cos x)}=\sqrt{\cos(\sin x+\cos x)\cdot\cos(1-\sin x\cos x)}.$$

Find $\sin^5 x+\cos^5 x$.

Solution. *We have that* $1-\sin x\cos x=1-0,5\sin(2x)\in[0,5;1,5]$*; we deduce that* $\sin x+\cos x\geq 0$ *and* $\sin x+\cos x=\sqrt{2}\sin(x+\frac{\pi}{4})\leq\sqrt{2}$*. Note that* $1-\sin x\cos x,\sin x+\cos x\in[0,\frac{\pi}{2}]$*, hence*

$$\cos\sqrt{(\sin x+\cos x)(1-\sin x\cos x)}\geq\cos\left(\frac{(\sin x+\cos x)+(1-\sin x\cos x)}{2}\right)\geq$$

$$\geq\frac{\cos(\sin x+\cos x)+\cos(1-\sin x\cos x)}{2}\geq\sqrt{\cos(\sin x+\cos x)\cdot\cos(1-\sin x\cos x)}.$$

Thus, this equation is equivalent to the following equation $\sin x+\cos x=1-\sin x\cos x$*. Therefore*

$$\sin x+\cos x=1-\frac{(\sin x+\cos x)^2-1}{2},$$

hence $\sin x+\cos x=1$ *and* $\sin^5 x+\cos^5 x=1$.

Problem 5. Let I be the incenter of triangle ABC and $AC=10$. A circle passes through the points I and B and intersects AB and BC correspondingly at points M and N. Find $AM+CN$.

Solution. *Note that* $\angle CNI+\angle AMI=\angle BMI+\angle AMI=180^\circ$*. Without loss of generality, we may assume that* $\angle CNI\geq 90^\circ$*. Let* K *be a point on* AC *side, such that* $\angle CKI=\angle CNI$*. Hence,* $\angle AKI=\angle AMI$*, thus* $\triangle AKI=\triangle AMI$ *and* $\triangle CKI=\triangle CNI$*. Therefore* $AM+CN=AK+CK=AC=10$.

Problem 6. Find the minimum value of the expression

$$\left(\sin x-\frac{1}{\sin^2 y}-2\right)^2+\left(\frac{\sin x}{\sin^2 y}+1\right)^2.$$

Solution. *We have that*

$$\left(\sin x-\frac{1}{\sin^2 y}-2\right)^2+\left(\frac{\sin x}{\sin^2 y}+1\right)^2=\left(\sin x+\frac{1}{\sin^2 y}-2\right)^2+\left(\frac{\sin x}{\sin^2 y}-1\right)^2+$$

$$+\frac{8}{\sin^2 y}\geq 0+0+8=8.$$

In the case of $\sin x = \sin y = 1$*, we have*

$$\left(\sin x - \frac{1}{\sin^2 y} - 2\right)^2 + \left(\frac{\sin x}{\sin^2 y} + 1\right)^2 = 8,$$

thus the minimum value of the expression is 8.

Problem 7. Let $ABCD$ be a quadrilateral with a circle inscribed in it. Denote by O the intersection point of the diagonals of $ABCD$. Assume that $\angle AOB < 90^\circ$. Let AA_1, BB_1, CC_1 and DD_1 be respectively the altitudes of triangles AOB and COD. Find the perimeter of quadrilateral $A_1B_1C_1D_1$, if $A_1B_1 + C_1D_1 = 27$.

Solution. *Let* $\angle AOB = \alpha$*. Note that the circle with the diameter* AB *passes through the points* A_1, B_1*. By the law of sines from triangle* AA_1B_1*, we deduce that* $A_1B_1 = AB\sin(90^\circ - \alpha) = AB\cos\alpha$*. In a similar way, we obtain that* $C_1D_1 = CD\cos\alpha$*,* $A_1D_1 = AD\cos\alpha, B_1C_1 = BC\cos\alpha$*, therefore* $A_1B_1 + C_1D_1 = AB\cos\alpha + CD\cos\alpha = BC\cos\alpha + AD\cos\alpha = B_1C_1 + A_1D_1$*. Thus, the perimeter of* $A_1B_1C_1D_1$ *is equal to* 54.

Problem 8. Evaluate the expression

$$\cos^2\frac{\pi}{18} + \frac{\cos^2\frac{\pi}{18}}{\left(4\cos\frac{\pi}{18} + \sqrt{3}\right)^2}.$$

Solution. *Note that*

$$\sin\left(\frac{\pi}{6} - \frac{\pi}{18}\right) = \sin\frac{\pi}{9},$$

therefore

$$\frac{\cos\frac{\pi}{18}}{4\cos\frac{\pi}{18} + \sqrt{3}} = \sin\frac{\pi}{18}.$$

Thus

$$\cos^2\frac{\pi}{18} + \frac{\cos^2\frac{\pi}{18}}{\left(4\cos\frac{\pi}{18} + \sqrt{3}\right)^2} = \cos^2\frac{\pi}{18} + \sin^2\frac{\pi}{18} = 1.$$

Problem 9. Let $ABCD$ be a convex quadrilateral, such that $\angle ACB = \angle DBC = \angle BAM$, where M is the intersection point of rays CB and DA. Moreover, $CD = 6$ and $MB = BC$. Find AD.

Solution. *Consider the escribed circle of triangle* BCD *and denote by* N *and* K*, respectively, the intersection points with the rays* CA *and* BA*. We have that* $\angle NCB = \angle DBC = \angle DNC$*, thus* $\angle NCB = \angle DNC$*, therefore* $BC \parallel ND$*. Let* $AB \cap ND = R$*. Note that* $\triangle MAB \sim \triangle DAR$ *and* $\triangle CAB \sim \triangle NAR$*, hence* $\frac{MB}{RD} = \frac{AB}{AR} = \frac{BC}{NR}$*. Thus* $NR = RD$*. We have* $\angle KAD = \angle BAM = \angle ACB = \angle NKB$*, thus* $NK \parallel AD$*. Summarizing*

the results, we deduce that $\triangle NKR = \triangle DAR$*, thus* $NK = AD$*. Therefore,* $NKDA$ *is a parallelogram, hence* $NC \parallel KD$*. Thus,* $AD = NK = CD$*, we obtain that* $AD = CD = 6$*.*

7.1.2 Problem Set 2

Problem 1. Let two circles be mutually externally tangent at point A. Let a be a tangent line to those circles at points B and C. Given that $AB = 20$ and $AC = 21$. Find BC.

Solution. *Let* $O_1B \perp a$ *and* $O_2C \perp a$*, where* O_1 *and* O_2 *are the centres of the circles. We have that* $O_1B \parallel O_2C$ *and* $\angle ABC + \angle ACB = \frac{1}{2}\angle AO_1B + \frac{1}{2}\angle AO_2C = 90^\circ$*, thus* $\angle BAC = 90^\circ$ *and* $BC^2 = AB^2 + AC^2 = 841$*, thus* $BC = 29$*.*

Problem 2. Find the maximum value of the expression $\cos x + \cos y + \sin x \sin y$.

Solution. *We have that*

$$1 \cdot \cos x + 1 \cdot \cos y + \sin x \cdot \sin y \leq \frac{1 + \cos^2 x}{2} + \frac{1 + \cos^2 y}{2} + \frac{\sin^2 x + \sin^2 y}{2} = 2.$$

Note that for $\cos x = \cos y = 1$ *we obtain that* $\cos x + \cos y + \sin x \cdot \sin y = 2$*. Thus, the greatest value of the given expression is equal to* 2*.*

Problem 3. The incircle of triangle ABC with the incenter I is tangent to sides AB and BC at points C_1 and A_1, respectively. The lines AI and A_1C_1 intersect at the point M. Given that $AC = 68$ and $\angle A = 30^\circ$. Find the area of triangle AMC.

Solution. *We have that* $BA_1 = BC_1$*, thus* $\angle BA_1C_1 = \angle BC_1A_1 = \frac{\angle A + \angle C}{2} = \angle MIC$*. Therefore* $\angle CA_1M = \angle MIC$*; hence, the points* C, A_1, M, I *are on the same circle. Thus, we deduce that* $\angle CMA = \angle CA_1I = 90^\circ$*. Let* N *be the midpoint of* AC *and* MK *be the altitude of triangle* AMC*. We have* $\angle MNC = 2\angle MAC = 30^\circ$ *and* $MN = \frac{AC}{2}$*, thus* $MK = \frac{MN}{2} = \frac{AC}{4} = 17$ *and* $(AMC) = 578$*.*

Problem 4. Evaluate the expression

$$\left(\frac{\sqrt{3}}{\sin\dfrac{\pi}{9}} + \frac{1}{\cos\dfrac{\pi}{9}}\right)\sec\frac{2\pi}{9}.$$

Solution. *We have that*

$$\left(\frac{\sqrt{3}}{\sin\frac{\pi}{9}} + \frac{1}{\cos\frac{\pi}{9}}\right) sec\frac{2\pi}{9} = 4 \cdot \frac{\sin\frac{\pi}{3}\cos\frac{\pi}{9} + \cos\frac{\pi}{3}\sin\frac{\pi}{9}}{\sin\frac{2\pi}{9} \cdot \cos\frac{2\pi}{9}} = 4 \cdot 2 \cdot \frac{\sin\frac{4\pi}{9}}{\sin\frac{4\pi}{9}} = 8.$$

Problem 5. Let I be the incenter of triangle ABC. Given that the perimeter of ABC is equal to 25 and $AC = 10$. A circle passes through the points I and B and intersects the continuation of the side AB and the side BC at points M and N, respectively. Find $BN - BM$.

Solution. *Let K be the symmetric point of N with respect to the ray CI. We have that $\angle AMI = \angle BMI = \angle BNI = \angle IKA$ and $\angle MAI = \angle KAI$, thus $\triangle MAI = \triangle KAI$. Hence $AM = AK$. Therefore $BN - BM = BC - CN - (AM - AB) = BC + AB - CK - AK = AB + BC + AC - 2AC = 25 - 20 = 5$.*

Problem 6. Let $ABCD$ be a convex quadrilateral. Given that $BC = 3, AD = 5$ and $MN = 1$, where the points M and N are midpoints of the diagonals AC and BD. Find the ratio of the area of triangle ABC to the area of triangle BCD.

Solution. *Let the point K be the midpoint of CD, then NK and MK are the mid-segments of the triangles BCD and ACD, respectively. Therefore, $NK \parallel BC$, $MK \parallel AD$ and $NK = \frac{BC}{2} = 1,5$, $MK = \frac{AD}{2} = 2,5$. We deduce that $MK = MN + NK$, thus the point N is on the segment MK. Hence $BC \parallel AD$. We obtain that $(ABD) = (BCD)$.*

Problem 7. Evaluate the expression

$$\frac{1}{2^{13}}(1+\tan 1^\circ)(1+\tan 2^\circ)\cdot\dots\cdot(1+\tan 44^\circ).$$

Solution. *Note that*

$$1 + tgk^\circ = \frac{\sin k^\circ + \cos k^\circ}{\cos k^\circ} = \sqrt{2}\cdot\frac{\cos(45^\circ - k^\circ)}{\cos k^\circ}.$$

Thus

$$\frac{1}{2^{13}}(1+tg1^\circ)(1+tg2^\circ)\cdot\dots\cdot(1+tg44^\circ) =$$

$$= \frac{(\sqrt{2})^{44}}{2^{13}}\cdot\frac{\cos 44^\circ}{\cos 1^\circ}\cdot\frac{\cos 43^\circ}{\cos 2^\circ}\cdot\dots\cdot\frac{\cos 1^\circ}{\cos 44^\circ} = 512.$$

Problem 8. Consider a point M inside of an isosceles triangle ABC ($AB = BC$). Given that $\angle MBA = 10^\circ$, $\angle MBC = 30^\circ$ and $BM = AC$. Find $36\sin^2\angle MCA$.

Solution. *Let the points D and A be on the same side of the line BC and $BD = AC$, $\angle DBC = \angle ACB$. Denote by K the intersection point of the rays BM and CD. We have that $\angle BDC = 70^\circ$ and $\angle DBK = 40^\circ$, therefore $\angle BDK = \angle BKD$, thus $BK = BD = AC = BM$. That means that the points M and K coincide, hence $\angle MCA = \angle KCA = 30^\circ$. Thus* $36\sin^2\angle MCA = 9$.

Problem 9. Consider a quadrilateral $ABCD$, such that $\angle A = \angle C = 60^\circ$ and $\angle B = 100^\circ$. Let the angle between the lines AI_2 and CI_1 be equal to n°, where I_1 and I_2 are the incenters of triangles ABD and CBD, respectively. Find n.

Solution. *Note that $\angle BO_1D = 90^\circ + \frac{1}{2}\angle A = 120^\circ$ and $\angle C = 60^\circ$, therefore O_1BCD is an inscribed quadrilateral. Hence $\angle DO_1C = \angle DBC$, thus $\angle CMD = \angle DBC + \frac{1}{2}\angle ADB$, where M is the intersection point of the lines BD and CO_1. In a similar way, we deduce that $\angle AND = \angle ABD + \frac{1}{2}\angle BDC$, where N is the intersection point of the lines BD and AO_2. We obtain that*

$$n^\circ = |\angle CMD - (180^\circ - \angle AND)| = |\angle CMD + \angle AND - 180^\circ| =$$

$$= |\angle B + \frac{1}{2}\angle D - 180^\circ| = 10^\circ.$$

7.1.3 Problem Set 3

Problem 1. Let H be the intersection point of the altitudes of acute triangle ABC. Given that $AH = 1$, $AB = 15$, $BC = 18$. Find CH.

Solution. *Consider the altitude BB_1. According to the Pythagorean theorem, we have $AH^2 - CH^2 = (AB_1^2 + B_1H^2) - (CB_1^2 + B_1H^2) = AB_1^2 - B_1C^2 = AB^2 - BC^2$, thus $CH^2 = AH^2 + BC^2 - AB^2 = 100$, then $CH = 10$.*

Problem 2. The incircle of triangle ABC with the incenter I is tangent to sides AB and BC at points C_1 and A_1, respectively. The lines AI and CI intersect the line A_1C_1 at the points M and N, respectively. Given that $AC = 12$ and $\angle B = 60^\circ$. Find MN.

Solution. *We have that $BA_1 = BC_1$, thus $\angle BA_1C_1 = \angle BC_1A_1 = \frac{\angle A + \angle C}{2} = \angle MIC$. Therefore $\angle CA_1M = \angle BA_1C_1 = \angle MIC$, hence the points C, I, A_1, M are on the same circle. Thus, we deduce that $\angle CMA = \angle CA_1I = 90^\circ$. Let K be the midpoint of AC. We have $AK = MK = KC$ and $\angle MKC = 2\angle MAC = \angle A$, in a similar way we obtain that $NK = AK$ and $\angle NKA = \angle C$. Therefore, $NK = MK$ and $\angle MKN = 180^\circ - \angle A - \angle C = \angle B = \angle 60^\circ$, thus MNK is an equilateral triangle. Hence $MN = MK = 6$.*

Problem 3. Evaluate the expression

$$\cos^2\frac{5\pi}{18} + \frac{\cos^2\frac{5\pi}{18}}{\left(4\cos\frac{5\pi}{18} - \sqrt{3}\right)^2}.$$

Solution. *Note that*

$$\sin\left(\frac{\pi}{6} + \frac{5\pi}{18}\right) = \sin\frac{5\pi}{9}.$$

Hence

$$\frac{\cos\frac{5\pi}{18}}{4\cos\frac{5\pi}{18}-\sqrt{3}}=\sin\frac{5\pi}{18}.$$

Thus

$$\cos^2\frac{5\pi}{18}+\frac{\cos^2\frac{5\pi}{18}}{\left(4\cos\frac{5\pi}{18}-\sqrt{3}\right)^2}=\cos^2\frac{5\pi}{18}+\sin^2\frac{5\pi}{18}=1.$$

Problem 4. Let $ABCDE$ be a pentagon inscribed in a circle. Given that $AB=CD$, $BC=2AB$, $AE=1$, $BE=4$, $CE=14$. Find DE.

Solution. *We have that $AB=CD$, thus arcs ABC and BCD are equal. Hence $\angle ABC=\angle BCD$. Note that $\triangle ABC=\triangle DCB$, hence $AC=BD$. By Ptolemy's theorem from the quadrilaterals $EABC$ and $EBCD$, we deduce that $AC\cdot EB=AE\cdot BC+AB\cdot EC$ and $EC\cdot BD=EB\cdot CD+BC\cdot ED$. Therefore*

$$\frac{14AB+BC}{4}=\frac{ED\cdot BC+4\cdot CD}{14},$$

then $ED=26$.

Problem 5. Evaluate the expression

$$32\sin\frac{\pi}{22}\sin\frac{3\pi}{22}\sin\frac{5\pi}{22}\sin\frac{7\pi}{22}\sin\frac{9\pi}{22}.$$

Solution. *We have that*

$$32\sin\frac{\pi}{22}\sin\frac{3\pi}{22}\sin\frac{5\pi}{22}\sin\frac{7\pi}{22}\sin\frac{9\pi}{22}=$$

$$=\frac{2\sin\frac{\pi}{22}\cos\frac{\pi}{22}2\sin\frac{3\pi}{22}\cos\frac{3\pi}{22}2\sin\frac{5\pi}{22}\cos\frac{5\pi}{22}2\sin\frac{7\pi}{22}\cos\frac{7\pi}{22}2\sin\frac{9\pi}{22}\cos\frac{9\pi}{22}}{\cos\frac{\pi}{22}\cos\frac{3\pi}{22}\cos\frac{5\pi}{22}\cos\frac{7\pi}{22}\cos\frac{9\pi}{22}}=$$

$$=\frac{\sin\frac{\pi}{11}\sin\frac{3\pi}{11}\sin\frac{5\pi}{11}\sin\frac{7\pi}{11}\sin\frac{9\pi}{11}}{\cos\frac{\pi}{22}\cos\frac{3\pi}{22}\cos\frac{5\pi}{22}\cos\frac{7\pi}{22}\cos\frac{9\pi}{22}}=$$

$$=\frac{\cos\left(\frac{\pi}{2}-\frac{\pi}{11}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{11}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{11}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{11}\right)\cos\left(\frac{\pi}{2}-\frac{9\pi}{11}\right)}{\cos\frac{\pi}{22}\cos\frac{3\pi}{22}\cos\frac{5\pi}{22}\cos\frac{7\pi}{22}\cos\frac{9\pi}{22}}=1.$$

Problem 6. Given that the sines of the angles of a quadrilateral are the terms of a geometric progression. Find the number of all the possible values of the common ratio of that geometric progression.

Solution. *Let* $\sin\alpha$, $\sin\beta$, $\sin\gamma$, $\sin\delta$ *be terms of a geometric progression, where* α, β, γ, δ *are the angles of some quadrilateral. We have that* $\alpha+\beta+\gamma+\delta=2\pi$, *thus* $0<\alpha,\beta,\gamma,\delta<\pi$. *We also have that* $\sin\alpha\sin\delta=\sin\beta\sin\delta$, *thus* $\cos(\alpha-\delta)=\cos(\beta-\gamma)$. *Therefore* $\alpha-\delta=\beta-\gamma$ *or* $\alpha-\delta=\gamma-\beta$. *Hence, we obtain that either* $\alpha+\gamma=\pi$ *or* $\alpha+\beta=\pi$. *We deduce that* $q^2=\dfrac{\sin\gamma}{\sin\alpha}=1$ *or* $q=\dfrac{\sin\beta}{\sin\alpha}=1$. *Thus,* $q=1$.

Problem 7. Find the greatest value of the following expression

$$4\sin x+48\sin x\cos x+3\cos x+14\sin^2 x.$$

Solution. *Note that*

$$f(x)=4\sin x+3\cos x+48\sin x\cos x+14\sin^2 x=4\sin x+3\cos x+24\sin 2x-7\cos 2x+$$

$$+7\leq\sqrt{4^2+3^2}\cdot\sqrt{\sin^2 x+\cos^2 x}+\sqrt{24^2+(-7)^2}\cdot\sqrt{\sin^2 2x+\cos^2 2x}+7=$$

$$=5+25+7=37.$$

For $x=arctg\dfrac{4}{3}$, $\sin x=\dfrac{4}{5}$, $\cos x=\dfrac{3}{5}$, $\sin 2x=\dfrac{24}{25}$, $\cos 2x=-\dfrac{7}{25}$, *we obtain that* $f(x)=37$. *Therefore, the greatest value of function* $f(x)$ *is equal to 37.*

Problem 8. Consider a quadrilateral $ABCD$, such that $\angle A=\angle C=60^\circ$ and $\angle B=100^\circ$. Let O_1 and O_2 be the circumcenters of triangles ABD and CBD, respectively. Given that the angle between the lines AO_2 and CO_1 is equal to n°. Find n.

Solution. *Note that* $\angle BO_1D=2\angle BAD=120^\circ$ *and* $\angle C=60^\circ$. *Therefore,* O_1BCD *is a cyclic quadrilateral. Hence* $\angle DO_1C=\angle DBC$, *thus* $\angle CMD=\angle DBC+\angle BDO_1=\angle DBC+30^\circ$. *Therefore,* M *is the intersection point of the lines* BD *and* CO_1. *In a similar way, we deduce* $\angle AND=\angle ABD+30^\circ$, *where* N *is the intersection point of the lines* BD *and* AO_2. *We obtain that* $n^\circ=|\angle CMD-(180^\circ-\angle AND)|=|\angle CMD+\angle AND-180^\circ|=|\angle B+60^\circ-180^\circ|=20^\circ$. *Thus,* $n=20$.

Problem 9. Consider an obtuse triangle ABC with non-equal sides, circumcenter O and radius R. Given that the segment CO intersects the side AB at the point E and the radiuses of inscribed circles of triangles ACE and BCE are equal. Find $\dfrac{10R}{AC+BC}$.

Solution. *Denote by* I_1, I_2 *and by* r_1 *the incenters and the inradiuses of triangles* ACE *and* BCE, *respectively. Let* $\angle A=\alpha$ *and* $\angle B=\beta$. *We have that* $\angle AOC=2\beta$, *thus* $\angle OCA=90^\circ-\beta$ *and*

$$AC = r_1 ctg\frac{\alpha}{2} + r_1 ctg\left(45^\circ - \frac{\beta}{2}\right).$$

In a similar way, we obtain that

$$BC = r_1 ctg\frac{\beta}{2} + r_1 ctg\left(45^\circ - \frac{\alpha}{2}\right).$$

Using law of sines for triangle ABC, we deduce that

$$\frac{AC}{BC} = \frac{\sin\beta}{\sin\alpha},$$

and

$$\frac{ctg\frac{\alpha}{2} + ctg\left(45^\circ - \frac{\beta}{2}\right)}{ctg\frac{\beta}{2} + ctg\left(45^\circ - \frac{\alpha}{2}\right)} = \frac{\sin\beta}{\sin\alpha}.$$

We deduce that

$$\frac{\sin\left(45^\circ + \frac{\alpha}{2} - \frac{\beta}{2}\right)\cos\frac{\alpha}{2}}{\sin\left(45^\circ - \frac{\beta}{2}\right)} = \frac{\sin\left(45^\circ + \frac{\beta}{2} - \frac{\alpha}{2}\right)\cos\frac{\beta}{2}}{\sin\left(45^\circ - \frac{\alpha}{2}\right)}.$$

Therefore

$$\frac{\sin\left(45^\circ + \alpha - \frac{\beta}{2}\right)}{\sin\left(45^\circ - \frac{\beta}{2}\right)} = \frac{\sin\left(45^\circ + \beta - \frac{\alpha}{2}\right)}{\sin\left(45^\circ - \frac{\alpha}{2}\right)}.$$

Hence

$$\cos\left(\frac{3\alpha}{2} - \frac{\beta}{2}\right) - \cos\left(\frac{3\beta}{2} - \frac{\alpha}{2}\right) = 2\sin\frac{\alpha-\beta}{2}.$$

Thus

$$2\cos\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2} = 1$$

which means that $\sin\alpha + \sin\beta = 1$. *Using the law of sines for triangle ABC, we obtain that*

$$\frac{10R}{AB+BC} = \frac{10R}{2R\sin\alpha + 2R\sin\beta} = 5.$$

7.1.4 Problem Set 4

Problem 1. Given that $3\sin\alpha+4\cos\alpha-7\cos\beta=12$. Find the value of the expression $3\sin\beta-4\cos\beta+25\cos\alpha$.

Solution. *We have that* $3\sin\alpha+4\cos\alpha=5(\frac{3}{5}\sin\alpha+\frac{4}{5}\cos\alpha)=5\sin(\alpha+\phi)$, *where* $\phi=\arccos\frac{3}{5}$. *Thus* $3\sin\alpha+4\cos\alpha-7\cos\beta\leq 5+7=12$. *Note that the equality holds true, if* $\alpha+\phi=\frac{\pi}{2}+2\pi n$, *where* $n\in\mathbb{Z}$ *and* $\cos\beta=-1$. *Therefore* $\cos\alpha=\sin\phi=\frac{4}{5}$ *and* $3\sin\beta-4\cos\beta+25\cos\alpha=4+20=24$.

Problem 2. Given that a circle intersects the sides of an angle at four points. Let M and N be the midpoints of the arcs (of the circle) which are outside of the angle. Denote by n° the angle between the line MN and the bisector of the given angle. Find n.

Solution. *Note that the line passing by the midpoints of two segments (located outside of the angle) of the circle cuts apart from the angle an isosceles triangle. Therefore* $n=90$, *as a bisector of an isosceles triangle is also an altitude.*

Problem 3. Let ABC be a triangle. Given that the median (of vertex B) is equal to 27. Consider a point N on that median, such that $BN=24$ and $\angle ANC=180^\circ-\angle ABC$. Find AC.

Solution. *Let us consider point K, which is symmetric to point N with respect to the midpoint M of side AC. We have that $ANCK$ is a parallelogram, therefore $\angle AKC=\angle ANC=180^\circ-\angle ABC$. We obtain that the points A,B,C,K are placed on a circle. Hence $AM\cdot MC=BM\cdot MK$, thus $AM^2=27\cdot 3$. Hence $AC=2AM=18$.*

Problem 4. Consider a triangle ABC. Let the sides AB and BC be tangent of a circle ω at points E and F, respectively. Given that ω intersects the side AC and M is such a point on the side AC that $AE:CF=AM:MC$. Assume that the line FM intersects ω at point K. Given also that $AE=14$. Find AK.

Solution. *Consider the tangent line AP of the circle ω. Consider also the line l parallel to FP and passing through the point C. Let $AP\cap l=N$. We have that $\angle PFC=\frac{\overset{\frown}{PF}}{2}=\angle NPF$ and $CN\parallel PF$, thus $PN=CF$. By Thales' theorem $\frac{AM_1}{M_1C}=\frac{AP}{PN}=\frac{AE}{CF}$, where $M_1=PF\cap AC$. Therefore $\frac{AM_1}{M_1C}=\frac{AM}{MC}$, thus $M_1\equiv M$ and $K\equiv P$. Hence $AK=AE=14$.*

Problem 5. Given that the points $M(-1,5),N(2,6),K(4,4),P(0,1)$ are on the district sides of a square. Find the area of the square.

Solution. *Note that N and P are on the parallel sides of the square. Consider a point $L(x,y)$, such that $ML \perp PN$, $ML = PN$ and points M, L are located on the different sides of line PN. Note that L is on the line which includes a side of the square and the point K. We have that $\overrightarrow{ML} \cdot \overrightarrow{NP} = 0$ and $ML = NP$, hence $2x+5y = 23$ and $(x+1)^2+(y-5)^2 = 29$. As $x > 0$, thus $x = 4$ and $y = 3$. Therefore $a = \rho(M, KL) = 5$, where a is the length of square's side and $S = 25$.*

Problem 6. Given that $\phi = \frac{\pi}{12} - \frac{1}{2}\arccos\frac{5\sqrt{3}+1}{10}$. Find the value of the following expression

$$\frac{1}{\sin\phi\cdot\sin\left(\phi+\frac{\pi}{6}\right)}+\frac{1}{\sin\left(\phi+\frac{\pi}{6}\right)\cdot\sin\left(\phi+\frac{2\pi}{6}\right)}+\cdots+\frac{1}{\sin\left(\phi+\frac{4\pi}{6}\right)\cdot\sin\left(\phi+\frac{5\pi}{6}\right)}.$$

Solution. *We have that*

$$\frac{1}{\sin\phi\sin\left(\phi+\frac{\pi}{6}\right)}+\cdots+\frac{1}{\sin\left(\phi+\frac{4\pi}{6}\right)\sin\left(\phi+\frac{5\pi}{6}\right)}=$$

$$=2\Bigg(\frac{\sin\frac{\pi}{6}}{\sin\phi\sin\left(\phi+\frac{\pi}{6}\right)}+\cdots+\frac{\sin\frac{\pi}{6}}{\sin\left(\phi+\frac{4\pi}{6}\right)\sin\left(\phi+\frac{5\pi}{6}\right)}\Bigg)=$$

$$=2\Bigg(\frac{\sin\left(\left(\phi+\frac{\pi}{6}\right)-\phi\right)}{\sin\phi\sin\left(\phi+\frac{\pi}{6}\right)}+\cdots+\frac{\sin\left(\left(\phi+\frac{5\pi}{6}\right)-\left(\phi+\frac{4\pi}{6}\right)\right)}{\sin\left(\phi+\frac{4\pi}{6}\right)\sin\left(\phi+\frac{5\pi}{6}\right)}\Bigg)=$$

$$=2\left(ctg\phi-ctg\left(\phi+\frac{\pi}{6}\right)+\cdots+ctg\left(\phi+\frac{4\pi}{6}\right)-ctg\left(\phi+\frac{5\pi}{6}\right)\right)=$$

$$=2\left(ctg\phi-ctg\left(\phi+\frac{5\pi}{6}\right)\right)=\frac{2\sin\frac{5\pi}{6}}{\sin\phi\sin\left(\phi+\frac{5\pi}{6}\right)}=$$

$$=\frac{2}{\cos\frac{5\pi}{6}-\cos\left(2\phi+\frac{5\pi}{6}\right)}=\frac{2}{-\frac{\sqrt{3}}{2}+\frac{5\sqrt{3}+1}{10}}=20.$$

Problem 7. Let in triangle ABC a line, which passes through its incenter and is parallel to the side AC, intersects sides AB and BC at points E, F, respectively.

Denote by D the midpoint of the side AC, by M the intersection point of rays ED and BC, by N the intersection point of rays FD and BA. Given that the parameter of ABC is equal to 22 and $AC = 10$. Find MN.

Solution. *We have that $\triangle NAD \sim \triangle NEF$ and $\triangle MCD \sim \triangle MFE$, hence $\frac{NA}{NE} = \frac{AD}{EF} = \frac{DC}{EF} = \frac{MC}{MF}$. Therefore $\frac{NE}{NA} = \frac{MF}{MC}$, thus $\frac{AE}{NA} = \frac{CF}{MC}$. We have that $EF \parallel AC$, thus $MN \parallel AC$. Let I be the incenter of triangle ABC and $EF = EI + IF = AE + CF$.*

Note that $\triangle EAD \sim \triangle ENM$, hence $\frac{EA}{EN} = \frac{AD}{MN}$. We have that $\frac{NA}{EN} = \frac{AD}{EF}$, thus $NA = \frac{EA}{EF} \cdot MN$. In a similar way, we obtain that $MC = \frac{CF}{EF} \cdot MN$, hence

$$NA + MC = \left(\frac{EA}{EF} + \frac{CF}{EF}\right) MN = MN.$$

We have that $\triangle ABC \sim \triangle NBM$, thus $\frac{22}{12 + 2MN} = \frac{10}{MN}$, thus $MN = 60$.

Problem 8. Evaluate the expression

$$4096\sqrt{3} \sin\frac{\pi}{27} \sin\frac{2\pi}{27} \sin\frac{4\pi}{27} \sin\frac{5\pi}{27} \sin\frac{7\pi}{27} \sin\frac{8\pi}{27} \sin\frac{10\pi}{27} \sin\frac{11\pi}{27} \sin\frac{13\pi}{27}.$$

Solution. *Let us prove that*

$$\sin\alpha \sin\left(\frac{\pi}{3} - \alpha\right) \sin\left(\frac{\pi}{3} + \alpha\right) = \frac{1}{4}\sin 3\alpha. \tag{7.1}$$

We have that

$$\sin\alpha \sin\left(\frac{\pi}{3} - \alpha\right) \sin\left(\frac{\pi}{3} + \alpha\right) = \frac{1}{2}\sin\alpha(\cos 2\alpha - \cos\frac{2\pi}{3}) =$$

$$= \frac{1}{2}(\sin\alpha\cos 2\alpha + \frac{1}{2}\sin\alpha) = \frac{1}{2}(\frac{1}{2}\sin 3\alpha - \frac{1}{2}\sin\alpha + \frac{1}{2}\sin\alpha) = \frac{1}{4}\sin 3\alpha.$$

By (7.1) we have that

$$\sin\frac{\pi}{27}\sin\frac{8\pi}{27}\sin\frac{10\pi}{27} = \frac{1}{4}\sin\frac{\pi}{9}$$

$$\sin\frac{2\pi}{27}\sin\frac{7\pi}{27}\sin\frac{11\pi}{27} = \frac{1}{4}\sin\frac{2\pi}{9}$$

$$\sin\frac{4\pi}{27}\sin\frac{5\pi}{27}\sin\frac{13\pi}{27} = \frac{1}{4}\sin\frac{4\pi}{9}.$$

Hence, we deduce that

$$4096\sqrt{3}\sin\frac{\pi}{27}\sin\frac{2\pi}{27}\sin\frac{4\pi}{27}\sin\frac{5\pi}{27}\sin\frac{7\pi}{27}\sin\frac{8\pi}{27}\sin\frac{10\pi}{27}\sin\frac{11\pi}{27}\sin\frac{13\pi}{27}=$$

$$=4096\sqrt{3}\frac{1}{64}\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{4\pi}{9}=64\sqrt{3}\frac{1}{4}\sin\frac{\pi}{3}=24.$$

Problem 9. Let all the vertices of a triangle ABC are on parabola $y=x^2$. Given that line AB is parallel to axes Ox and that the abscissa of point C is located in between of the abscissas of points A and B. Let CH be the altitude of ABC and $\tan\angle ACB=0.01$. Find $\dfrac{CH-1}{AB}$.

Solution. *Let* $A\left(-\frac{a}{2},\frac{a^2}{4}\right)$, $B\left(\frac{a}{2},\frac{a^2}{4}\right)$, $C(x_0,x_0^2)$, *where* $a>0$ *and* $-\frac{a}{2}<x_0<\frac{a}{2}$. *From the right triangles* ACH *and* BCH, *we obtain that* $\tan\angle ACH=\dfrac{AH}{CH}$ *and* $\tan\angle BCH=\dfrac{BH}{CH}$. *Therefore*

$$\tan\angle ACB=\frac{\tan\angle ACH+\tan\angle BCH}{1-\tan\angle ACH\cdot\tan\angle BCH}=\frac{AB\cdot CH}{CH^2-AH\cdot BH}=$$

$$=\frac{AB\cdot CH}{CH^2-\left(\frac{a^2}{4}-x_0^2\right)}=\frac{AB\cdot CH}{CH^2-CH}.$$

Hence, we obtain that

$$\frac{CH-1}{AB}=\frac{1}{\tan\angle ACB}=100.$$

7.1.5 Problem Set 5

Problem 1. The medians of triangle ABC intersect at point G. Given that $\angle AGB=90^\circ$ and $AB=20$. Find the length of the median corresponding to vertex C.

Solution. *Let* M *be the midpoint of* AB *side. We have that* $GM=\dfrac{AB}{2}$ *and* $CG=2GM$, *thus* $CM=CG+GM=20+10=30$.

Problem 2. Let $ABCD$ be a trapezoid. A circle is inscribed inside the trapezoid and touches the bases AD and BC at points M and N, respectively. Given that $AM=9$, $MD=12$ and $BN=4$. Find NC.

Solution. *Let O be the centre of the inscribed circle of the trapezoid. Assume that the inscribed circle touches the legs AB and CD at the points E and K, respectively. We have that $\angle COD = 180^\circ - \angle OCD - \angle ODC = 180^\circ - \frac{\angle C}{2} - \frac{\angle D}{2} = 90^\circ$. In a similar way, we deduce that $\angle AOB = 90^\circ$. On the other hand, we have that $AE = AM = 9$, $BE = BN = 4$ and $KD = MD = 12$. Therefore $AE \cdot EB = OE^2 = OK^2 = CK \cdot KD$. Hence $NC = CK = \frac{9 \cdot 4}{12} = 3$.*

Problem 3. Find the greatest value of the expression

$$2\cos 3x - 3\sqrt{3}\sin x - 3\cos x.$$

Solution. *We have that*

$$2\cos 3x - 3\sqrt{3}\sin x - 3\cos x = 2\cos 3x - 6\sin\left(x + \frac{\pi}{6}\right) \leq 2 + 6 = 8.$$

If $x = -\frac{2\pi}{3}$, then $2\cos 3x - 6\sin\left(x + \frac{\pi}{6}\right) = 2 + 6 = 8$. Therefore, the greatest value of the given expression is equal to 8.

Problem 4. Consider a trapezoid, such that the base angles (corresponding to the longer base) are equal to 15° and 75°. Given that the length of the segment connecting the midpoints of its diagonals is equal to 20. Find the altitude of the trapezoid.

Solution. *Let $\angle A = 75^\circ$, $\angle D = 15^\circ$ and points M, N be the midpoints of bases BC, AD, respectively. Let $ME \parallel AB$, $MF \parallel CD$, $MH \perp AD$, where E, H, N points are on base AD. Note that $\angle MEF = \angle A = 75^\circ$ and $\angle MFE = \angle D = 15^\circ$, thus $\angle EMF = 90^\circ$. On the other hand, $EN = AN - AE = AN - BM = DN - MC = DN - DF = NF$. Therefore, $MN = NF$. Hence $\angle MNH = 30^\circ$ and $MH = \frac{MN}{2} = \frac{AD - BC}{2} = 10$.*

Problem 5. Let ABC be an acute-angled triangle. Denote by H the intersection point of the altitudes AA_1 and BB_1. Let M be a random point on the side AC and ω be the circumcircle of triangle MA_1C. Let MN be a diameter of ω. Denote by n° the angle between the lines NH and MB. Find n.

Solution. *Let K be the second intersection point of line NH and circle ω. We have that $\angle A_1KN = \angle A_1CN = \angle MCN - \angle MCA_1 = 90^\circ - \angle ACB = \angle A_1BH$. Therefore $\angle A_1KN = \angle A_1BH$, thus points H, B, A_1, K are on the same circle. Hence, we deduce that $\angle HKB = \angle HA_1B = 90^\circ$. Therefore, points B, K, M are on the same line and $n = 90$.*

Problem 6. Evaluate the expression

$$11\sin^2\frac{3\pi}{11} + \left(2\sin\frac{\pi}{11} - \sin\frac{3\pi}{11} - 2\sin\frac{5\pi}{11}\right)^2.$$

Solution. *We have that*

$$11\sin^2\frac{3\pi}{11}+\left(2\sin\frac{\pi}{11}-\sin\frac{3\pi}{11}-2\sin\frac{5\pi}{11}\right)^2=12\sin^2\frac{3\pi}{11}+4\sin^2\frac{\pi}{11}+4\sin^2\frac{5\pi}{11}-$$

$$-4\sin\frac{\pi}{11}\sin\frac{3\pi}{11}+4\sin\frac{3\pi}{11}\sin\frac{5\pi}{11}-8\sin\frac{\pi}{11}\sin\frac{5\pi}{11}=6-6\cos\frac{6\pi}{11}+2-2\cos\frac{2\pi}{11}+2-$$

$$-2\cos\frac{10\pi}{11}-2\cos\frac{2\pi}{11}+2\cos\frac{4\pi}{11}+2\cos\frac{2\pi}{11}-2\cos\frac{8\pi}{11}-4\cos\frac{4\pi}{11}+4\cos\frac{6\pi}{11}=$$

$$=10-\left(2\cos\frac{2\pi}{11}+2\cos\frac{4\pi}{11}+2\cos\frac{6\pi}{11}+2\cos\frac{8\pi}{11}+2\cos\frac{10\pi}{11}\right)=10-$$

$$-\frac{-\sin\frac{\pi}{11}+\sin\frac{3\pi}{11}-\sin\frac{3\pi}{11}+\sin\frac{5\pi}{11}-\cdots-\sin\frac{9\pi}{11}+\sin\frac{11\pi}{11}}{\sin\frac{\pi}{11}}=$$

$$=10-\frac{-\sin\frac{\pi}{11}+\sin\pi}{\sin\frac{\pi}{11}}=11.$$

Problem 7. Let A and B be points on the parabola $y=x^2$. Let C be a point on that parabola, such that the tangent line to a parabola passing through point C is parallel to AB. Given that the median corresponding to vertex C of triangle ABC is equal to 81. Find the area of triangle ABC.

Solution. *Consider the points $A(x_1,x_1^2)$, $B(x_2,x_2^2)$, $C(x_0,x_0^2)$. The equation of a tangent line passing through point C is $y=k(x-x_0)+x_0^2$, where $k=\dfrac{x_2^2-x_1^2}{x_2-x_1}=x_2+x_1$. Let D be a point on that line with abscissa x_2, we have that*

$$(ABC)=(ABD)=\frac{1}{2}|x_2-x_1||x_2^2-((x_1+x_2)(x_2-x_0)+x_0^2)|=$$

$$=\frac{1}{2}|x_2-x_1||x_0(x_0-x_1-x_2)+x_1x_2|. \tag{7.2}$$

Note that parabola $y=x^2$ and line $y=(x_1+x_2)(x-x_0)+x_0^2$ have only one intersection point. Therefore, the discriminant of the following quadratic equation

$$x^2-(x_1+x_2)x-x_0^2+x_0(x_1+x_2)=0$$

is equal to 0. Hence, $(x_1+x_2-2x_0)^2=0$. Thus, we obtain that

$$x_0 = \frac{x_1 + x_2}{2}. \tag{7.3}$$

Therefore, by (7.89) and (7.90) we deduce that

$$(ABC) = \frac{1}{8}|x_2 - x_1| \cdot (x_2 - x_1)^2.$$

Let M be the midpoint of AB, thus we have that

$$81 = CM = \left(\frac{x_1^2 + x_2^2}{2} - \left(\frac{x_1 + x_2}{2}\right)^2\right) = \left(\frac{x_2 - x_1}{2}\right)^2.$$

Hence, we obtain that

$$(ABC) = \frac{1}{8} \cdot 9 \cdot 2 \cdot 4 \cdot 81 = 729.$$

Problem 8. Let

$$2\cos^2\left(2\phi + \frac{5\pi}{6}\right) = \sqrt{3}\left(\sin\left(2\phi + \frac{5\pi}{6}\right) - \cos\left(2\phi + \frac{5\pi}{6}\right)\right).$$

Find the value of the following expression

$$\frac{\cos\left(\phi + \frac{\pi}{6}\right)}{\sin\phi \sin\left(\phi + \frac{\pi}{6}\right)\sin\left(\phi + \frac{2\pi}{6}\right)} + \frac{\cos\left(\phi + \frac{2\pi}{6}\right)}{\sin\left(\phi + \frac{\pi}{6}\right)\sin\left(\phi + \frac{2\pi}{6}\right)\sin\left(\phi + \frac{3\pi}{6}\right)} +$$

$$\frac{\cos\left(\phi + \frac{3\pi}{6}\right)}{\sin\left(\phi + \frac{2\pi}{6}\right)\sin\left(\phi + \frac{3\pi}{6}\right)\sin\left(\phi + \frac{4\pi}{6}\right)} + \frac{\cos\left(\phi + \frac{4\pi}{6}\right)}{\sin\left(\phi + \frac{3\pi}{6}\right)\sin\left(\phi + \frac{4\pi}{6}\right)\sin\left(\phi + \frac{5\pi}{6}\right)}.$$

Solution. *We have that*

$$\frac{\cos\left(x + \frac{\pi}{6}\right)}{\sin x \sin\left(x + \frac{\pi}{6}\right)\sin\left(x + \frac{2\pi}{6}\right)} = \frac{2}{\sqrt{3}} \cdot \frac{\cos\left(x + \frac{\pi}{6}\right)\sin\left(x + \frac{2\pi}{6} - x\right)}{\sin x \sin\left(x + \frac{\pi}{6}\right)\sin\left(x + \frac{2\pi}{6}\right)} =$$

$$= \frac{2}{\sqrt{3}}\left(ctgx \cdot ctg\left(x + \frac{\pi}{6}\right) - ctg\left(x + \frac{\pi}{6}\right) \cdot ctg\left(x + \frac{2\pi}{6}\right)\right) = \frac{2}{\sqrt{3}}\left(ctgx \cdot ctg\left(x + \frac{\pi}{6}\right) +\right.$$

$$\left. 1 - \left(ctg\left(x + \frac{\pi}{6}\right) \cdot ctg\left(x + \frac{2\pi}{6}\right) + 1\right)\right) =$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\cos\frac{\pi}{6}}{\sin x\sin\left(x+\frac{\pi}{6}\right)} - \frac{\cos\frac{\pi}{6}}{\sin\left(x+\frac{\pi}{6}\right)\sin\left(x+\frac{2\pi}{6}\right)}\right) =$$

$$= \frac{1}{\sin x\sin\left(x+\frac{\pi}{6}\right)} - \frac{1}{\sin\left(x+\frac{\pi}{6}\right)\sin\left(x+\frac{2\pi}{6}\right)}.$$

Therefore, we obtain that

$$\frac{\cos\left(x+\frac{\pi}{6}\right)}{\sin x\sin\left(x+\frac{\pi}{6}\right)\sin\left(x+\frac{2\pi}{6}\right)} = \frac{1}{\sin x\sin\left(x+\frac{\pi}{6}\right)} - \frac{1}{\sin\left(x+\frac{\pi}{6}\right)\sin\left(x+\frac{2\pi}{6}\right)}.$$

In the last equality, setting the following values $x = \phi$, $x = \phi + \frac{\pi}{6}$, $x = \phi + \frac{2\pi}{6}$, $x = \phi + \frac{3\pi}{6}$, *and summing up the obtained four equalities, we deduce that*

$$\frac{\cos\left(\phi+\frac{\pi}{6}\right)}{\sin\phi\sin\left(\phi+\frac{\pi}{6}\right)\sin\left(\phi+\frac{2\pi}{6}\right)} + \frac{\cos\left(\phi+\frac{2\pi}{6}\right)}{\sin\left(\phi+\frac{\pi}{6}\right)\sin\left(\phi+\frac{2\pi}{6}\right)\sin\left(\phi+\frac{3\pi}{6}\right)} +$$

$$\frac{\cos\left(\phi+\frac{3\pi}{6}\right)}{\sin\left(\phi+\frac{2\pi}{6}\right)\sin\left(\phi+\frac{3\pi}{6}\right)\sin\left(\phi+\frac{4\pi}{6}\right)} + \frac{\cos\left(\phi+\frac{4\pi}{6}\right)}{\sin\left(\phi+\frac{3\pi}{6}\right)\sin\left(\phi+\frac{4\pi}{6}\right)\sin\left(\phi+\frac{5\pi}{6}\right)} =$$

$$= \frac{1}{\sin\phi\sin\left(\phi+\frac{\pi}{6}\right)} - \frac{1}{\sin\left(\phi+\frac{\pi}{6}\right)\sin\left(\phi+\frac{2\pi}{6}\right)} + \dots + \frac{1}{\sin\left(\phi+\frac{3\pi}{6}\right)\sin\left(\phi+\frac{4\pi}{6}\right)} -$$

$$- \frac{1}{\sin\left(\phi+\frac{4\pi}{6}\right)\sin\left(\phi+\frac{5\pi}{6}\right)} = \frac{1}{\sin\phi\sin\left(\phi+\frac{\pi}{6}\right)} - \frac{1}{\sin\left(\phi+\frac{4\pi}{6}\right)\sin\left(\phi+\frac{5\pi}{6}\right)} =$$

$$= \frac{2\left(\cos\left(2\phi+\frac{\pi}{6}\right) - \cos\left(2\phi+\frac{3\pi}{2}\right)\right)}{\left(\cos\frac{5\pi}{6} - \cos\left(2\phi+\frac{5\pi}{6}\right)\right)\left(\cos\frac{\pi}{2} - \cos\left(2\phi+\frac{5\pi}{6}\right)\right)} =$$

$$= \frac{4\sqrt{3}\sin\left(2\phi+\frac{5\pi}{6}\right)}{\cos\left(2\phi+\frac{5\pi}{6}\right)\left(2\cos\left(2\phi+\frac{5\pi}{6}\right)+\sqrt{3}\right)} = 4.$$

Problem 9. Let ABC be an isosceles triangle. Denote by O the centre of a circle that intersects sides AB and BC at points M,N and P,Q, respectively, such that point M is between points A, N and point Q is between points P, C. Given that the distance of point O and the altitude BH of triangle ABC are equal to $\sqrt{6}+\sqrt{2}$ and $\angle B = 30^\circ$. Find $|AM+BP-CQ-BN|$.

Solution. *Let OE, OF, OD be perpendicular to AB, BC, AC sides, respectively. We have that $ME = EN$ and $PF = FQ$, thus*

$$|AM+BP-CQ-BN| = |AE+BF-CF-BE| = 2|AE-CF|.$$

According to Carnot's theorem, we obtain that

$$AE^2+BF^2+CD^2 = CF^2+BE^2+AD^2.$$

Therefore

$$(AE-CF)(AE+CF)+(BF-BE)(BF+BE) = (AD-CD)AC.$$

$$|AE-CF|\cdot(AE+CF+BF+BE) = HD\cdot AC.$$

$$2|AE-CF| = HD\cdot\frac{AC}{AB} = (\sqrt{6}+\sqrt{2})\cdot 2\cdot\cos 75^\circ = (\sqrt{6}+\sqrt{2})\cdot 2\cdot\frac{\sqrt{6}-\sqrt{2}}{4} = 2.$$

7.1.6 Problem Set 6

Problem 1. Let AA_1 and BB_1 be the altitudes of an acute triangle ABC. Let O be the circumcenter of triangle ABC. Given that $\angle C = 60^\circ$ and that the distance of point O and line AA_1 is equal to 15. Find the distance of point O and line BB_1.

Solution. *Let $AA_1 \cap BB_1 = H$, $OM \perp AA_1$, $M \in AA_1$ and $ON \perp BB_1$, $N \in BB_1$. The midpoints of BC and AC sides are points E and F. We have that $\angle CBB_1 = 30^\circ$, thus $\frac{HB}{2} = HA_1$. On the other hand, $\frac{HB}{2} = OF$, hence $HA_1 = OF$. In a similar way, one can prove that $HB_1 = OE$. Therefore, $HM = |HA_1 - OE| = |HB_1 - OF| = HN$. Hence, $ON = OM = 15$.*

Problem 2. Let α,β,γ be the angles of triangle ABC. Given that $4\sin\alpha + 4\sin\beta = 9+8\cos\gamma$. Find $64(\cos^2\gamma - \sin^2\alpha - \sin^2\beta)$.

Solution. *We have that*

$$\left(4\cos\frac{\gamma}{2} - \cos\frac{\alpha-\beta}{2}\right)^2 + \sin^2\frac{\alpha-\beta}{2} = 0.$$

Hence, $\cos\frac{\gamma}{2} = \frac{1}{4}$ *and* $\sin\frac{\alpha-\beta}{2} = 0$. *Thus,* $\alpha = \beta$, $\gamma = 180^\circ - 2\alpha$. *We deduce that* $\sin\alpha = \frac{1}{4}$. *Therefore,*

$$64(\cos^2\gamma - \sin^2\alpha - \sin^2\beta) = 64\left(\left(2\cos^2\alpha - 1\right)^2 - \frac{1}{16} - \frac{1}{16}\right) = 64\left(\frac{49}{64} - \frac{1}{8}\right) = 41.$$

Problem 3. Consider a right-angled triangle ABC. Let O be the centre of a semicircle that is inscribed to ABC, such that O is on the hypotenuse AB. Let D be a point of that semicircle, such that $BD = BC$. Given that $\cos\angle A = \frac{12}{13}$. Find $13\cos\angle BOD$.

Solution. *Let* $\angle A = \alpha$, $\angle BOD = \phi$ *and* $OD = r$. *We have* $BD = BC = r + rtg\alpha$. *Using the law of cosines for triangle BOD, we obtain that*

$$a^2 = r^2 + \frac{r^2}{\cos^2\alpha} - \frac{2r^2}{\cos\alpha}\cos\phi.$$

Thus

$$(r + rtg\alpha)^2 = r^2 + \frac{r^2}{\cos^2\alpha} - \frac{2r^2}{\cos\alpha}\cos\phi.$$

We obtain that,

$$2\cos\phi = \cos\alpha - 2\sin\alpha = \frac{12}{13} - \frac{10}{13}$$

and

$$\cos\phi = \frac{1}{13}.$$

Therefore $13\cos\phi = 1$.

Problem 4. Let acute triangle ABC is inscribed to a circle. Let l be a tangent line passing through point C. Given that the altitude CC_1 is equal to 29. Denote by a the distance of point A and line l. Denote by b the distance of point B and line l. Find the value of the product ab.

Solution. *Let* $M, N \in l$ *and* $AM \perp l$, $BN \perp l$. *We have that* $\triangle BCC_1 \sim \triangle CAM$. *Then* $\frac{AM}{CC_1} = \frac{AC}{BC}$, *hence* $AM = CC_1 \cdot \frac{AC}{BC}$. *In a similar way, we deduce* $\triangle ACC_1 \sim \triangle CBN$. *Thus,* $BN = CC_1 \cdot \frac{BC}{AC}$. *Therefore, we deduce that*

$$AM \cdot BN = CC_1 \cdot \frac{AC}{BC} \cdot CC_1 \cdot \frac{BC}{AC} = CC_1^2 = 841.$$

Problem 5. Let $ABCD$ be inscribed and circumscribed quadrilateral. Let the inscribed circle touches the sides AB and CD at points M and N. Given that $CN = 14$, $DN = 8$, $AM = 12$. Find BM.

Solution. *Let* I *be the centre of the circle inscribed to quadrilateral ABCD. We have that* $\angle AMI = 90^\circ$, $\angle CNI = 90^\circ$ *and* $\angle IAM = \frac{1}{2}\angle BAD = 90^\circ - \frac{1}{2}\angle BCD =$

$90^\circ - \angle ICN = \angle CIN$, thus $\triangle AIM \sim \triangle ICN$. Hence, $IM^2 = AM \cdot CN$. In a similar way, we obtain that $\triangle BMI \sim \triangle IND$, thus $IM^2 = BM \cdot DN$. Therefore, $AM \cdot CN = BM \cdot DN$. We deduce that,

$$BM = \frac{12 \cdot 14}{8} = 21.$$

Problem 6. Given that

$$\cos\alpha + \cos\beta + \cos\gamma + \cos\alpha\cos\beta\cos\gamma = \frac{1273}{845},$$

$$\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\alpha\cos\gamma = \frac{2532}{4225}.$$

Find the value of the following expression

$$\frac{4225}{3}|\sin\alpha\sin\beta\sin\gamma|.$$

Solution. *We have that*

$$(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) = 1 + (\cos\alpha + \cos\beta + \cos\gamma + \cos\alpha\cos\beta\cos\gamma) +$$

$$+ \cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha = 1 + \frac{1273}{845} + \frac{2532}{4225} = \frac{13122}{4225}$$

and

$$(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma) = 1 - \frac{1273}{845} + \frac{2532}{4225} = \frac{392}{4225}.$$

Therefore,

$$\sin^2\alpha\sin^2\beta\sin^2\gamma = (1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma) =$$

$$= \frac{2268^2}{4225^2}.$$

Thus,

$$\frac{4225}{3}|\sin\alpha\sin\beta\sin\gamma| = 756.$$

Problem 7. Let quadrilateral $ABCD$ be, such that $\angle BAC = 55^\circ$, $\angle DAC = 10^\circ$, $\angle BCA = \angle DCA = 25^\circ$ and $\angle ADB = n^\circ$. Find n.

Solution. *Let I be the centre of the inscribed circle of triangle BCD. We have that*

$$\angle BID = 90^\circ + \frac{1}{2}\angle BCD = 115^\circ.$$

Note that $\angle BAD + \angle BID = 65^\circ + 115^\circ = 180^\circ$, hence ABID is an inscribed quadrilateral. Therefore, $\angle IBC = \angle IBD = \angle IAD = 10^\circ$. On the other hand, $\angle ADB = \angle AIB = \angle IBC + \angle ICB = 35^\circ$. Thus $n = 35$.

Problem 8. Let M be the smallest number, such that the following inequality

$$8\cos x\cos y\cos z(\tan x + \tan y + \tan z) \leq M,$$

holds true for any positive numbers x, y, z, where $x + y + z = \frac{\pi}{2}$. Find M.

Solution. *Note that*

$$2\cos x\cos y\cos z(\tan x + \tan y + \tan z) = \cos x\cos y\cos z(\tan x + \tan y) +$$

$$+ \cos x\cos y\cos z(\tan y + \tan z) + \cos x\cos y\cos z(\tan x + \tan z) =$$

$$= \cos z\sin(x + y) + \cos x\sin(y + z) + \cos y\sin(x + z) =$$

$$= \cos z\sin\left(\frac{\pi}{2} - z\right) + \cos x\sin\left(\frac{\pi}{x} - x\right) + \cos y\sin\left(\frac{\pi}{2} - y\right) = \cos^2 x + \cos^2 y + \cos^2 z.$$

Therefore,

$$8\cos x\cos y\cos z(\tan x + \tan y + \tan z) = 4\cos^2 x + 4\cos^2 y + 4\cos^2 z = 8 + 2\cos 2x + 2\cos 2y -$$

$$-4\sin^2 z = 8 + 4\sin z\cos(x - y) - 4\sin^2 z = 8 - (2\sin z - \cos(x - y))^2 + \cos^2(x - y) \leq 9.$$

If $x = y = z = \frac{\pi}{6}$, we have that

$$8\cos x\cos y\cos z(\tan x + \tan y + \tan z) = 9.$$

Thus, the smallest value of M is equal to 9.

Problem 9. Consider a triangle ABC. Let the segments AA_1, BB_1, CC_1 intersect at the same point. The segments AA_1, B_1C_1 intersect at point P, and the segments CC_1, A_1B_1 intersect at point Q. Given that $\angle ABP = 75^\circ$, $\angle PBB_1 = 30^\circ$, $\angle QBB_1 = 15^\circ$ and $\angle ABC = n^\circ$. Find n.

Solution. *Denote by R the intersection point of BB_1 and A_1C_1. According to Ceva's theorem, from triangle $A_1B_1C_1$, we deduce that*

$$\frac{C_1P}{PB_1} \cdot \frac{B_1Q}{QA_1} \cdot \frac{A_1R}{RC_1} = 1. \tag{7.4}$$

Denote by (XYZ) the area of triangle XYZ, we have that

$$\frac{C_1P}{PB_1} = \frac{(C_1BP)}{(PBB_1)} = \frac{\frac{1}{2}C_1B \cdot BP \sin 75^\circ}{\frac{1}{2}BP \cdot BB_1 \sin 30^\circ} = \frac{C_1B \sin 75^\circ}{BB_1 \sin 30^\circ}.$$

Therefore, we obtain that

$$\frac{C_1P}{PB_1} = \frac{C_1B \sin 75^\circ}{BB_1 \sin 30^\circ}.$$

In a similar way, we obtain that

$$\frac{B_1Q}{QA_1} = \frac{BB_1 \sin 15^\circ}{BA_1 \sin \alpha}$$

and

$$\frac{A_1R}{RC_1} = \frac{BA_1 \sin(\alpha + 15^\circ)}{C_1B \sin 105^\circ},$$

where $\alpha = \angle QBC$. *From the obtained equalities and* (7.89)*, we deduce that,*

$$\sin 15^\circ \sin 75^\circ \sin(\alpha + 15^\circ) = \sin 30^\circ \sin \alpha \sin 105^\circ.$$

Thus,

$$\sin 15^\circ \sin(\alpha + 15^\circ) = \sin 30^\circ \sin \alpha.$$

Hence, we obtain that

$$\cos \alpha - \cos(\alpha + 30^\circ) = \cos(\alpha - 30^\circ) - \cos(\alpha + 30^\circ).$$

Therefore, $\alpha = 30^\circ - \alpha$ *and* $n = 75 + 30 + 15 + 15 = 135$.

7.1.7 Problem Set 7

Problem 1. The altitudes of triangle ABC intersect at point H. Given that $\angle C = 45^\circ$ and $AB = 20$. Find CH.

Solution. *Let O be the circumcenter of triangle ABC and M be the midpoint of side AB. We have that* $\angle MOB = \frac{1}{2} \cdot \angle AOB = \frac{1}{2} \cdot 2\angle C = 45^\circ$, $\angle OMB = 90^\circ$ *and* $CH = 2OM = 2MB = AB = 20$.

Problem 2. Find the greatest value of the following equation

$$(|\sin \alpha - \cos \beta| + |\cos \alpha + \sin \beta|)^2.$$

Solution. *We have that*

$$|\sin\alpha - \cos\beta| \leq |\sin\alpha| + |\cos\beta|,$$

and

$$|\cos\alpha + \sin\beta| \leq |\cos\alpha| + |\sin\beta|.$$

Hence, we deduce that

$$|\sin\alpha - \cos\beta| + |\cos\alpha + \sin\beta| \leq \sqrt{(|\sin\alpha| + |\cos\alpha|)^2} + \sqrt{(|\cos\beta| + |\sin\beta|)^2}$$

$$= \sqrt{1 + |\sin 2\alpha|} + \sqrt{1 + |\sin 2\beta|} \leq 2\sqrt{2}.$$

Thus, we obtain that

$$(|\sin\alpha - \cos\beta| + |\cos\alpha + \sin\beta|)^2 \leq 8.$$

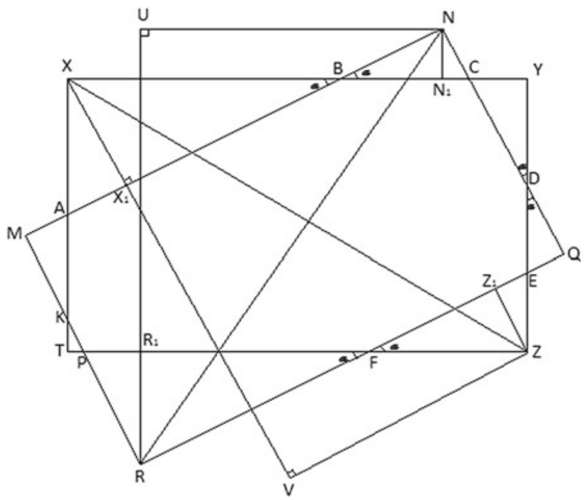
If $\alpha = \dfrac{\pi}{4}$ *and* $\beta = \dfrac{3\pi}{4}$, *then*

$$(|\sin\alpha - \cos\beta| + |\cos\alpha + \sin\beta|)^2 = 8.$$

Therefore, the greatest value of the given expression is equal to 8.

Problem 3. Let $ABCDEFPK$ be a octagon created by the intersection of two equal rectangles. Given that $AB + CD + EF + PK = 23$. Find the perimeter of $ABCDEFPK$.

Solution. *Let* $MN = XY$. *Let us prove that* $AB + EF = BC + FP$.



Note that if $\angle ABX = \alpha$, *then* $\angle NBC = \alpha$, $\angle CNB = \angle CYD = 90^\circ$. *Hence,* $\angle CDY = \angle QDE = \alpha$, $\angle DEQ = \angle ZEF = 90^\circ - \alpha$, $\angle ZFE = \angle RFP = \alpha$. *Therefore, triangles* AXB, BNC, FZE, PRF *are pairwise similar. Thus, we obtain that*

$$\frac{AB}{XX_1} = \frac{EF}{ZZ_1} = \frac{BC}{NN_1} = \frac{PF}{RR_1} = k. \tag{7.5}$$

Note that, $\angle RNM = \angle XZT$. *If we denote* $\angle RNM = \beta$, *then* $\angle XZV = \beta + \alpha = \angle RNU$ *and* $XZ = RN$. *Hence,* $\triangle XZV = \triangle RNU$, $XV = RU$ *or* $XX_1 + MR + ZZ_1 = NN_1 + YZ + RR_1$. *Thus, we deduce that* $XX_1 + ZZ_1 = NN_1 + RR_1$. *From this equation and from* (7.5)*, it follows that* $AB + EF = BC + PF$. *In a similar way, one can prove that* $CD + PK = DE + KA$. *Therefore,*

$$AB + BC + CD + DE + EF + FP + PK + KA = 2(AB + CD + EF + PK) = 46.$$

Problem 4. Let CD be a bisector of triangle ABC. Given that $AB = 1.5CD$, $\angle A = 1.5\angle C$. Assume $\angle B = n^\circ$. Find n.

Solution. *For triangle* ABC *let us choose point* E *on side* BC*, such that* $\angle EAC = \angle ACB$. *We denote by* F *the intersection point of rays* ED *and* CA.

Let $\angle C = 2\alpha$, $CD = 2a$, *then* $\angle DAE = \angle DCE = \alpha$. *Thus, points* A, D, E, C *are on the same circle. We have that* $\angle DEA = \angle DCA = \alpha$ *and* $\angle AFE = \angle CAE - \angle FEA = \alpha$. *Hence,* $AD = DE = x$ *and* $FA = AE = EC = y$. *Note that,* $\triangle FAD = \triangle CED$. *Thus, we obtain that* $FD = CD = 2a$. *Let* $BE = z$. *As* $\triangle FAD \sim \triangle FEC$, *hence* $y^2 = x(2a + x)$. *On the other hand, according to the bisector's property from triangle* DBC*, we deduce that* $z = \dfrac{3a - x}{2a} \cdot y$. *Note that,* $\triangle DBE \sim \triangle CBA$. *Therefore,* $\dfrac{z}{3a} = \dfrac{3a - x}{y + z}$. *We have that*

$$\frac{y^2(5a - x)}{4a^2} = 3a.$$

Hence, we obtain that

$$12a^3 = (5a - x)(x^2 + 2ax).$$

Therefore, $(x - a)(x^2 - 2ax - 12a^2) = 0$ *and* $x < 3a$. *It follows that* $x = a$, $BD = DC = 2a$. *Thus,* $DE \perp BC$ *and* $\angle B = 30^\circ$.

Problem 5. Let A_1, A_2, A_3, A_4 be points on one of the sides of angle A and B_1, B_2, B_3, B_4 be points on the other side of angle A, such that triangle AA_iB_i is covered by triangle $AA_{i+1}B_{i+1}$, for $i = 1, 2, 3$. Given that $A_iB_iB_{i+1}A_{i+1}$, $i = 1, 2, 3$, are simultaneously inscribed and circumscribed quadrilaterals. Let $A_1B_1 = 1$ and $A_4B_4 = 8$. Find A_3B_3.

Solution. *Let* O_i *be the centre of the inscribed circle of quadrilateral* $A_iA_{i+1}B_{i+1}B_i$ *and* r_i *be the radius of that circle, where* $i = 1, 2, 3$. *Note that*

$$\angle O_iA_iB_i = \frac{\angle A_{i+1}A_iB_i}{2} = \frac{\angle A_{i+1}B_{i+1}B_{i+2}}{2} = \angle O_{i+1}B_{i+1}A_{i+1}.$$

In a similar way, one can obtain that $\angle O_iB_iA_i = O_{i+1}A_{i+1}B_{i+1}$, *for* $i = 1,2,3$. *Thus, we deduce that* $\triangle O_1A_1B_1 \sim \triangle O_2B_2A_2$. *Hence,* $\frac{A_1B_1}{A_2B_2} = \frac{r_1}{r_2}$. *In a similar way, one can prove that* $\triangle O_1A_2B_2 \sim \triangle O_2B_3A_1$. *Therefore,*

$$\frac{A_2B_2}{A_3B_3} = \frac{r_1}{r_2} = \frac{A_1B_1}{A_2B_2}.$$

We have obtained that A_1B_1, A_2B_2, A_3B_3 *is a geometric progression. In a similar way, one can prove that* A_2B_2, A_3B_3, A_4B_4 *is a geometric progression. As* $A_1B_1 = 1$, $A_4B_4 = 8$, *thus* $A_3B_3 = 4$.

Problem 6. Evaluate the expression

$$\frac{1+4\cos\frac{\pi}{7}+2\cos\frac{2\pi}{7}}{\cos^3\frac{\pi}{7}}.$$

Solution. *We have that*

$$\frac{1+4\cos\frac{\pi}{7}+2\cos\frac{2\pi}{7}}{\cos^3\frac{\pi}{7}} = \frac{\sin\frac{\pi}{7}+\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}+2\sin\frac{2\pi}{7}}{\sin\frac{\pi}{7}\cos^3\frac{\pi}{7}} =$$

$$= \frac{2\left(\sin\frac{3\pi}{7}+2\sin\frac{2\pi}{7}\right)}{\sin\frac{2\pi}{7}\cos^2\frac{\pi}{7}} = \frac{4\left(\sin\frac{3\pi}{7}+2\sin\frac{2\pi}{7}\right)}{\left(\sin\frac{3\pi}{7}+\sin\frac{\pi}{7}\right)\cos\frac{\pi}{7}} =$$

$$= \frac{8\left(\sin\frac{3\pi}{7}+2\sin\frac{2\pi}{7}\right)}{\sin\frac{4\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{2\pi}{7}} = 8.$$

Problem 7. Let $ABCD$ be inscribed and circumscribed quadrilateral. Given that the radiuses of the inscribed and circumscribed circles are equal to 7 and 12, respectively. Find the product of the diagonals of $ABCD$.

Solution. *Let* $AB = a$, $BC = b$, $CD = c$, $AD = d$, $AC = e$, $BD = f$. *We have that*

$$\sqrt{abcd} = S_{ABCD} = S_{ABC} + S_{ACD} = \frac{abe}{4R} + \frac{cde}{4R}.$$

Therefore,

$$\frac{4R}{e}\cdot\frac{4R}{f}=\frac{ab+cd}{\sqrt{abcd}}\cdot\frac{ad+bc}{\sqrt{abcd}}. \tag{7.6}$$

Note that

$$\frac{ab+cd}{\sqrt{abcd}}\cdot\frac{ad+bc}{\sqrt{abcd}}=\frac{(a^2+c^2)bd+(b^2+d^2)ac}{abcd}=\frac{(a+c)^2(ac+bd)}{abcd}-4=\frac{ef}{r^2}-4. \tag{7.7}$$

From (7.6) *and* (7.7)*, we obtain that*

$$\frac{48^2}{ef}=\frac{ef}{49}-4.$$

Therefore, $ef=448$*.*

Problem 8. Let $A=\sin\frac{\pi}{81}\cdot\sin\frac{2\pi}{81}\cdot\cdots\cdot\sin\frac{80\pi}{81}$ and $B=\sin\frac{\pi}{27}\cdot\sin\frac{2\pi}{27}\cdot\cdots\cdot\sin\frac{26\pi}{27}$. Find the value of the following expression $\sqrt[7]{\frac{3B}{A}}$.

Solution. *We have that*

$$\sin\alpha\sin\left(\frac{\pi}{3}-\alpha\right)\sin\left(\frac{\pi}{3}+\alpha\right)=\frac{1}{2}\sin\alpha\left(\cos 2\alpha-\cos\frac{2\pi}{3}\right)=$$

$$=\frac{1}{2}\left(\sin\alpha\cos 2\alpha+\frac{1}{2}\sin\alpha\right)=\frac{1}{4}\left(\sin 3\alpha-\sin\alpha+\sin\alpha\right)=\frac{1}{4}\sin 3\alpha.$$

Therefore,

$$\sin\alpha\sin\left(\frac{\pi}{3}-\alpha\right)\sin\left(\frac{\pi}{3}+\alpha\right)=\frac{1}{4}\sin 3\alpha. \tag{7.8}$$

According to (7.8)*, we obtain that*

$$\sin\frac{\pi}{81}\cdot\sin\frac{26\pi}{81}\cdot\sin\frac{28\pi}{81}=\frac{1}{4}\sin\frac{\pi}{27},$$

$$\sin\frac{2\pi}{81}\cdot\sin\frac{25\pi}{81}\cdot\sin\frac{29\pi}{81}=\frac{1}{4}\sin\frac{2\pi}{27},$$

$$\dots$$

$$\sin\frac{13\pi}{81}\cdot\sin\frac{14\pi}{81}\cdot\sin\frac{40\pi}{81}=\frac{1}{4}\sin\frac{13\pi}{27}.$$

Multiplying all these equations, we deduce that

$$\sin\frac{\pi}{81}\cdot\sin\frac{2\pi}{81}\cdot\dots\cdot\sin\frac{40\pi}{81}=\frac{\sqrt{3}}{2}\cdot\left(\frac{1}{4}\right)^{13}\sin\frac{\pi}{27}\cdot\dots\cdot\sin\frac{13\pi}{27}.$$

$$A=\left(\sin\frac{\pi}{81}\cdot\sin\frac{2\pi}{81}\cdot\dots\cdot\sin\frac{80\pi}{81}\right)^2=\frac{3}{4^{14}}\left(\sin\frac{\pi}{27}\cdot\sin\frac{2\pi}{27}\cdot\dots\cdot\sin\frac{13\pi}{27}\right)^2=$$

$$=\frac{3}{4^{14}}\cdot\sin\frac{\pi}{27}\cdot\sin\frac{2\pi}{27}\cdot\dots\cdot\sin\frac{26\pi}{27}=\frac{3}{4^{14}}\cdot B.$$

Thus, we obtain that

$$\sqrt[7]{\frac{3B}{A}}=\sqrt[7]{4^{14}}=16.$$

Problem 9. Let AB and BC be the arcs constructed outside of triangle ABC, such that the sum of their angle measures is equal to 360°. Let M,N,K be given points on sides AB,BC,AC, respectively, such that $MK \parallel BC$ and $NK \parallel AB$. Given that E and F are such points on arcs AB and BC, respectively, that $\angle AME=\angle CNF=60^\circ$. Assume $\angle EKF=n^\circ$. Find n.

Solution. *Consider segment AEB, then we obtain a homotopy with the centre K and coefficient $\frac{BC}{AB}$. Assume that the image of segment AEB with respect to that homotopy be segment $A'E'B'$. Let us now cut segment $A'E'B'$ and attach to segment BCF, such that points A', B and B', C coincide, respectively. Here, we assume that points E' and F are on different sides of line BC. We have that*

$$\frac{A'M'}{M'B'}=\frac{AM}{MB}=\frac{AK}{CK}=\frac{BN}{NC}.$$

Therefore, point M', N coincides and $\angle CNF=60^\circ=\angle EMA=\angle E'M'A'$. Thus, it follows that point M' belongs to segment $E'F$.

From the assumptions of the problem, we have that the union of segments CFB and $A'E'B'$ is a circle. Hence, $BN\cdot M'B'=NF\cdot M'E'$. Thus, $BN\cdot MB=NF\cdot ME$. We deduce that

$$\frac{ME}{NK}=\frac{ME}{MB}=\frac{BN}{NF}=\frac{MK}{NF}$$

and

$$\angle EMK=60^\circ+\angle B=\angle KNF.$$

Hence, $\triangle EMK\sim\triangle KNF$. We have that

$$\angle EKF=\angle EKM+\angle MKN+\angle NKF=\angle EKM+\angle B+\angle MEK=180^\circ-60^\circ=120^\circ.$$

Therefore, $n=120$.

7.1.8 Problem Set 8

Problem 1. Consider a trapezoid with bases equal to 3 and 7. Given that the sum of the squares of its diagonals is equal to 100. Find the sum of the squares of its legs.

Solution. *Let the diagonals AC and BD of trapezoid $ABCD$ intersect at point O and the bases $BC = 3$, $AD = 7$. Consider the parallelogram $DBCE$. Note that*

$$AC^2 + BD^2 = AC^2 + CE^2 = 100.$$

On the other hand,

$$AE = AD + DE = AD + BC = 10.$$

Therefore,

$$AE^2 = AC^2 + CE^2.$$

It follows that

$$\angle ACE = 90^\circ.$$

Thus, we obtain that

$$\angle AOD = \angle ACE = 90^\circ.$$

According to the Pythagorean theorem, we deduce that

$$AB^2 + CD^2 = (AO^2 + BO^2) + (CO^2 + DO^2) = (AO^2 + OD^2) + (BO^2 + CO^2) =$$

$$= AD^2 + BC^2 = 49 + 9 = 58.$$

Hence, $AB^2 + CD^2 = 58$.

Problem 2. Evaluate the following expression

$$\sqrt{2}\left(\frac{\cos 44^\circ}{\cos 1^\circ} + \frac{\cos 43^\circ}{\cos 2^\circ} + \cdots + \frac{\cos 1^\circ}{\cos 44^\circ}\right) - \tan 1^\circ - \tan 2^\circ - \cdots - \tan 44^\circ.$$

Solution. *We have that*

$$\frac{\cos(45^\circ - \alpha)}{\cos\alpha} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\tan\alpha.$$

Therefore,

$$\sqrt{2}\left(\frac{\cos 44^\circ}{\cos 1^\circ} + \frac{\cos 43^\circ}{\cos 2^\circ} + \cdots + \frac{\cos 1^\circ}{\cos 44^\circ}\right) - \tan 1^\circ - \tan 2^\circ - \cdots - \tan 44^\circ =$$

$$= \sqrt{2}\Big(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\tan 1^\circ+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\tan 2^\circ+\cdots+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\tan 44^\circ\Big)-$$

$$-\tan 1^\circ-\tan 2^\circ-\cdots-\tan 44^\circ=44.$$

Problem 3. Let ABC be a triangle, such that $BC=5\sqrt{6}+5\sqrt{2}$, $\angle A=30^\circ$ and $AB>BC$, $AC>BC$. Consider points M and N on sides AB and AC, respectively, such that $MB=BC=CN$. Find the distance of the midpoints of CM and BN.

Solution. *Let points K, E and F be midpoints of segments BC, CM and BN. Note that EK and KF are the mid-segments of triangles CMB and CNB. Therefore, $EK \parallel MB$, $FK \parallel CN$. On the other hand,*

$$EK=\frac{MB}{2}=\frac{BC}{2}=KF.$$

Thus, it follows that

$$EK=KF=2.5(\sqrt{6}+\sqrt{2}).$$

Hence, $\angle EKF=\angle A=30^\circ$. From triangle EKF, by law of cosines, we obtain that

$$EF=EK\sqrt{2-2\cos 30^\circ}=2.5(\sqrt{6}+\sqrt{2})\sqrt{2-\sqrt{3}}=5.$$

Problem 4. Find the minimum value of the expression

$$\frac{64}{\sin^6 2\alpha}-\frac{1}{\sin^6\alpha}-\frac{1}{\cos^6\alpha}.$$

Solution. *We have that*

$$\frac{64}{\sin^6 2\alpha}-\frac{1}{\sin^6\alpha}-\frac{1}{\cos^6\alpha}=\frac{64}{\sin^6 2\alpha}-\frac{\sin^6\alpha+\cos^6\alpha}{\sin^6\alpha\cos^6\alpha}=\frac{64}{\sin^6 2\alpha}-$$

$$-\frac{\sin^4\alpha-\sin^2\alpha\cos^2\alpha+\cos^4\alpha}{\sin^6\alpha\cos^6\alpha}=\frac{64}{\sin^6 2\alpha}-\frac{(\sin^2\alpha+\cos^2\alpha)^2-3\sin^2\alpha\cos^2\alpha}{\sin^6\alpha\cos^6\alpha}=$$

$$=\frac{64}{\sin^6 2\alpha}-\frac{1}{\sin^6\alpha\cos^6\alpha}+\frac{3}{\sin^4\alpha\cos^4\alpha}=\frac{48}{\sin^4 2\alpha}.$$

Therefore, the smallest value of the given expression is equal to 48.

For example, for $\alpha=\frac{\pi}{4}$.

Problem 5. Let D be a point chosen on the side AC of triangle ABC, such that $\angle ABD=90^\circ$, $\angle DBC=18^\circ$, $CD=10$. Given that $AC=10\sqrt{5}$. Find BD.

Solution. *We have that* $\sin 36^\circ=\cos 54^\circ$. *Hence,* $2\sin 18^\circ\cos 18^\circ=4\cos^3 18^\circ-3\cos 18^\circ$. *Thus, it follows that*

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

We obtain that

$$\sin 54^\circ = \frac{\sqrt{5}+1}{4}.$$

Consider the following three cases:

a) If $BD > 10$*, then from triangle ABD we deduce that*

$$\sin\angle A = \frac{BD}{AD} > \frac{10}{10(\sqrt{5}-1)} = \frac{\sqrt{5}+1}{4} = \sin 54^\circ.$$

Thus, $\angle A > 54^\circ$.

On the other hand, $\angle C > \angle DBC = 18^\circ$. *Thus,* $\angle A + \angle B + \angle C > 54^\circ + 108^\circ + 18^\circ = 180^\circ$. *This leads to a contradiction.*

b) If $BD < 10$*, then* $\angle A < 54^\circ$ *and* $\angle C < 18^\circ$. *Therefore,* $\angle A + \angle B + \angle C < 180^\circ$. *This leads to a contradiction.*

c) If $BD = 10$*, then* $\angle A = 54^\circ$ *and* $\angle C = 18^\circ$. *This case is possible.*

Problem 6. Let $ABCD$ be a cyclic quadrilateral, such that the bisectors of the angles A and D intersect on the side BC. Given that $AB = 3$, $CD = 7$. Find BC.

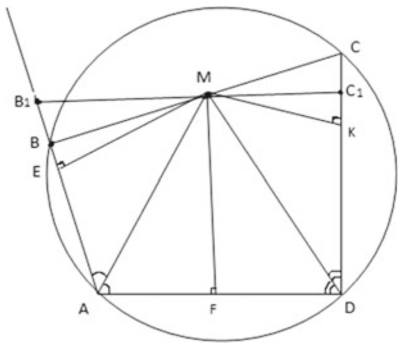
Solution. *Consider the following two cases:*

a) If $BC \parallel AD$. *Let* M *be the intersection point of bisectors of angles A and D.*

We have that $\angle BAM = \angle DAM = \angle AMB$*, thus* $\angle BAM = \angle AMB$. *Hence,* $BM = AB$.

In a similar way, we obtain that $CM = CD$. *Thus,* $BC = BM + CM = AB + CD = 10$.

b) If $BC \nparallel AD$.



Let B_1C_1 *be a parallel line to AD passing through the point* M*, see the picture above. Let ME, MF and MK be perpendicular to lines ME, MF and MK. According to the angle bisector theorem, we have that* $ME = MF$ *and* $MF = MK$. *Thus,* $ME = MK$.

Note that, $\angle MB_1A = 180^\circ - \angle BAD = \angle BCD$. *From the equality of triangles* B_1ME *and* CMK*, we deduce that* $B_1M = CM$. *In a similar way, we obtain that* $BM = C_1M$. *Therefore,* $\triangle BMB_1 = \triangle C_1MC$. *It follows that,* $BB_1 = CC_1$.

Thus, $BC = BM + MC = C_1M + B_1M = C_1D + AB_1 = CD - CC_1 + AB + BB_1 = AB + CD = 10$.

Problem 7. Let $ABCD$ be a convex quadrilateral, such that $\angle BAC + \angle ADB = \angle BCA + \angle CDB$ and $\angle BAC \neq \angle CDB$. Given that $AB = 3$, $BC = 4$ and $CD = 8$. Find AD.

Solution. *Let* ω *be the circumscribed circle of triangle* ADC. *From the condition* $\angle BAC \neq \angle CDB$*, it follows that* B *is not on* ω. *Let us denote by* B_1 *the intersection point of* ω *and the ray* DB. *Note that*

$$\angle BAB_1 = |\angle BAD - \angle B_1AD| = |\angle B_1AC - \angle BAC| = |\angle BDC - \angle BAC| =$$

$$= |\angle ADB - \angle BCA| = \angle BCB_1.$$

On the other hand,

$$\frac{BB_1}{BD} = \frac{(ABB_1)}{(ABD)} = \frac{AB \sin \angle BAB_1}{AD \sin \angle BAD},$$

and

$$\frac{BB_1}{BD} = \frac{CB \sin \angle BCB_1}{CD \sin \angle BCD}.$$

Using these equations and the conditions $\angle BAB_1 = \angle BCB_1$, $\angle BAD + \angle BCD = 180^\circ$*, we deduce that*

$$\frac{AB}{AD} = \frac{CB}{CD}.$$

Therefore, $AD = 6$.

Problem 8. Evaluate the expression

$$\frac{1}{\sqrt{2}\cos 18^\circ}\left(\frac{1}{\sin^3 9^\circ} - \frac{1}{\cos^3 9^\circ}\right) - 40\sqrt{5}.$$

Solution. *We have that*

$$\frac{1}{\sqrt{2}\cos 18^\circ}\left(\frac{1}{\sin^3 9^\circ} - \frac{1}{\cos^3 9^\circ}\right) - 40\sqrt{5} =$$

$$= \frac{8\sqrt{2}(\cos 9^\circ \cos 45^\circ - \sin 9^\circ \sin 45^\circ)\left(1 + \frac{1}{2}\sin 18^\circ\right)}{\sqrt{2}\cos 18^\circ \cdot \sin^3 18^\circ} - 40\sqrt{5} =$$

$$= \frac{8\sqrt{2}\cos 54^\circ\left(1+\frac{1}{2}\sin 18^\circ\right)}{\cos 18^\circ \cdot \sin^3 18^\circ} - 40\sqrt{5} = \frac{16\left(1+\frac{1}{2}\sin 18^\circ\right)}{\sin^2 18^\circ} - 40\sqrt{5} =$$

$$= \frac{16(7+\sqrt{5})}{3-\sqrt{5}} - 40\sqrt{5} = 4(7+\sqrt{5})(3+\sqrt{5}) - 40\sqrt{5} = 104.$$

Here, we have used that (see the solution of Problem 5)

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

Problem 9. Let M, N, K ne points chosen on the sides AB, BC, AC of triangle ABC, respectively. Given that triangle MNK is an acute triangle and that $h_m + h_n + h_k - \min(h_m, h_n, h_k) = \min(h_a, h_b, h_c)$, where h_a, h_b, h_c and h_m, h_n, h_k are the altitudes of triangles ABC and MNK, respectively. Find all possible values of the ratio $\frac{(ABC)}{(MNK)}$, where we denote by (ABC) the area of triangle ABC.

Solution. *At first, let us prove the following lemma:*

Lemma 7.1. *If M is a point inside of triangle ABC and C_1, A_1, B_1 are points on sides AB, BC and AC, respectively, then $MA_1 + MB_1 + MC_1 \geq \min(h_a, h_b, h_c)$, where h_a, h_b, h_c are the altitudes of triangle ABC. The equality sign holds true, when ABC is an equilateral triangle and $MA_1 \perp BC$, $MB_1 \perp AC$, $MC_1 \perp AB$.*

Proof. *We have that*

$$MA_1 + MB_1 + MC_1 \geq \frac{2(MBC)}{BC} + \frac{2(MAC)}{AC} + \frac{2(MAB)}{AB} \geq$$

$$\geq \frac{2(MBC)+2(MAC)+2(MAB)}{\max(AB, BC, AC)} = \frac{2(ABC)}{\max(AB, BC, AC)} = \min(h_a, h_b, h_c).$$

The equality sign holds true, if and only if $AB = BC = AC$ and $MA_1 = \frac{2(MBC)}{BC}$, $MB_1 = \frac{2(MAC)}{AC}$, $MC_1 = \frac{2(MAB)}{AB}$. Hence, $MA_1 \perp BC$, $MB_1 \perp AC$, $MC_1 \perp AB$.

Let H be the intersection point of the altitudes MM_1, NN_1 and KK_1 of triangle MNK. Note that H is inside of triangle MNK and according to the lemma

$$HM_1 + HN_1 + HK_1 \geq \min(h_m, h_n, h_k), \tag{7.9}$$

and

$$HM + HN + HK \geq \min(h_a, h_b, h_c). \tag{7.10}$$

Summing up these inequalities, we obtain that

$$h_m + h_n + h_k \geq \min(h_m, h_n, h_k) + \min(h_a, h_b, h_c). \tag{7.11}$$

According to the assumptions of the problem in (7.14), *it holds true the equality sign. Therefore, in* (7.21) *and* (7.13) *it holds true the equality signs also.*

According to the lemma triangles ABC, MNK *are equilateral, and from the condition* $HM \perp AB$, *we obtain that* $NK \parallel AB$. *In a similar way, we deduce that* $MN \perp AC$, $MK \perp BC$. *Therefore,* M, N, K *are the midpoints of the sides of triangle* ABC. *Hence, it follows that*

$$\frac{(ABC)}{(MNK)} = 4.$$

7.1.9 Problem Set 9

Problem 1. Let BH be an altitude of acute triangle ABC and O be its circumcenter. Given that the line HO passes through the midpoint of BC side and $BC = 20\sqrt{2}$, $AC = 35$. Find AB.

Solution. *Denote by* M *the midpoint of* BC. *We have that* $OM \perp BC$, *thus median* HM *of triangle* HBC *is simultaneously its altitude. Hence, we deduce that* $\angle C = \angle HBC = 45^\circ$. *According to the law of cosines, we obtain that* $AB^2 = 800 + 1225 - 1400$. *Therefore,* $AB = 25$.

Problem 2. Find the smallest value of the following expression

$$\tan^2\alpha + \frac{4}{\cos\alpha} + 40.$$

Solution. *We have that*

$$\tan^2\alpha + \frac{4}{\cos\alpha} + 40 = \frac{1}{\cos^2\alpha} + \frac{4}{\cos\alpha} + 4 + 35 = \left(\frac{1}{\cos\alpha} + 2\right)^2 + 35 \geq 35.$$

Note that, if $\alpha = \dfrac{2\pi}{3}$, *then*

$$\tan^2\alpha + \frac{4}{\cos\alpha} + 40 = 35.$$

Therefore, the smallest value of the given expression is equal to 35.

Problem 3. Let $ABCDE$ be a convex pentagon, such that its perimeter is equal to 25, $DE = 5$ and $EB = BD$. Given that the inscribed circles of triangles ABE, BCD touch diagonals EB, BD at points M, N, respectively, such that $BM = BN$. Find $AE + BC$.

Solution. *We have that*

$$BM = \frac{AB+BE-AE}{2},$$

and

$$BN = \frac{BD+BC-CD}{2}.$$

Hence, we deduce that either $AB+BE-AE = BD+BC-CD$ *or* $AB+CD = BC+AE$. *On the other hand, given that* $(AB+CD)+(AE+BC) = 25-5 = 20$. *Therefore, we obtain that* $BC+AE = 10$.

Problem 4. Let the lines including the legs AB and CD of trapezoid $ABCD$ intersect at point M. Given that $AB = 14$, $CD = 16$, $\angle AMD = 60^\circ$. Find the distance between the midpoints of the bases of trapezoid $ABCD$.

Solution. *Consider the parallel lines to the legs of trapezoid ABCD passing through the midpoint of the smaller base. Then, we consider the intersection points of those lines and the larger base of the trapezoid. One can easily deduce that the segment connecting the midpoints of the bases is a median of a triangle with sides 14 and 16 with an angle between those sides equal to* 60°. *Therefore, we have that*

$$m = \sqrt{\frac{2\cdot 14^2 + 2\cdot 16^2 - (14^2+16^2-14\cdot 16)}{4}} = 13.$$

Problem 5. Evaluate the expression

$$\frac{\cos 2^\circ \cos^2 89^\circ}{\cos 3^\circ}\left(\frac{1}{\cos 2^\circ \cos 3^\circ} + \frac{1}{\cos 3^\circ \cos 4^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ}\right).$$

Solution. *Note that*

$$\cos 89^\circ\left(\frac{1}{\cos 2^\circ \cos 3^\circ} + \frac{1}{\cos 3^\circ \cos 4^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ}\right) =$$

$$= \frac{\sin(3^\circ - 2^\circ)}{\cos 2^\circ \cos 3^\circ} + \cdots + \frac{\sin(89^\circ - 88^\circ)}{\cos 88^\circ \cos 89^\circ}$$

$$= \tan 3^\circ - \tan 2^\circ + \cdots + \tan 89^\circ - \tan 88^\circ = \tan 89^\circ - \tan 2^\circ = \frac{\sin 87^\circ}{\cos 2^\circ \cos 89^\circ} = \frac{\cos 3^\circ}{\cos 2^\circ \cos 89^\circ}.$$

Thus, it follows that

$$\frac{\cos 2^\circ \cos^2 89^\circ}{\cos 3^\circ}\left(\frac{1}{\cos 2^\circ \cos 3^\circ} + \frac{1}{\cos 3^\circ \cos 4^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ}\right) = 1.$$

Problem 6. Let $ABCD$ be a circumscribed quadrilateral and its diagonals intersect at point M. Given that $\frac{BC}{AD} = \frac{1}{2}$, $\frac{BM}{DM} = \frac{1}{3}$, $\frac{AM}{MC} = \frac{5}{3}$. Find $\frac{AB}{CD}$.

Solution. *Denote by L, N, P, K the intersection points of the inscribed circle of $ABCD$ and the sides AB, BC, CD, DA, respectively.*

Lemma 7.2. *The diagonal BD is divided by segment NK into two parts with a ratio $BN : DK$, (counting from the vertex B)*

Let R be the intersection point of line BC and the parallel line to NK passing through point D. We have that

$$\angle DKN = \frac{\smile NPK}{2} = \angle KNR,$$

and $DR \parallel NK$. Thus, we obtain that $KD = NR$.

According to Thales' theorem, it follows that

$$BM_1 : M_1D = BN : NR = BN : DK,$$

where M_1 is the intersection point of segments NK and BD.

On the other hand, according to the lemma segments LP and NK intersect the diagonal BD at the same point. Through this point passes also the diagonal AC. According to the lemma and the assumptions of the problem, we deduce that $BN = BL = x$, $DP = DK = 3x$, $AL = AK = 5y$, $CN = CP = 3y$ and $2(x+3y) = 5y+3x$. Hence, it follows that $y = x$, $AB = CD = 6x$. Therefore, $AB : CD = 1$.

Problem 7. Given that the circumradius of triangle ABC is equal to the length of one of its bisectors. Given that the circumcenter of triangle ABC is inside of the inscribed circle of ABC. Let the greatest angle of triangle is equal to n°. Find n.

Solution. *Let O and I be the circumcenter and the incenter of triangle ABC, respectively. Given that the bisector AM is equal to AO. We have that $IO \le r \le IM$, where r is the radius of the inscribed circle of triangle ABC. According to the triangle inequality, we have that $AO \le AI + IO$. On the other hand, $AO = AM = AI + IM \ge AI + IO$. Therefore, $AO = AI + IO$ and points M, O coincide. Hence, $n = 90$.*

Problem 8. Let $\alpha_1 + \alpha_2 + \cdots + \alpha_{2015} = \frac{\pi}{2}$, where $\alpha_1 > 0$, $\alpha_2 > 0, \ldots, \alpha_{2015} > 0$. Calculate the following expression

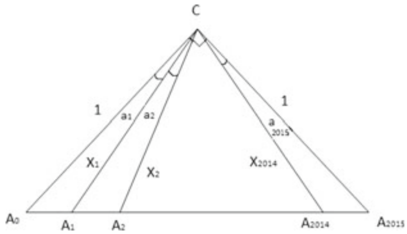
$$\frac{\sin\alpha_1}{\sin\alpha_1 + \cos\alpha_1} + \frac{\sin\alpha_2}{(\sin\alpha_1 + \cos\alpha_1)} \times \frac{1}{(\sin(\alpha_1+\alpha_2) + \cos(\alpha_1+\alpha_2))} + \cdots +$$

$$+ \frac{\sin\alpha_{2014}}{(\sin(\alpha_1+\alpha_2+\cdots+\alpha_{2013}) + \cos(\alpha_1+\alpha_2+\cdots+\alpha_{2013}))} \times$$

$$\times \frac{1}{(\sin(\alpha_1+\alpha_2+\cdots+\alpha_{2014}) + \cos(\alpha_1+\alpha_2+\cdots+\alpha_{2014}))} +$$

$$+\frac{\sin\alpha_{2015}}{\sin(\alpha_1+\alpha_2+\cdots+\alpha_{2014})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{2014})}.$$

Solution. *Consider the following figure.*



We have that

$$(A_0CA_i)+(A_{2015}CA_i)=\frac{1}{2}.$$

Hence,

$$x_i=\frac{1}{\sin(\alpha_1+\cdots+\alpha_i)+\cos(\alpha_1+\cdots+\alpha_i)}, i=1,2,\ldots,2014.$$

On the other hand,

$$(A_0CA_1)+(A_1CA_2)+\cdots+(A_{2014}CA_{2015})=\frac{1}{2}.$$

Therefore,

$$\frac{\sin\alpha_1}{\sin\alpha_1+\cos\alpha_1}+\frac{\sin\alpha_2}{(\sin\alpha_1+\cos\alpha_1)}\times\frac{1}{(\sin(\alpha_1+\alpha_2)+\cos(\alpha_1+\alpha_2))}+\cdots+$$

$$+\frac{\sin\alpha_{2014}}{(\sin(\alpha_1+\alpha_2+\cdots+\alpha_{2013})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{2013}))}\times$$

$$\times\frac{1}{(\sin(\alpha_1+\alpha_2+\cdots+\alpha_{2014})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{2014}))}+$$

$$+\frac{\sin\alpha_{2015}}{\sin(\alpha_1+\alpha_2+\cdots+\alpha_{2014})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{2014})}=1.$$

Problem 9. Let $ABCDEF$ be a convex hexagon, such that $\angle A=90^\circ$, $\angle C=120^\circ$, $\angle E=150^\circ$ and $\angle CDE=2\angle BDF$. Given that $AB-BC=10$ and $CD-DE=20$. Find $\sqrt{3}(EF-FA)$.

Solution. *Consider a rotation with the centre D and the rotation angle EDC, such that the rotation direction coincides with the direction of passing through points A,*

B, C. Let this rotation maps point E to point E_1 and point F to point F_1. Note that E_1 belongs to side CD and $CE_1 = CD - DE_1 = CD - ED = 20$.

Denote by K_1 the intersection point of lines BC and E_1F_1. We have that $\angle BCD = 120^\circ$ and $\angle CE_1K_1 = 30^\circ$. Thus, it follows that $\angle CK_1E_1 = 90^\circ$. Therefore,

$$CK_1 = \frac{CE_1}{2} = 10.$$

Note that $DF = DF_1$ and $\angle FDB = \angle FDE + \angle BDC = \angle BDC + \angle F_1DC = \angle F_1DB$. We obtain that $\triangle BDF = \triangle BDF_1$. Hence, $BF = BF_1$.

We have that, $BK_1 = BC + CK_1 = BC + 10 = AB$. Thus, $\triangle FAB = \triangle F_1K_1B$. Hence, $AF = F_1K_1$.

Therefore, $E_1K_1 = 10\sqrt{3}$ and $E_1K_1 = E_1F_1 - K_1F_1 = EF - AF$. We have obtained that

$$\sqrt{3}(EF - FA) = 30.$$

7.1.10 Problem Set 10

Problem 1. Find the greatest value of the expression

$$2(\sin x + 2\cos x)(\cos x + 2\sin x).$$

Solution. *We have that*

$$2(\sin x + 2\cos x)(\cos x + 2\sin x) = 2(2(\sin^2 x + \cos^2 x) + 5\sin x\cos x) = 4 + 5\sin 2x.$$

Therefore, the greatest value of the given expression is equal to 9.

Problem 2. Let ABC be a right triangle, such that the radius of the inscribed circle is equal to 10. Given that CD is the altitude to the hypotenuse. Find

$$(AD + CD - AC)^2 + (BD + CD - BC)^2.$$

Solution. *Note that triangles ACD, BCD and ABC are pairwise similar. Denote by r_1, r_2 and r the inradiuses of triangles ACD, BCD and ABC, respectively. We have that*

$$\frac{(ACD)}{(ABC)} = \frac{r_1^2}{r^2},$$

and

$$\frac{(BCD)}{(ABC)} = \frac{r_2^2}{r^2}.$$

Thus, it follows that $r_1^2 + r_2^2 = r^2$. *On the other hand,*

$$r_1 = \frac{AD + CD - AC}{2},$$

and

$$r_2 = \frac{BD + CD - BC}{2}.$$

Hence, we obtain that

$$(AD + CD - AC)^2 + (BD + CD - BC)^2 = 4r^2 = 400.$$

Problem 3. Let ABC be a triangle, such that $AC = 195$, $\angle B = 120^\circ$. Let BD be a bisector. Given that $AB \cdot CD = 10920$. Find BD.

Solution. *Let AE be the bisector of angle $\angle A$ and EF, EP, EK be perpendicular to AB, BD, AC, respectively. We have that $\angle FBE = 60^\circ = \angle PBE$. Thus, it follows that $EF = EP$. On the other hand, from the condition $\angle EAF = \angle EAK$, we deduce that $EK = EF$. Hence, $EP = EK$. Therefore, $\angle BDE = \angle CDE$. According to the property of angle bisector, from triangles BDC and ABC, we obtain that*

$$\frac{BD}{DC} = \frac{BE}{EC} = \frac{AB}{AC}.$$

Thus, it follows that

$$BD = \frac{AB \cdot CD}{AC} = \frac{10920}{195} = 56.$$

Problem 4. Let M and N be the midpoints of sides AD and BC of quadrilateral $ABCD$. Given that $AB = 6$, $BC = 4$, $CD = 8$, $AD = 14$. Find the possible greatest value of MN.

Solution. *Let P, K, E, F be the midpoints of sides AB, CD, AC, BD. Note that*

$$EN = \frac{AB}{2} = 3, \ FM = \frac{AB}{2} = 3, \ NF = \frac{CD}{2} = 4, \ EM = \frac{CD}{2} = 4.$$

Thus, it follows that $ENFM$ is a parallelogram.

We have that,

$$MN^2 + EF^2 = 50.$$

Therefore, to find the possible greatest value of MN is equivalent to find the possible smallest value of EF. Note that $EF^2 + PK^2 = 106$, hence to find the possible smallest value of EF is equivalent to find the possible greatest value of PK.

On the other hand,

$$PK \leq PE + EK = 9.$$

Note that, $PK = 9$, if $ABCD$ is a trapezoid with bases BC and AD.
Therefore, the greatest value of MN is equal to 5.

Problem 5. Find the value of the expression

$$\left(3\cos^2\frac{\pi}{82}-\sin^2\frac{\pi}{82}\right)\left(3\cos^2\frac{3\pi}{82}-\sin^2\frac{3\pi}{82}\right)\left(3\cos^2\frac{9\pi}{82}-\sin^2\frac{9\pi}{82}\right)\left(3\cos^2\frac{27\pi}{82}-\sin^2\frac{27\pi}{82}\right).$$

Solution. *We have that*

$$\sin 3x = \sin x(3\cos^2 x - \sin^2 x).$$

Thus, it follows that

$$\left(3\cos^2\frac{\pi}{82}-\sin^2\frac{\pi}{82}\right)\left(3\cos^2\frac{3\pi}{82}-\sin^2\frac{3\pi}{82}\right)\left(3\cos^2\frac{9\pi}{82}-\sin^2\frac{9\pi}{82}\right)\left(3\cos^2\frac{27\pi}{82}-\sin^2\frac{27\pi}{82}\right)=$$

$$=\frac{\sin\frac{3\pi}{82}\left(3\cos^2\frac{3\pi}{82}-\sin^2\frac{3\pi}{82}\right)\left(3\cos^2\frac{9\pi}{82}-\sin^2\frac{9\pi}{82}\right)\left(3\cos^2\frac{27\pi}{82}-\sin^2\frac{27\pi}{82}\right)}{\sin\frac{\pi}{82}}=$$

$$=\frac{\sin\frac{9\pi}{82}\left(3\cos^2\frac{9\pi}{82}-\sin^2\frac{9\pi}{82}\right)\left(3\cos^2\frac{27\pi}{82}-\sin^2\frac{27\pi}{82}\right)}{\sin\frac{\pi}{82}}=\frac{\sin\frac{81\pi}{82}}{\sin\frac{\pi}{82}}=1.$$

Problem 6. Let N be a given point on side CD of quadrilateral $ABCD$ and M be a given point inside of triangle ABD. Given that $\angle BND \neq 90°$ and $\angle BDC = 40°$, $\angle BMN = 80°$, $MN = MD$. Let $\angle MBN = n°$. Find n.

Solution. *We have that $\angle BMN = 80°$, thus $\angle MBN < 100°$. Hence, $\angle DBN < \angle MBN = 100°$. On the other hand, $\angle BND > 40°$. Let us choose point K on diagonal BD, such that $\angle KND = 40°$. Let l be perpendicular line DN and which passes through the midpoint of line segment DN. Consider the intersection points M and K of line l and circumcircle ω of triangle BKN. Note that the circumcenter of triangle BND is on circle ω and line l. From the condition $\angle BND \neq 90°$, it follows that the circumcenter of triangle BND is the point M. Therefore, $\angle MBN = 50°$. Hence, we obtain that $n = 50$.*

Problem 7. Find the value of the expression

$$\frac{\sin\frac{\pi}{16}}{\sin\frac{\pi}{16}+\cos\frac{\pi}{16}}+\frac{\sin\frac{\pi}{16}}{\left(\sin\frac{\pi}{16}+\cos\frac{\pi}{16}\right)\left(\sin\frac{2\pi}{16}+\cos\frac{2\pi}{16}\right)}+\cdots$$

$$\cdots+\frac{\sin\frac{\pi}{16}}{\left(\sin\frac{6\pi}{16}+\cos\frac{6\pi}{16}\right)\left(\sin\frac{7\pi}{16}+\cos\frac{7\pi}{16}\right)}+\frac{\sin\frac{\pi}{16}}{\sin\frac{7\pi}{16}+\cos\frac{7\pi}{16}}.$$

Solution. ***Lemma 7.3.*** *Let* $\alpha_1+\alpha_2+\cdots+\alpha_n=\frac{\pi}{2}$*, where* $\alpha_1>0$*,* $\alpha_2>0,\ldots,$ $\alpha_n>0$*. Prove that*

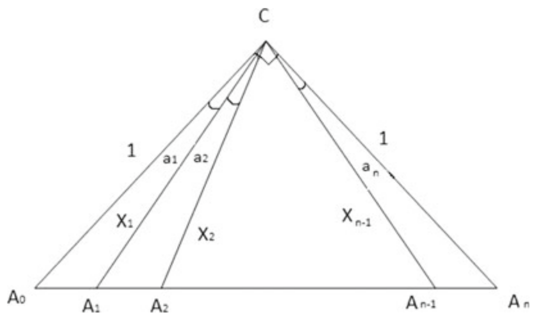
$$\frac{\sin\alpha_1}{\sin\alpha_1+\cos\alpha_1}+\frac{\sin\alpha_2}{(\sin\alpha_1+\cos\alpha_1)}\times\frac{1}{(\sin(\alpha_1+\alpha_2)+\cos(\alpha_1+\alpha_2))}+\cdots+$$

$$+\frac{\sin\alpha_{n-1}}{(\sin(\alpha_1+\alpha_2+\cdots+\alpha_{n-2})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{n-2}))}\times$$

$$\times\frac{1}{(\sin(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{n-1}))}+$$

$$+\frac{\sin\alpha_{2015}}{\sin(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})}=1.$$

Proof. *Consider the following figure.*



We have that

$$(A_0CA_i)+(A_nCA_i)=\frac{1}{2}.$$

Hence,

$$x_i=\frac{1}{\sin(\alpha_1+\cdots+\alpha_i)+\cos(\alpha_1+\cdots+\alpha_i)},i=1,2,\ldots,n-1.$$

On the other hand,

$$(A_0CA_1)+(A_1CA_2)+\cdots+(A_{n-1}CA_n)=\frac{1}{2}.$$

Therefore,

$$\frac{\sin\alpha_1}{\sin\alpha_1+\cos\alpha_1}+\frac{\sin\alpha_2}{(\sin\alpha_1+\cos\alpha_1)}\times\frac{1}{(\sin(\alpha_1+\alpha_2)+\cos(\alpha_1+\alpha_2))}+\cdots+$$

$$+\frac{\sin\alpha_{2n-1}}{(\sin(\alpha_1+\alpha_2+\cdots+\alpha_{n-2})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{n-2}))}\times$$

$$\times\frac{1}{(\sin(\alpha_1+\alpha_2+\cdots+\alpha_{n-1})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{n-1}))}+$$

$$+\frac{\sin\alpha_{2015}}{\sin(\alpha_1+\alpha_2+\cdots+\alpha_{2014})+\cos(\alpha_1+\alpha_2+\cdots+\alpha_{2014})}=1.$$

This ends the proof of the lemma.

According to the lemma, if $\alpha_1=\alpha_2=\cdots=\alpha_8=\dfrac{\pi}{16}$, *then we obtain that*

$$\frac{\sin\frac{\pi}{16}}{\sin\frac{\pi}{16}+\cos\frac{\pi}{16}}+\frac{\sin\frac{\pi}{16}}{\left(\sin\frac{\pi}{16}+\cos\frac{\pi}{16}\right)\left(\sin\frac{2\pi}{16}+\cos\frac{2\pi}{16}\right)}+\cdots$$

$$\cdots+\frac{\sin\frac{\pi}{16}}{\left(\sin\frac{6\pi}{16}+\cos\frac{6\pi}{16}\right)\left(\sin\frac{7\pi}{16}+\cos\frac{7\pi}{16}\right)}+\frac{\sin\frac{\pi}{16}}{\sin\frac{7\pi}{16}+\cos\frac{7\pi}{16}}=1.$$

Problem 8. Let ABC be an acute triangle, such that the circumradius and inradius are equal to 26 and 10, respectively. Given that $\angle A=60^\circ$. Let S be the area of the triangle that has vertices at the midpoints of minor arcs AB, AC and major arc BC. Find $\sqrt{3}S$.

Solution. *Let M, N be the midpoints of the minor arcs AB, AC and K be the midpoint of major arc BC. Let I be the incenter of triangle ABC. Note that*

$$\angle KNM=\frac{1}{2}\left(\frac{360^\circ-2\angle A}{2}-\angle C\right)=\frac{1}{2}\angle B.$$

In a similar way, we obtain that

$$\angle KMN=\frac{1}{2}\angle C.$$

Hence, triangles KMN and IBC are similar. Thus, it follows that

$$\frac{(KMN)}{(IBC)} = \frac{MN^2}{BC^2}.$$

According to the law of sines, it follows that

$$\frac{MN}{BC} = \frac{2\cdot 26\sin\angle MKN}{2\cdot 26\sin 60^\circ} = 1.$$

We have that

$$(IBC) = \frac{BC\cdot 10}{2} = 26\cdot 10\cdot \sin 60^\circ = 130\sqrt{3},$$

and

$$\sqrt{3}(IBC) = 390.$$

Problem 9. Let $DABC$ be a tetrahedron, such that $\angle DAC = \angle DBC$, $\angle ADB = \angle ACB$ and $AC + AD = 25$. Find $BC + BD$.

Solution. *Let us prove that $AD = BC$ and $AC = BD$.*

Denote $AD = a$, $AC = b$, $BC = c$, $BD = d$, $\angle DAC = \alpha$, $\angle ADB = \beta$. Consider point D_1 on the plane ABC, such that points D_1, C are on the same side of AB line and $AD_1 = AD = a$, $BD_1 = BD = d$. Thus, $\triangle ADB = \triangle AD_1B$. Hence, $\angle AD_1B = \beta = \angle ACB$. We deduce that points A, D_1, C, B are on the same circle. Therefore,

$$\angle D_1AC = \angle D_1BC = \gamma.$$

According to the law of cosines, we have that

$$a^2 + d^2 - 2ad\cos\beta = AB^2 = b^2 + c^2 - 2bc\cos\beta.$$

Thus, it follows that

$$(2bc - 2ad)\cos\beta = b^2 + c^2 - a^2 - d^2. \tag{7.12}$$

In a similar way, we obtain that

$$(2cd - 2ab)\cos\alpha = c^2 + d^2 - a^2 - b^2. \tag{7.13}$$

$$(2cd - 2ab)\cos\gamma = c^2 + d^2 - a^2 - b^2. \tag{7.14}$$

Consider the following cases:

a) If $cd = ab$, then from (7.13), it follows that $c^2 + d^2 = a^2 + d^2$. Hence, $(c+d)^2 = (a+b)^2$. Thus, either $a = c$, $b = d$ or $a = d$, $b = c$. In the second case, from (7.21), it follows that $a = b = c = d$.

This ends the proof of the statement.

b) If $cd \neq ab$, then from (7.13) and (7.14), we deduce that $\alpha = \gamma$. Therefore, $CD = CD_1$.

(Note that if points D_1 and C coincide, the form (7.13), it follows that $a = b = c = d$. This leads to a contradiction.)

We have obtained that the plane ABC is the plane passing through the midpoint of the segment DD_1 and perpendicular to it, as $DA = D_1A$, $DB = D_1B$ and $DC = D_1C$. On the other hand, the plane ABC is passing through the point D_1. This leads to a contradiction.

Hence, $BC + BD = AC + AD = 25$.

7.1.11 Problem Set 11

Problem 1. Let M be the midpoint of the leg CD of the trapezoid $ABCD$. Given that $AB = BD$, $\angle ACB = 30^\circ$ and $\angle MBC = n^\circ$. Find n.

Solution. *Let us consider parallelogram $DBCN$. Note that $ABCN$ is an isosceles trapezoid, as $AB = BD = CN$. Thus, it follows that, $\angle MBC = \angle NBC = \angle ACB = 30^\circ$. Therefore, $n = 30$.*

Problem 2. Given that

$$\frac{1}{\sin\dfrac{\pi}{12}} - \frac{1}{\cos\dfrac{\pi}{12}} = \sqrt{n}.$$

Find n.

Solution. *We have that*

$$\frac{1}{\sin\dfrac{\pi}{12}} - \frac{1}{\cos\dfrac{\pi}{12}} = \frac{2\left(\cos\dfrac{\pi}{12} - \sin\dfrac{\pi}{12}\right)}{\sin\dfrac{\pi}{6}} =$$

$$= 4\sqrt{2}\sin\left(\frac{\pi}{4} - \frac{\pi}{12}\right) = \sqrt{8}.$$

Therefore, $n = 8$.

Problem 3. Let $ABCD$ be a cyclic quadrilateral. Given that $\angle BAC = \angle BCA = 75^\circ$ and $BD = 20$. Find the area of $ABCD$.

Solution. *Let M be the intersection point of diagonals BD, AC and R be the circumradius of $ABCD$. Note that $\angle AMD = \angle MDC + \angle MCD = \angle BAC + \angle ACD = \angle BCA + \angle ACD = \angle BCD$. Hence, we obtain that $\angle AMD = \angle BCD$.*

According to the law of sines, it follows that $\frac{AC}{\sin 30^\circ} = 2R$ *and*

$$BD = 2R\sin\angle BCD = 2R\sin\angle AMD.$$

We have that

$$(ABCD) = \frac{1}{2}AC\cdot BD\sin\angle AMD = BD\cdot\frac{1}{4}\cdot 2R\sin\angle AMD = \frac{BD^2}{4} = 100.$$

Problem 4. Consider a triangle ABC, such that $\angle A = 52^\circ$, $\angle B = 109^\circ$ and $(ABC) = 100$. Let BD be the altitude of triangle ABC. Find $AB\cdot CD$.

Solution. *Let us choose a point E on the ray AB, outside the segment AB. Note that* $\angle EBC = 71^\circ = \angle CBD$. *Let DF be one of the bisectors of triangle BDC, then AF is one of the bisectors of triangle ABC. According to the angle bisector property, it follows that*

$$\frac{AB}{AC} = \frac{BF}{FC} = \frac{BD}{DC}.$$

Thus, it follows that

$$AB\cdot CD = AC\cdot BD = 2(ABC) = 200.$$

Problem 5. Consider triangle ABC and point T. Given that $\angle ATB = \angle BTC = \angle ATC = 120^\circ$, $AC = 3$, $BC = 4$, $\angle ACB = 90^\circ$. Find $\frac{9BT+7CT}{AT}$.

Solution. *According to the Pythagorean theorem, we obtain that* $AB = 5$. *On the other hand, according to the law of cosines, from triangles ATC, BTC, ATB, it follows that*

$$AT^2 + AT\cdot BT + BT^2 = 25,$$

$$BT^2 + BT\cdot CT + CT^2 = 16,$$

and

$$AT^2 + AT\cdot CT + CT^2 = 9.$$

We deduce that

$$AT^3 - BT^3 = 25(AT - BT),$$

$$BT^3 - CT^3 = 16(BT - CT),$$

and

$$CT^3 - AT^3 = 9(CT - AT).$$

Summing up the last equations, we obtain that

$$9BT + 7CT = 16AT.$$

Therefore,

$$\frac{9BT + 7CT}{AT} = 16.$$

Problem 6. Find the possible smallest value of the expression

$$20 + 16\sin\left(\frac{\pi}{6} - \alpha\right) \cdot \sin\alpha \cdot \sin\left(\frac{\pi}{6} + \alpha\right).$$

Solution. *Note that*

$$A = 20 + 16\sin\left(\frac{\pi}{6} - \alpha\right) \cdot \sin\alpha \cdot \sin\left(\frac{\pi}{6} + \alpha\right) = 20 + 8\sin\alpha\left(\cos 2\alpha - \cos\frac{\pi}{3}\right) =$$

$$= 20 + 8\sin\alpha\cos 2\alpha - 4\sin\alpha = 20 + 4\sin 3\alpha - 8\sin\alpha.$$

Therefore, $A = 20 + 4\sin 3\alpha - 8\sin\alpha \geq 20 - 4 - 8 = 8.$

On the other hand, if $\alpha = \frac{\pi}{2}$, *then* $A = 8.$

Thus, the smallest possible value of the given expression is equal to 8.

Problem 7. Let $ABCD$ be a convex quadrilateral. Given that $BC = CD$, $\angle BAD = 30^\circ$, $AC = 20$ and $AB + AD = 10\sqrt{2}(\sqrt{3}+1)$. Find the area of $ABCD$.

Solution. *Let CH be the altitude of triangle BCD. Denote by M the intersection point of the ray HC and the circumcircle of triangle ABD. We have that $\angle BMD = 150^\circ$ and $\angle MBD = \angle MDB = 15^\circ$. According to Ptolemy's theorem, it follows that*

$$AB + AD = AM \cdot \frac{BD}{BM} = AM \cdot \frac{\sin 30^\circ}{\sin 15^\circ} = 2AM \cdot \cos(45^\circ - 30^\circ) = AM \cdot \frac{\sqrt{2}(\sqrt{3}+1)}{2}.$$

Therefore, $AM = 20 = AC$. *Hence, points M and C coincide.*

Thus, $ABCD$ is an inscribed quadrilateral. Denote by R the radius of that circle. Let N be the intersection point of diagonals AC and BD. We have that

$$(ABCD) = \frac{1}{2}AC \cdot BD\sin\angle BNA = 10 \cdot BD\sin\angle ADC =$$

$$= 10 \cdot R\sin\angle ADC = 10 \cdot \frac{AC}{2} = 100.$$

Problem 8. Find the value of the expression

$$\frac{\sin\frac{\pi}{40}}{\cos\frac{3\pi}{40}} + \frac{\sin\frac{3\pi}{40}}{\cos\frac{9\pi}{40}} + \frac{\sin\frac{9\pi}{40}}{\cos\frac{27\pi}{40}} + \frac{\sin\frac{27\pi}{40}}{\sin\frac{81\pi}{40}}.$$

Solution. *Note that,*

$$\frac{\sin\frac{\pi}{40}}{\cos\frac{3\pi}{40}}+\frac{\sin\frac{3\pi}{40}}{\cos\frac{9\pi}{40}}+\frac{\sin\frac{9\pi}{40}}{\cos\frac{27\pi}{40}}+\frac{\sin\frac{27\pi}{40}}{\sin\frac{81\pi}{40}}=\frac{2\sin\frac{\pi}{40}\cos\frac{\pi}{40}}{2\cos\frac{\pi}{40}\cos\frac{3\pi}{40}}+\frac{2\sin\frac{3\pi}{40}\cos\frac{3\pi}{40}}{2\cos\frac{3\pi}{40}\cos\frac{9\pi}{40}}+$$

$$+\frac{2\sin\frac{9\pi}{40}\cos\frac{9\pi}{40}}{2\cos\frac{9\pi}{40}\cos\frac{27\pi}{40}}+\frac{2\sin\frac{27\pi}{40}\cos\frac{27\pi}{40}}{2\cos\frac{27\pi}{40}\cos\frac{81\pi}{40}}=\frac{\sin\left(\frac{3\pi}{40}-\frac{\pi}{40}\right)}{2\cos\frac{\pi}{40}\cos\frac{3\pi}{40}}+\cdots+\frac{\sin\left(\frac{81\pi}{40}-\frac{27\pi}{40}\right)}{2\cos\frac{27\pi}{40}\cos\frac{81\pi}{40}}=$$

$$=\frac{1}{2}\left(\tan\frac{3\pi}{40}-\tan\frac{\pi}{40}+\tan\frac{9\pi}{40}-\tan\frac{3\pi}{40}+\tan\frac{27\pi}{40}-\tan\frac{9\pi}{40}+\tan\frac{81\pi}{40}-\tan\frac{27\pi}{40}\right)=$$

$$=\frac{1}{2}\left(\tan\frac{81\pi}{40}-\tan\frac{\pi}{40}\right)=0.$$

Problem 9. Let $ABCDEF$ be a cyclic hexagon, such that $AB=DE$, $BC=EF$, $CD=AF$, $AB\neq BC$. Given that the minor arc AF is equal to 60°. For how many points M of the minor arc AF can it hold true the following equation?

$$MC+MD=MA+MB+ME+MF.$$

Solution. *We have that* $\overset{\frown}{AB}=\overset{\frown}{DE}$ *and* $\overset{\frown}{BC}=\overset{\frown}{EF}$. *Therefore,* $\overset{\frown}{ABC}=\overset{\frown}{FED}=120^\circ$. *Let* E_1 *and* B_1 *be points on the circumcircle of* $ABCDEF$ *hexagon, such that* $\overset{\frown}{AFE_1}=120^\circ$ *and* $\overset{\frown}{FAB_1}=120^\circ$. *Given that* $MC=MA+ME_1$ *and* $MD=MF+MB_1$. *Hence, according to the assumptions of the problem, it follows that*

$$MB+ME=MB_1+ME_1. \tag{7.15}$$

Note that BE *and* B_1E_1 *are diameters, thus from* (7.21) *it follows that*

$$MB^2+2MB\cdot ME+ME^2=MB_1^2+2MB_1\cdot ME_1+ME_1^2,$$

or

$$MB\cdot ME=MB_1\cdot ME_1. \tag{7.16}$$

From (7.21) *and* (7.16), *we deduce that* $\angle MBE=\angle ME_1B_1$. *Therefore,*

$$\frac{60^\circ-\overset{\frown}{MA}+\overset{\frown}{FE}}{2}=\frac{\overset{\frown}{MA}+60^\circ}{2},$$

$$\overset{\frown}{MA}=\frac{\overset{\frown}{FE}}{2}.$$

Thus, we obtain that point M *is unique.*

7.1.12 Problem Set 12

Problem 1. Let ABC be an isosceles triangle, such that $\angle B = 120^\circ$. Let M be a point on base AC, such that $\angle MBC = 30^\circ$. Given that the inradius of triangle ABM is equal to $15+5\sqrt{3}$. Find the value of the inradius of triangle BMC.

Solution. *Denote by r_1, r_2 the inradiuses of triangles ABM and BMC. Let $BM = a$. Note that $\angle ABM = 90^\circ$ and $\angle MAB = \angle MCB = 30^\circ$. Thus, it follows that $MC = BM = a$, $BC = AB = \sqrt{3}a$, $AM = 2a$.*

Therefore

$$r_2 = \frac{(BMC)}{a+\frac{\sqrt{3}a}{2}} = \frac{\frac{\sqrt{3}a^2}{4}}{a+\frac{\sqrt{3}a}{2}} = \frac{\sqrt{3}(2-\sqrt{3})}{2}a,$$

and

$$r_1 = \frac{a+\sqrt{3}a-2a}{2} = \frac{\sqrt{3}-1}{2}a.$$

Hence, we obtain that

$$\frac{r_2}{r_1} = \frac{\sqrt{3}(2-\sqrt{3})}{\sqrt{3}-1} = \frac{3-\sqrt{3}}{2}.$$

Thus, it follows that

$$r_2 = \frac{3-\sqrt{3}}{2}r_1 = \frac{3-\sqrt{3}}{2}\cdot 5(3+\sqrt{3}) = 15.$$

Problem 2. Let $0 \le \alpha < \frac{\pi}{6}$. Find the smallest possible value of the expression

$$\tan\alpha + \tan(2\alpha) - \tan(3\alpha).$$

Solution. *We have that*

$$\tan\alpha + \tan(2\alpha) - \tan(3\alpha) = \frac{\sin 2\alpha}{\cos 2\alpha} - \frac{\sin 3\alpha\cos\alpha - \sin\alpha\cos 3\alpha}{\cos\alpha\cos 3\alpha} =$$

$$= \tan\alpha\tan 2\alpha\tan 3\alpha \ge 0.$$

Note that, if $\alpha = 0$, then $\tan\alpha\tan 2\alpha\tan 3\alpha = 0$.

Therefore, the smallest value of the expression $\tan\alpha\tan 2\alpha\tan 3\alpha$ *is equal to 0.*

Problem 3. Let the inscribed circle of quadrilateral $ABCD$ be tangent to sides AB and CD at points M and N, respectively. Given that $AB + CD = 25$ and $AM \cdot MB = CN \cdot ND = 36$. Find the area of quadrilateral $ABCD$.

Solution. *Note that, if I is the incenter of quadrilateral $ABCD$, then*

$$\angle AIB+\angle CID=\left(180^\circ-\frac{\angle A}{2}-\frac{\angle B}{2}\right)+\left(180^\circ-\frac{\angle C}{2}-\frac{\angle D}{2}\right)=180^\circ.$$

Without loss of generality, one can assume that

$$\angle AIB\le 90^\circ\le\angle CID.$$

Let us choose point I_1 on line segment MI, such that $\angle AI_1B=90^\circ$, thus it follows that

$$AM\cdot MB=MI_1^2\le MI^2.$$

In a similar way, one can prove that

$$NI^2\le CN\cdot ND.$$

Hence, we obtain that

$$AM\cdot MB\le MI^2=NI^2\le CN\cdot ND.$$

Therefore

$$MI=\sqrt{AM\cdot MB}=6,$$

and

$$(ABCD)=\frac{1}{2}(AB+BC+CD+AD)\cdot MI=(AB+CD)MI=150.$$

Problem 4. Let the distance between the centres of circles ω_1, ω_2, with radiuses equal to 10 and 17, be equal to 21. Given that circles ω_1, ω_2 intersect at points A and B. Let l be a line passing through point B and intersecting circles ω_1, ω_2 at points M and N. Find the greatest possible value of the area of triangle AMN.

Solution. *Let O_1 and O_2 be the centres of circles ω_1, ω_2, respectively. Denote by E, F the midpoints of chords MB and BN, respectively. We have that $MN=2EF\le 2O_1O_2$ and $h\le AB$, where h is the distance of point A and line MN. Therefore*

$$(AMN)=\frac{MN\cdot h}{2}\le O_1O_2\cdot AB=2(AO_1BO_2)=4(O_1BO_2)=4\cdot 84=336.$$

We deduce that

$$(AMN)\le 336.$$

Note that, if $MN\parallel O_1O_2$, then $MN=2O_1O_2$ and $h=AB$. Hence, $(AMN)=336$.

Thus, it follows that the greatest possible value of the area of triangle AMN is equal to 336.

Problem 5. Find the greatest possible value of the expression

$$\sin x \sin y + \cos y \cos z + \sin z \sin x.$$

Solution. *Note that*

$$\sin x \sin y + \cos y \cos z + \sin z \sin x \leq \frac{\sin^2 x + \sin^2 y}{2} + \frac{\cos^2 y + \cos^2 z}{2} + \frac{\sin^2 z + \sin^2 x}{2} =$$

$$= 1 + \sin^2 x \leq 2.$$

If $x = y = z = \frac{\pi}{2}$, *then* $\sin x \sin y + \cos y \cos z + \sin z \sin x = 2$. *Therefore, the greatest possible value of the given equation is equal to 2.*

Problem 6. Let O be the circumcenter of quadrilateral $ABCD$. We denote by M the intersection point of the diagonals of quadrilateral $ABCD$. Given that $AD = 20$, $BD = 21$ and $AB = AM = 13$. Let the circumcircle of triangle AMD intersects line segments AB and CD at points N, K, respectively. We denote by S the area of pentagon $ONBCK$. Find the value of $\frac{507}{700}S$.

Solution. *Triangle ABD is an acute triangle; thus, point O is inside of triangle ABD. Let AH be the altitude of triangle ABD, then*

$$AH = \frac{2(ABD)}{BD} = 12.$$

Therefore $BM = 2BH = 10$ *and* $DM = 11$. *From the condition* $DM > MB$, *it follows that point O is inside of triangle AMD. We deduce that*

$$(ONBCK) = (ONBM) + (MBC) + (OMCK).$$

Note that

$$\triangle AMD \sim \triangle BMC.$$

Hence

$$(BMC) = \frac{10^2}{13^2} \cdot (AMD) = \frac{100}{169} \cdot 66 = \frac{6600}{169}.$$

We have that $\angle ADB = \angle MNB$. *Thus, it follows that*

$$\triangle BAD \sim \triangle BMN.$$

Therefore

$$\frac{MN}{20} = \frac{10}{13}.$$

We deduce that

$$MN = \frac{200}{13}.$$

In a similar way, we obtain that

$$MK = \frac{200}{13}.$$

On the other hand

$$\angle NMB = \angle A, \quad \angle OBD = 90^\circ - \angle A.$$

Hence $OB \perp MN$. In a similar way, we obtain that $OC \perp KM$.

According to the law of sines, from triangle ABD, we deduce that $OB = \frac{65}{6}$.

Thus, it follows that

$$(ONBCK) = \frac{MN \cdot OB}{2} + \frac{6600}{169} + \frac{KM \cdot OC}{2} =$$

$$= MN \cdot OB + \frac{6600}{169} = \frac{200}{13} \cdot \frac{65}{6} + \frac{6600}{169} = \frac{104300}{507},$$

and

$$\frac{507}{700}(ONBCK) = 149.$$

Problem 7. Find the value of the sum

$$\frac{1}{\sin \frac{\pi}{2^{2016}-1}} \cdot \sum_{k=0}^{2015} \left(\sin \frac{2^k \pi}{2^{2016}-1} \cdot \left(8\cos^3 \frac{2^k \pi}{2^{2016}-1} - 8\cos \frac{2^k \pi}{2^{2016}-1} + 1 \right) \right).$$

Solution. *Note that*

$$\sin\alpha(8\cos^3\alpha - 8\cos\alpha + 1) = \sin\alpha - 4\sin 2\alpha \cdot \sin^2\alpha =$$

$$= \sin\alpha - 2\sin 2\alpha(1 - \cos 2\alpha) = \sin\alpha - 2\sin 2\alpha + \sin 4\alpha.$$

Thus, it follows that

$$\frac{\pi}{2^{2016}-1} = \phi.$$

Therefore

$$\sum_{k=0}^{2015}(\sin(2^k\phi)(8\cos^3(2^k\phi)-8\cos(2^k\phi)+1))=\sin\phi-2\sin 2\phi+\sin 4\phi+\sin 2\phi-$$

$$-2\sin 4\phi+\sin 8\phi+\cdots+\sin(2^{2015}\phi)-2\sin(2^{2016}\phi)+\sin(2^{2017}\phi)=\sin\phi-\sin 2\phi-$$

$$-\sin(2^{2016}\phi)+\sin(2^{2017}\phi)=\sin\phi-\sin 2\phi-\sin(\pi+\phi)+\sin(2\pi+2\phi)=2\sin\phi.$$

Hence

$$\frac{1}{\sin\frac{\pi}{2^{2016}-1}}\cdot\sum_{k=0}^{2015}\left(\sin\frac{2^k\pi}{2^{2016}-1}\cdot\left(8\cos^3\frac{2^k\pi}{2^{2016}-1}-8\cos\frac{2^k\pi}{2^{2016}-1}+1\right)\right)=2.$$

Problem 8. Let $ABCD$ be a convex quadrilateral, such that $R_A\cdot R_C=R_B\cdot R_D$, where R_A, R_B, R_C, R_D are circumradiuses of triangles DAB, ABC, BCD, CDA, respectively. Find the number of possible values of the expression $\frac{R_A}{R_B}$.

Solution. *Let*

$$\angle BAC=\alpha,\ \angle ABD=\beta,\ \angle ACD=\gamma,\ \angle BDC=\delta.$$

Thus, it follows that $\alpha+\beta=\gamma+\delta$.

According to the law of sines, from triangles ABC and BCD, it follows that

$$2R_B\sin\alpha=BC=2R_C\sin\delta.$$

Therefore

$$\frac{\sin\alpha}{\sin\delta}=\frac{R_C}{R_B}.$$

In a similar way, we obtain that

$$\frac{\sin\beta}{\sin\gamma}=\frac{R_D}{R_A}.$$

Hence, it follows that

$$\frac{\sin\alpha}{\sin\delta}=\frac{R_C}{R_B}=\frac{R_D}{R_A}=\frac{\sin\beta}{\sin\gamma}.$$

We deduce that

$$\sin\alpha\sin\gamma=\sin\beta\sin\delta,$$

and

$$\cos(\alpha-\gamma)-\cos(\alpha+\gamma)=\cos(\beta-\delta)-\cos(\beta+\delta).$$

On the other hand

$$\cos(\alpha-\gamma)=\cos(\delta-\beta).$$

Thus

$$\cos(\alpha+\gamma)=\cos(\beta+\delta).$$

Note that

$$(\alpha+\gamma)+(\beta+\delta)<2\pi.$$

Therefore

$$\alpha+\gamma=\beta+\delta.$$

On the other hand

$$\alpha+\beta=\gamma+\delta.$$

Thus $\alpha=\delta$, *this means that ABCD is a cyclic quadrilateral. Hence*

$$\frac{R_A}{R_B}=1.$$

Problem 9. Let two circles (considered with their interior parts) with centres O_1, O_2 do not have a common interior point and do not have any point outside the square with side length $15+10\sqrt{2}$. Given that the sum of their diameters is equal to $20+10\sqrt{2}$. Find $(2-\sqrt{2})\cdot\dfrac{AD}{O_1O_2}$.

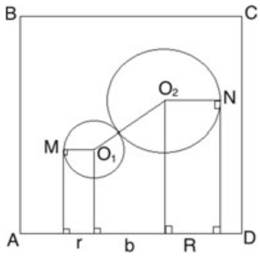
Solution. *Let r and R be the radiuses of circles with centres* O_1 *and* O_2*, respectively. Assume that the considered square is denoted by ABCD. Let us denote by a and b the projections of line segment* O_1O_2 *on lines AB and AD, respectively. We have that*

$$a^2+b^2=O_1O_2^2.$$

Thus, without loss of generality we may assume that

$$b\geq\frac{O_1O_2}{\sqrt{2}}.$$

Consider the following figure.



We have that

$$AD = 15+10\sqrt{2} \geq r+R+b \geq 10+5\sqrt{2}+\frac{O_1O_2}{\sqrt{2}} \geq$$

$$\geq 10+5\sqrt{2}+\frac{r+R}{\sqrt{2}} = 10+5\sqrt{2}+5\sqrt{2}+5 = 15+10\sqrt{2}.$$

Thus, it follows that

$$O_1O_2 = r+R, \ AD = \frac{2+\sqrt{2}}{2}(r+R),$$

and

$$(2-\sqrt{2}) \cdot \frac{AD}{O_1O_2} = 1.$$

Problem 10. Let $\alpha, \beta, \gamma, \delta$ be acute angles, such that

$$4(\tan\alpha+\tan\beta+\tan\gamma+\tan\delta)\cos\alpha\cos\beta\cos\gamma\cos\delta = 3\sqrt{3}.$$

Find the value of the expression

$$(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta)^2.$$

Solution. *Let*

$$M = \cos\alpha\cos\beta\cos\gamma\cos\delta(\tan\alpha+\tan\beta+\tan\gamma+\tan\delta).$$

Note that

$$M = \sin(\alpha+\beta)\cos\gamma\cos\delta+\sin(\gamma+\delta)\cos\alpha\cos\beta.$$

On the other hand

$$\cos x\cos y = \frac{1}{2}(\cos(x+y)+\cos(x-y)) \leq \frac{1}{2}(1+\cos(x+y)) = \cos^2\frac{x+y}{2}.$$

Therefore

$$M \le \sin(\alpha+\beta)\cos^2\frac{\gamma+\delta}{2}+\sin(\gamma+\delta)\cos^2\frac{\alpha+\beta}{2}=$$

$$=2\cos\frac{\alpha+\beta}{2}\cos\frac{\gamma+\delta}{2}\sin\frac{\alpha+\beta+\gamma+\delta}{2}\le$$

$$\le 2\cos^2\frac{\alpha+\beta+\gamma+\delta}{4}\sin\frac{\alpha+\beta+\gamma+\delta}{2}=$$

$$\frac{4}{\sqrt{3}}\left(\sqrt[4]{\cos^2\frac{\alpha+\beta+\gamma+\delta}{4}\cdot\cos^2\frac{\alpha+\beta+\gamma}{4}\cdot\cos^2\frac{\alpha+\beta+\gamma+\delta}{4}\cdot 3\sin^2\frac{\alpha+\beta+\gamma+\delta}{4}}\right)^2\le$$

$$\frac{4}{\sqrt{3}}\left(\frac{\cos^2\frac{\alpha+\beta+\gamma+\delta}{4}+\cos^2\frac{\alpha+\beta+\gamma+\delta}{4}+\cos^2\frac{\alpha+\beta+\gamma+\delta}{4}+3\sin^2\frac{\alpha+\beta+\gamma+\delta}{4}}{4}\right)^2$$

$$=\frac{3\sqrt{3}}{4}.$$

According to the assumption of the problem $M=\frac{3\sqrt{3}}{4}$. *Therefore* $\gamma=\delta$, $\alpha=\beta$, $\frac{\alpha+\beta}{2}=\frac{\gamma+\delta}{2}$ *and*

$$\cos^2\frac{\alpha+\beta+\gamma+\delta}{4}=3\sin^2\frac{\alpha+\beta+\gamma+\delta}{4}.$$

Thus, it follows that

$$\alpha=\beta=\gamma=\delta=\frac{\pi}{6},$$

and

$$(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta)^2=12.$$

Problem 11. Let $ABCD$ be a cyclic quadrilateral, such that $BC=CD$ and $\angle BAD=30^\circ$. Let M, N be points on sides AB, AD, respectively, such that $MN=BM+DN$. Given that $\angle MCN=n^\circ$. Find n.

Solution. *Let us choose a point* K *on ray* MB, *such that* $MK=MN$. *Note that* $BK=MK-MB=MN-MB=ND$. *On the other hand,* $BC=CD$ *and* $\angle KBC=\angle ADC$. *Therefore* $\triangle KBC=\triangle NDC$.

From the last equation, we deduce that $\angle BCK=\angle DCN$ *and* $CK=CN$. *Hence* $\triangle KMC=\triangle NMC$. *Thus*

$$n^\circ=\angle MCK=\angle MCB+\angle BCK=\angle MCB+\angle DCN.$$

We obtain that

$$n^\circ = \frac{1}{2}\angle BCD = \frac{1}{2}\cdot 150^\circ = 75^\circ.$$

Therefore $n = 75$.

Problem 12. Let $ABCD$ be a tetrahedron that does not have points outside a unit cube and $AB \cdot CD \cdot d = 2$, where d is the distance between lines AB and CD. Find the number of possible values of the total surface area of $ABCD$.

Solution. *Denote by M and N the midpoints of edges AB and CD, respectively. At first, let us prove the following lemma.*

Lemma 7.3. *If tetrahedron $ABCD$ does not have points outside a unit cube, then*

$$AB^2 + CD^2 + 2MN^2 \leq 6,$$

where M, N are the midpoints of edges AB, CD.

Proof. *Consider the symmetric points of A, B, C, D, M, N with respect to the centre O of the unit cube. Let us denote these points by A', B', C', D', M', N', respectively. Consider parallelograms $D'CDC'$, $ABA'B'$ and $MNM'N'$. The sum of the squares of adjacent sides is not more than the square of the greatest diagonal. Thus, it follows that*

$$CD^2 + NN'^2 \leq 3,$$

and

$$AB^2 + MM'^2 \leq 3.$$

Therefore, either

$$CD^2 + AB^2 + NN'^2 + MM'^2 \leq 6,$$

or

$$CD^2 + AB^2 + 2MN^2 + 2MN'^2 \leq 6.$$

We obtain that

$$AB^2 + CD^2 + 2MN^2 \leq 6.$$

Hence

$$6 \geq AB^2 + CD^2 + 2MN^2 \geq 3\sqrt[3]{AB^2 \cdot CD^2 \cdot 2MN^2}.$$

Thus, we deduce that

$$2 \geq AB \cdot CD \cdot MN \geq AB \cdot CD \cdot d.$$

On the other hand, we have that $AB \cdot CD \cdot d = 2$.

Therefore

$$CD^2 + NN'^2 = 3,$$

$$AB^2 + MM'^2 = 3,$$

$$AB = CD = \sqrt{2} \cdot MN, \quad MN = d.$$

From the first two equations, it follows that, the vertices of tetrahedron are also the vertices of the cube.

From the condition, $AB = CD$, it follows that points C, D and points A, B are not opposite vertices of the cube.

If C and D are the endpoints of one of the edges of the cube, then A and B are also the endpoints of one of the edges of the cube. In this case, $MN \neq d$.

Hence, $CD = \sqrt{2}$, $AB = \sqrt{2}$ and $MN = 1$. Thus, $ABCD$ is a regular tetrahedron with the edges equal to $\sqrt{2}$.

Therefore, the number of possible values of the total surface area of ABCD is equal to 1.

7.1.13 Problem Set 13

Problem 1. Let there exist n $(n \geq 4)$ line segments, such that any four among them are the sides of a quadrilateral and any three of them are not the sides of a triangle. Find the greatest possible value of n.

Solution. *Let us prove that the greatest possible value of n is equal to 4.*

We provide the following example for $n = 4$. Consider line segments with lengths equal to 1, 2, 3, 5. Note that with these line segments one can construct a quadrilateral, but it is not possible to construct a triangle (with these line segments).

Now, let us prove that if five line segments are such that any four among them are the sides of a quadrilateral, then there are at least three line segments among them that are the sides of a triangle.

We proceed the proof by contradiction argument. Consider line segments with lengths $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$ and such that there are no three line segments that are the sides of some triangle. We have that

$$a_5 \geq a_4 + a_3 \geq a_4 + a_2 + a_1.$$

Thus, it follows that

$$a_5 \geq a_4 + a_2 + a_1.$$

This leads to a contradiction, as a_1, a_2, a_4, a_5 are the sides of some quadrilateral.

This ends the solution.

Problem 2. Find the smallest possible value of the expression

$$2\sin x\cos y+2\sin y\cos z+2\sin z\cos x+10.$$

Solution. *Using that* $2ab\geq -a^2-b^2$, *we deduce that*

$$2\sin x\cos y+2\sin y\cos z+2\sin z\cos x+10\geq -\sin^2 x-\cos^2 y-\sin^2 y-$$

$$-\cos^2 z-\sin^2 z-\cos^2 x+10=7.$$

Note that, if $x=y=z=\dfrac{3\pi}{4}$, *then*

$$2\sin x\cos y+2\sin y\cos z+2\sin z\cos x+10=3\cdot 2\cdot\frac{\sqrt{2}}{2}\cdot\left(-\frac{\sqrt{2}}{2}\right)+10=7.$$

Therefore, the smallest possible value of the given expression is equal to 7.

Problem 3. Let the area of convex quadrilateral $ABCD$ be equal to 100 and M be the intersection point of its diagonals. Denote by AA_1, DD_1, BB_1, CC_1 the altitudes of triangles AMD and BMC. Given that $\angle AMD=60^\circ$. Find the area of quadrilateral $A_1B_1C_1D_1$.

Solution. *Note that points* A_1, B_1, C_1, D_1 *are on rays MD, MC, MB, MA, respectively. From triangle* MAA_1, *we deduce that*

$$MA_1=\frac{MA}{2}.$$

In a similar way, we obtain that

$$MC_1=\frac{MC}{2},\quad MB_1=\frac{MB}{2},\quad MD_1=\frac{MD}{2}.$$

Therefore

$$A_1C_1=MA_1+MC_1=\frac{MA}{2}+\frac{MC}{2}=\frac{AC}{2},$$

and

$$B_1D_1=\frac{BD}{2}.$$

Thus, it follows that

$$(A_1B_1C_1D_1)=\frac{1}{2}\cdot A_1C_1\cdot B_1D_1\sin 60^\circ=\frac{1}{4}(ABCD)=25.$$

Hence

$$(A_1B_1C_1D_1)=25.$$

Problem 4. Let angles α, β, γ be such that

$$|\sin\alpha| = \sin\beta\cos\gamma, \quad |\cos\alpha| = \sin\gamma\cos\beta.$$

Find

$$|\sin\alpha| + |\cos\alpha| + 2|\sin\beta| + 2|\cos\beta| + 4|\sin\gamma| + 4|\cos\gamma|.$$

Solution. *We have that*

$$|\sin\alpha| + |\cos\alpha| = \sin(\beta+\gamma) \leq 1.$$

Note that $|\sin\alpha| + |\cos\alpha| \geq \sin^2\alpha + \cos^2\alpha = 1$. *Hence,* $|\sin\alpha| + |\cos\alpha| = 1$. *Thus, it follows that either* $|\sin\alpha| = 1$ *or* $|\cos\alpha| = 1$.

If $|\sin\alpha| = 1$, *then* $|\sin\beta| = 1$, $|\cos\gamma| = 1$ *and* $|\cos\beta| = 0$, $|\sin\gamma| = 0$. *Therefore,*

$$|\sin\beta| + |\cos\beta| = 1,$$

$$|\sin\gamma| + |\cos\gamma| = 1.$$

In a similar way, if $|\cos\alpha| = 1$, *then we obtain that*

$$|\sin\beta| + |\cos\beta| = 1,$$

$$|\sin\gamma| + |\cos\gamma| = 1.$$

Hence, we deduce that

$$|\sin\alpha| + |\cos\alpha| + 2|\sin\beta| + 2|\cos\beta| + 4|\sin\gamma| + 4|\cos\gamma| = 7.$$

Problem 5. Let N be a point on median BM of triangle ABC, such that $\angle ABM = \angle MNC$. Find the ratio of the median of triangle MNC passing through point M to the median of triangle NBC passing through point N.

Solution. *Denote by* E, F *the midpoints of line segments* BC *and* NC. *Let* K *be the symmetric point of point* N *with respect to point* M. *We have that* $AM = MC$ *and* $MN = MK$. *Therefore, quadrilateral* $ANCK$ *is a parallelogram. We have that*

$$\angle AKN = \angle KNC = \angle ABM.$$

Thus, it follows that triangle ABK *is an isosceles triangle. Hence*

$$CN = AK = AB.$$

According to the triangle mid-segment theorem

$$ME = \frac{AB}{2} = \frac{CN}{2} = NF,$$

and

$$\angle EMN = \angle ABM = \angle MNC.$$

Therefore

$$\triangle EMN = \triangle FNM.$$

We deduce that

$$\frac{MF}{NE} = 1.$$

Problem 6. Let $ABCD$ be a given rhombus and M be such a point that $MA = 27$, $MB = 12$, $AB = 18$, $\angle MAB = \angle MBC$. Find the smallest possible value of MD.

Solution. *Note that*

$$\frac{MA}{BC} = \frac{27}{18} = \frac{18}{12} = \frac{AB}{BM}.$$

We have that

$$\angle MAB = \angle MBC.$$

Thus, it follows that

$$\triangle MAB \sim \triangle CBM.$$

Thus, it follows that

$$\frac{MC}{BM} = \frac{2}{3}.$$

Therefore $MC = 8$. By triangle inequality (points M, C, D), we have that

$$MC + MD \geq CD.$$

We deduce that $MD \geq 10$.

Now, let us consider triangles ABM and MBC, such that points A and B are on the different sides of line BM. Note that

$$\angle ABM = \angle BMC.$$

Hence, point M is on the side CD of rhombus $ABCD$. Thus

$$MD = CD - MC = 18 - 8 = 10.$$

Therefore, the smallest possible value of MD is equal to 10.

Problem 7. Let $ABCD$ be a cyclic quadrilateral, such that

$$2\tan\frac{\angle D}{2} = \tan\frac{\angle A}{2}.$$

Given that $CD - AB = 5$. Find $AD + BC$.

Solution. *At first, let us prove the following lemma.*

Lemma 7.4. *Let ABCD be a cyclic quadrilateral, then*

$$\frac{CD-AB}{BC+AD}=\frac{\tan\frac{\angle A}{2}-\tan\frac{\angle D}{2}}{\tan\frac{\angle A}{2}+\tan\frac{\angle D}{2}}.$$

Proof. *Denote by R the radius of the circumscribed circle of quadrilateral $ABCD$. Denote by $\alpha=\angle BAC=\angle BDC$, $\beta=\angle CAD$ and $\gamma=\angle ADB$. According to the law of sines, we obtain that*

$$\frac{CD-AB}{BC+AD}=\frac{2R\sin\beta-2R\sin\gamma}{2R\sin\alpha+2R\sin(180^\circ-\alpha-\gamma-\beta)}=\frac{2\sin\frac{\beta-\gamma}{2}\cos\frac{\beta+\gamma}{2}}{2\sin\frac{\alpha+\beta+\alpha+\gamma}{2}\cos\frac{\beta+\gamma}{2}}=$$

$$=\frac{\sin\frac{\alpha+\beta-(\alpha+\gamma)}{2}}{\sin\frac{\alpha+\beta+\alpha+\gamma}{2}}=\frac{\tan\frac{\angle A}{2}-\tan\frac{\angle D}{2}}{\tan\frac{\angle A}{2}+\tan\frac{\angle D}{2}}.$$

This ends the proof of the lemma.

According to the lemma, we have that

$$\frac{5}{BC+AD}=\frac{2\tan\frac{\angle D}{2}-\tan\frac{\angle D}{2}}{2\tan\frac{\angle D}{2}+\tan\frac{\angle D}{2}}.$$

Thus, it follows that $BC+AD=15$.

Problem 8. Find the value of the following expression

$$4096\cos\frac{\pi}{17}\cos\frac{2\pi}{17}\cos\frac{3\pi}{17}\cos\frac{4\pi}{17}\cos\frac{5\pi}{17}\cos\frac{6\pi}{17}\cos\frac{7\pi}{17}\cos\frac{8\pi}{17}.$$

Solution. *Note that*

$$2\cos\alpha=\frac{\sin 2\alpha}{\sin\alpha}.$$

Thus, it follows that

$$4096\cos\frac{\pi}{17}\cos\frac{2\pi}{17}\cos\frac{3\pi}{17}\cos\frac{4\pi}{17}\cos\frac{5\pi}{17}\cos\frac{6\pi}{17}\cos\frac{7\pi}{17}\cos\frac{8\pi}{17}=$$

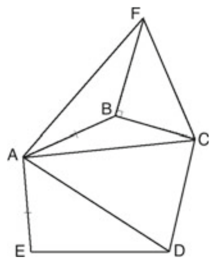
$$= 16 \cdot \frac{\sin\frac{2\pi}{17}}{\sin\frac{\pi}{17}} \cdot \frac{\sin\frac{4\pi}{17}}{\sin\frac{2\pi}{17}} \cdot \frac{\sin\frac{6\pi}{17}}{\sin\frac{3\pi}{17}} \cdot \frac{\sin\frac{8\pi}{17}}{\sin\frac{4\pi}{17}} \cdot \frac{\sin\frac{10\pi}{17}}{\sin\frac{5\pi}{17}} \cdot \frac{\sin\frac{12\pi}{17}}{\sin\frac{6\pi}{17}} \cdot \frac{\sin\frac{14\pi}{17}}{\sin\frac{7\pi}{17}} \cdot \frac{\sin\frac{16\pi}{17}}{\sin\frac{8\pi}{17}} =$$

$$= 16 \cdot \frac{\sin\frac{10\pi}{17}}{\sin\frac{7\pi}{17}} \cdot \frac{\sin\frac{12\pi}{17}}{\sin\frac{5\pi}{17}} \cdot \frac{\sin\frac{14\pi}{17}}{\sin\frac{3\pi}{17}} \cdot \frac{\sin\frac{16\pi}{17}}{\sin\frac{\pi}{17}} = 16.$$

Therefore $\sin\alpha = \sin(\pi - \alpha)$.

Problem 9. Let $ABCDE$ be a convex pentagon, such that $AB = AE$, $BC = 3$, $CD = 5$, $DE = 4$ and $\angle BAE = 2\angle CAD$. Find $(ABCD) + (ACDE) - 1.5(ABCDE)$.

Solution. *Consider the following figure.*



Let F *be such point that* $AF = AD$ *and* $BF = ED$. *Therefore*

$$\triangle AFB = \triangle ADE.$$

Thus, it follows that

$$\angle FAB = \angle DAE.$$

We obtain that

$$\angle FAC = \angle FAB + \angle BAC = \angle DAE + \angle BAC = \angle CAD.$$

Hence

$$\triangle FAC = \triangle DAC.$$

Therefore $CF = CD$. *We have that*

$$(ABCD) + (ACDE) - 1.5(ABCDE) = (ABC) + (ACD) + (ACD) + (AED) -$$

$$-1.5(ABC) - 1.5(ACD) - 1.5(AED) = 0.5((ACD) - (ABC) - (AED)) =$$

$$= 0.5((AFC) - (ABC) - (ABF)) = 0.5(BFC) = 3.$$

Problem 10. Let $ABCD$ be a convex quadrilateral, such that

$$\frac{CD-AB}{BC+AD}=\frac{\tan\frac{\angle A}{2}-\tan\frac{\angle D}{2}}{\tan\frac{\angle A}{2}+\tan\frac{\angle D}{2}}.$$

Given that $\angle A+\angle C=n^\circ$. Find n.

Solution. *Let us prove that ABCD is a cyclic quadrilateral. Proof by contradiction argument. Without loss of generality, one can assume that $\angle ACD>\angle ABD$.*

Denote by C_1 the intersection point of circumcircle of triangle ABD and ray CD. According to the lemma (see Problem 7), we have that

$$\frac{C_1D-AB}{BC_1+AD}=\frac{\tan\frac{\angle A}{2}-\tan\frac{\angle D}{2}}{\tan\frac{\angle A}{2}+\tan\frac{\angle D}{2}}.$$

Thus, it follows that

$$\frac{C_1D-AB}{BC_1+AD}=\frac{CD-AB}{BC+AD}=\delta,$$

where $|\delta|<1$.

Therefore

$$C_1D-AB=\delta(BC_1+AD),$$

and

$$CD-AB=\delta(BC+AD).$$

Hence

$$CC_1=C_1D-CD=\delta(BC_1-BC).$$

We obtain that

$$CC_1=|\delta||BC-BC_1|\leq|\delta|\cdot CC_1.$$

Thus $|\delta|\geq 1$. *This leads to a contradiction.*

Therefore, we deduce that $n=180$.

Problem 11. Let a, b be real numbers, such that the following inequality

$$\sum_{i=0}^{n}|\sin(2^i x)|\leq a+bn,$$

holds true for any x and for any non-negative integer n. Find the smallest possible value of $20a^2+16b^2$.

Solution. *If $n=0$, $x=\dfrac{\pi}{2}$, then $a\geq 1$.*

If $x=\dfrac{\pi}{3}$, then for any positive integer n, we have that

$$a+bn\geq \frac{\sqrt{3}}{2}(n+1).$$

Thus, it follows that

$$b\geq \frac{\sqrt{3}}{2}.$$

Therefore

$$20a^2+16b^2\geq 20+12=32.$$

Now, by mathematical induction (with respect to n), let us prove that

$$\sum_{i=0}^{n}|\sin(2^i x)|\leq 1+\frac{\sqrt{3}}{2}n. \tag{7.17}$$

In this case, we have that

$$20a^2+16b^2\geq 20+12=32.$$

Hence, the smallest possible value of $20a^2+16b^2$ is equal to 32.

Basis. *If $n=0$, then*

$$\sum_{i=0}^{n}|\sin(2^i x)|=|\sin x|\leq 1=1+\frac{\sqrt{3}}{2}\cdot 0.$$

Inductive step. *If for $n\leq k$, it holds true*

$$\sum_{i=0}^{n}|\sin(2^i x)|\leq 1+\frac{\sqrt{3}}{2}n,$$

then this inequality holds true for $n=k+1$.

Consider two cases:

a) If $|\sin x|\leq \dfrac{\sqrt{3}}{2}$, then

$$|\sin x|+|\sin 2x|+\cdots+|\sin 2^{k+1}x|\leq \frac{\sqrt{3}}{2}+|\sin 2x|+\cdots+|\sin 2^k\cdot 2x|\leq$$

$$\leq \frac{\sqrt{3}}{2}+1+\frac{\sqrt{3}}{2}k=1+\frac{\sqrt{3}}{2}(k+1).$$

b) If $|\sin x| > \frac{\sqrt{3}}{2}$*, then note that*

$$2|\sin x| + |\sin 2x| = \frac{2}{\sqrt{3}}\sqrt{(3-3|\cos x|)(1+|\cos x|^3)} \le \frac{2}{\sqrt{3}}\sqrt{\left(\frac{6}{4}\right)^4} = \frac{3\sqrt{3}}{2}.$$

Therefore, if $|\sin x| > \frac{\sqrt{3}}{2}$*, then*

$$|\sin x| + |\sin 2x| < \sqrt{3}.$$

Hence, if $k \ge 1$*, then*

$$|\sin x| + |\sin 2x| + \cdots + |\sin 2^{k+1}x| < \sqrt{3} + |\sin 4x| + \cdots + |\sin 4x \cdot 2^{k-1}| \le$$

$$\le \sqrt{3} + 1 + \frac{\sqrt{3}}{2}(k-1) = 1 + \frac{\sqrt{3}}{2}(k+1).$$

This ends the proof of the statement for $n = k+1$*. Therefore,* (7.17) *holds true.*

Problem 12. Let two balls with centres O_1, O_2 have no common inner point. Given that the sum of the diameters is equal to $15+5\sqrt{3}$ and that none of them has a point outside the cube $ABCDA_1B_1C_1D_1$ having a side length $10+5\sqrt{3}$. Find

$$(3-\sqrt{3}) \cdot \frac{AB}{O_1O_2}.$$

Solution. *Denote by r and R the radiuses of balls with centres* O_1*,* O_2*, respectively. Denote by a, b, c the projections of line segment* O_1O_2 *on lines AB, AD,* AA_1*, respectively.*

We have that

$$a^2 + b^2 + c^2 = O_1O_2^2.$$

Let $\max(a,b,c) = a$*, then* $a \ge \frac{O_1O_2}{\sqrt{3}}$.

Note that

$$AB \ge a + r + R \ge \frac{O_1O_2}{\sqrt{3}} + r + R \ge \frac{1+\sqrt{3}}{\sqrt{3}}(r+R) = \frac{1+\sqrt{3}}{\sqrt{3}} \cdot \frac{5\sqrt{3}(\sqrt{3}+1)}{2} = 10 + 5\sqrt{3}.$$

Hence, using the condition $AB = 10+5\sqrt{3}$*, we deduce that*

$$O_1O_2 = r + R, \quad AB = \frac{1+\sqrt{3}}{\sqrt{3}}(R+r).$$

Therefore

$$(3-\sqrt{3})\frac{AB}{O_1O_2} = (3-\sqrt{3})\cdot\frac{1+\sqrt{3}}{\sqrt{3}} = 2.$$

7.1.14 Problem Set 14

Problem 1. Let $ABCD$ be a trapezoid with bases BC and AD, such that $BC = 4$ and $AD = 9$. Given that $\frac{AB}{CD} = \frac{2}{3}$. Find AC.

Solution. *Let $AC = x$, $\angle ACB = \alpha$. We have that $\angle CAD = \angle ACB = \alpha$. From triangles ABC and ACD, according to the law of cosines, it follows that*

$$AB^2 = 16 + x^2 - 8x\cos\alpha,$$

and

$$CD^2 = 81 + x^2 - 18x\cos\alpha.$$

We obtain that

$$\frac{4}{9} = \frac{16+x^2-8x\cos\alpha}{81+x^2-18x\cos\alpha}.$$

Therefore $x = 6$. Hence $AC = 6$.

Problem 2. Let $SABC$ be a tetrahedron. Given that $SA = BC = 8.5$, $SB = AC = 5$, $SC = AB = \frac{1}{2}\sqrt{261}$. Find the volume of $SABC$.

Solution. *Let us consider points M, N, P on plane ABC, such that quadrilaterals ABMC, ABCN, ACBP are parallelograms. Therefore, points A, B, C are the midpoints of line segments NP, MP, MN, respectively.*

Denote by $(XYZT)$ the volume of tetrahedron $XYZT$. We have that

$$(SABC) = \frac{1}{4}(SMNP),$$

as

$$(ABC) = \frac{1}{4}(MNP).$$

On the other hand, $SC = AB = CN = CM$. Thus, it follows that $SN \perp SM$. In a similar way, we obtain that $SN \perp SP$ and $SP \perp SM$.

Hence, $SM \perp (SNP)$ and

$$(SMNP) = \frac{1}{3}(SNP)\cdot SM = \frac{1}{6}SM\cdot SN\cdot SP.$$

Let $SM = x$, $SN = y$, $SP = z$. We have that

$$x^2 + y^2 = 261,$$

$$x^2 + z^2 = 100,$$

$$y^2 + z^2 = 289.$$

Therefore, we obtain that $x = 6$, $y = 15$, $z = 8$ *and* $(SMNP) = 120$.

Problem 3. Let ABC be a triangle, such that $\cos\angle A = 0.8$, $\cos\angle B = 0.28$. Given that points M, N are on sides AB, AC, respectively, such that $AM = MN = NC = 8$. Find BC.

Solution. *Note that*

$$\cos 2\angle A = 2\cos^2\angle A - 1 = 0.28 = \cos\angle B.$$

Thus, it follows that $2\angle A = \angle B$. *On the other hand*

$$\angle BMN = 2\angle A = \angle B.$$

Let us consider an isosceles trapezoid $MBCK$ *with bases* MB *and* KC. *We have that*

$$\angle KNC = \angle ANM = \angle MAN = \angle NCK.$$

Therefore

$$\angle KNC = \angle NCK = \angle A.$$

Let $BC = x$, *then*

$$KN = KC = MK - MN = x - 8.$$

From triangle NKC, *we deduce that*

$$\cos\angle KNC = \frac{4}{x-8},$$

$$0.8 = \frac{4}{x-8}.$$

Hence $BC = 13$.

Problem 4. Find the greatest value of the expression

$$3\tan x - \frac{5}{\cos x} + 15,$$

where $x \in \left(0, \frac{\pi}{2}\right)$.

Solution. *We have that*

$$3\tan x-\frac{5}{\cos x}+15=\frac{3\sin x+4\cos x}{\cos x}-\frac{5}{\cos x}+11=\frac{5\sin(x+\phi)}{\cos x}-\frac{5}{\cos x}+11,$$

where $\phi=\arccos\frac{3}{5}$.

Thus, it follows that

$$3\tan x-\frac{5}{\cos x}+15\leq 11.$$

If $x=\arcsin\frac{3}{5}$, *then*

$$3\tan x-\frac{5}{\cos x}+15=\frac{5\sin\frac{\pi}{2}}{\cos\left(\arcsin\frac{3}{5}\right)}-\frac{5}{\cos\left(\arcsin\frac{3}{5}\right)}+11=11.$$

Therefore, if $x\in\left(0,\frac{\pi}{2}\right)$, *then the greatest value of the expression* $3\tan x-\frac{5}{\cos x}+15$ *is equal to 11.*

Problem 5. Let $ABCD$ be a cyclic quadrilateral, such that $\angle A=30°$, $AC=5(\sqrt{6}-\sqrt{2})$ and $BC=CD$. Find $AB+AD$.

Solution. *Let us choose points* M *and* N *on rays* AB *and* AD*, respectively, such that* $AM=AN=AB+AD$*. We have that* $\angle MBC=\angle ADC$, $BC=CD$ *and* $BM=AD$*. Thus, it follows that* $\triangle ADC=\triangle MBC$*. Therefore* $MC=AC$*. In a similar way, we obtain that* $CN=AC$*. From triangle* AMN*, according to the theorem of sines, we obtain that*

$$\frac{AN}{\sin 75°}=2AC.$$

Hence

$$AB+AD=2AC\sin 75°=2\cdot 5(\sqrt{6}-\sqrt{2})\cdot(\sin 30°\cos 45°+\cos 30°\sin 45°)=10.$$

Problem 6. Find the value of the expression

$$\frac{\sin^2\frac{2\pi}{3^{100}-1}}{\cos\frac{2\cdot 3\pi}{3^{100}-1}}+\frac{\sin^2\frac{2\cdot 3\pi}{3^{100}-1}}{\cos\frac{2\cdot 3^2\pi}{3^{100}-1}}+\cdots+\frac{\sin^2\frac{2\cdot 3^{99}\pi}{3^{100}-1}}{\cos\frac{2\cdot 3^{100}\pi}{3^{100}-1}}.$$

Solution. *Note that*

$$\frac{\sin^2\alpha}{\cos 3\alpha} = \frac{1}{4}\Big(\frac{1}{\cos 3\alpha} - \frac{1}{\cos\alpha}\Big). \tag{7.18}$$

We have that

$$\frac{1}{4}\Big(\frac{1}{\cos 3\alpha} - \frac{1}{\cos\alpha}\Big) = \frac{\cos\alpha - \cos 3\alpha}{4\cos\alpha\cos 3\alpha} =$$

$$= \frac{\sin\alpha\cdot\sin 2\alpha}{2\cos\alpha\cdot\cos 3\alpha} = \frac{\sin^2\alpha}{\cos 3\alpha}.$$

According to (7.18), *we have that*

$$\frac{\sin^2\dfrac{2\pi}{3^{100}-1}}{\cos\dfrac{2\cdot 3\pi}{3^{100}-1}} + \frac{\sin^2\dfrac{2\cdot 3\pi}{3^{100}-1}}{\cos\dfrac{2\cdot 3^2\pi}{3^{100}-1}} + \cdots + \frac{\sin^2\dfrac{2\cdot 3^{99}\pi}{3^{100}-1}}{\cos\dfrac{2\cdot 3^{100}\pi}{3^{100}-1}} =$$

$$= \frac{1}{4}\left(\frac{1}{\cos\dfrac{2\cdot 3^{100}\pi}{3^{100}-1}} - \frac{1}{\cos\dfrac{2\pi}{3^{100}-1}}\right) = 0.$$

Problem 7. Consider a quadrilateral $ABCD$, such that $AB = 4$, $AC = 6$, $AD = 9$ and $\angle BAC = \angle DAC = 30^\circ$. Find the square of the circumradius of triangle BCD.

Solution. *Note that $\triangle ABC \sim \triangle ACD$, as $\angle BAC = \angle CAD$ and $\dfrac{AB}{AC} = \dfrac{AC}{AD}$. Therefore $\angle BCA = \angle CDA$ and $\angle ABC = \angle ACD$. Hence, we obtain that*

$$\angle ABC + \angle CDA = \angle BCD.$$

Thus, it follows that $\angle BCD = 150^\circ$.

From triangle BCD, according to the law of sines, we have that

$$2R = \frac{BD}{\sin\angle BCD}.$$

We deduce that $R = BD$, where R is the circumradius of triangle BCD.

On the other hand, from triangle ABD, according to the law of cosines, we have that

$$BD^2 = 61.$$

Therefore $R^2 = 61$.

Problem 8. Let $ABCD$ be a circumscribed quadrilateral, such that $\angle BAC = 16^\circ$, $\angle DAC = 44^\circ$, $\angle ADC = 32^\circ$. Given that $\angle ACB = n^\circ$. Find n.

Solution. *Denote by I_1, I_2 the incenters of triangles ABC, ADC, respectively, and by B_1, D_1 the points of tangency of AC side and these circles. We have that*

$$AB_1 = \frac{AB + AC - BC}{2} = \frac{AD + AC - CD}{2} = AD_1.$$

Thus, it follows that $CB_1 = CD_1$. Therefore

$$\frac{I_1B_1}{I_2D_1} = \frac{AB_1 \tan 8^\circ}{AB_1 \tan 22^\circ},$$

and

$$\frac{I_1B_1}{I_2D_1} = \frac{CB_1 \tan \frac{n^\circ}{2}}{CB_1 \tan 52^\circ}.$$

We deduce that

$$\tan \frac{n^\circ}{2} = \tan 8^\circ \tan 52^\circ \tan 68^\circ = \tan 8^\circ \frac{\tan 60^\circ - \tan 8^\circ}{1 + \tan 60^\circ \tan 8^\circ} \cdot \frac{\tan 60^\circ + \tan 8^\circ}{1 - \tan 60^\circ \tan 8^\circ} =$$

$$= \frac{3\tan 8^\circ - \tan^3 8^\circ}{1 - 3\tan^2 8^\circ} = \tan 24^\circ.$$

Hence, we obtain that $n = 48$.

Problem 9. Given that tetrahedron $ABCD$ does not have any point outside $3 \times 4 \times 5$ rectangular parallelepiped. Find the greatest possible value of the expression

$$DA \cdot BC + DB \cdot AC + DC \cdot AB.$$

Solution. *At first, let us prove the following lemma.*

Lemma 7.5. *If triangle XYZ does not have any point outside $3 \times 4 \times 5$ rectangular parallelepiped, then*

$$XY^2 + YZ^2 + ZX^2 \leq 100.$$

Proof. *Let us choose the coordinate system, such that the vertices of $3 \times 4 \times 5$ rectangular parallelepiped are*

$$(3,0,0), (0,0,0), (0,4,0), (3,4,0), (3,0,5), (0,0,5), (0,4,5), (3,4,5).$$

Let $X(x_1, y_1, z_1)$, $Y(x_2, y_2, z_2)$, $Z(x_3, y_3, z_3)$. We have that

$$XY^2 + YZ^2 + ZX^2 = (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 + (y_1 - y_2)^2 +$$

$$+ (y_2 - y_3)^2 + (y_3 - y_1)^2 + (z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 \leq$$

$$\leq 2(\max(x_1,x_2,x_3)-\min(x_1,x_2,x_3))^2+2(\max(y_1,y_2,y_3)-\min(y_1,y_2,y_3))^2+$$

$$+2(\max(z_1,z_2,z_3)-\min(z_1,z_2,z_3))^2 \leq 2\cdot 3^2+2\cdot 4^2+2\cdot 5^2=100.$$

This ends the proof of the lemma.

Note that, using the lemma for triangles DAC, DAB, DBC and ABC, it follows that

$$DA\cdot BC+DB\cdot AC+DC\cdot AB \leq \frac{DA^2+BC^2}{2}+\frac{DB^2+AC^2}{2}+\frac{DC^2+AB^2}{2}=$$

$$=\frac{DA^2+DC^2+AC^2}{4}+\frac{DC^2+DB^2+BC^2}{4}+\frac{DA^2+DB^2+AB^2}{4}+$$

$$+\frac{AB^2+BC^2+AC^2}{4} \leq 100.$$

Therefore

$$DA\cdot BC+DB\cdot AC+DC\cdot AB \leq 100.$$

On the other hand, for points $A(3,0,0)$, $B(0,4,0)$, $C(0,0,5)$ and $D(3,4,5)$, we have that

$$DA\cdot BC+DB\cdot AC+DC\cdot AB=100.$$

Hence, the greatest possible value of the expression

$$DA\cdot BC+DB\cdot AC+DC\cdot AB$$

is equal to 100.

Problem 10. Find the value of the expression

$$\tan\frac{\pi}{9}\tan\frac{2\pi}{9}+\tan\frac{2\pi}{9}\tan\frac{4\pi}{9}-\tan\frac{\pi}{9}\tan\frac{4\pi}{9}.$$

Solution. *We have that*

$$\tan\frac{\pi}{9}\tan\frac{2\pi}{9}+\tan\frac{2\pi}{9}\tan\frac{4\pi}{9}=\tan\frac{2\pi}{9}\cdot\frac{\sin\frac{5\pi}{9}}{\cos\frac{\pi}{9}\cos\frac{4\pi}{9}}=\tan\frac{2\pi}{9}\tan\frac{4\pi}{9}\cdot\frac{1}{\cos\frac{\pi}{9}}.$$

Thus, it follows that

$$\tan\frac{\pi}{9}\tan\frac{2\pi}{9}+\tan\frac{2\pi}{9}\tan\frac{4\pi}{9}-\tan\frac{\pi}{9}\tan\frac{4\pi}{9}=\tan\frac{4\pi}{9}\left(\frac{2\sin\frac{\pi}{9}}{\cos\frac{2\pi}{9}}-\frac{\sin\frac{\pi}{9}}{\cos\frac{\pi}{9}}\right)=$$

$$= \tan\frac{\pi}{9}\tan\frac{4\pi}{9}\cdot\frac{2\cos\frac{\pi}{9}-\cos\frac{2\pi}{9}}{\cos\frac{2\pi}{9}} = \tan\frac{\pi}{9}\tan\frac{4\pi}{9}\cdot\frac{\cos\frac{\pi}{9}+2\sin\frac{\pi}{18}\sin\frac{\pi}{6}}{\cos\frac{2\pi}{9}} =$$

$$= \tan\frac{\pi}{9}\tan\frac{4\pi}{9}\cdot\frac{\cos\frac{\pi}{9}+\cos\frac{4\pi}{9}}{\cos\frac{2\pi}{9}} = \tan\frac{\pi}{9}\tan\frac{4\pi}{9}\cdot\frac{2\cos\frac{5\pi}{18}\cos\frac{\pi}{6}}{\cos\frac{2\pi}{9}} =$$

$$= \sqrt{3}\tan\frac{\pi}{9}\tan\frac{2\pi}{9}\cdot\tan\frac{4\pi}{9} = 3.$$

Note that

$$\tan\left(\frac{\pi}{3}-\alpha\right)\tan\alpha\tan\left(\frac{\pi}{3}+\alpha\right) = \frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha} = \tan(3\alpha).$$

If $\alpha = \frac{\pi}{9}$, *then we obtain that*

$$\tan\frac{\pi}{9}\tan\frac{2\pi}{9}\tan\frac{4\pi}{9} = \tan\frac{\pi}{3} = \sqrt{3}.$$

Problem 11. Let $ABCD$ be a quadrilateral, such that $\angle A = 75^\circ$, $\angle B = 45^\circ$, $\angle C = 120^\circ$. Given that lines AB, CD intersect at point E and lines AD, BC intersect at point F. Denote by M, N, P, Q the intersection points of the altitudes of triangles EAD, EBC, FCD, FAB, respectively. Find $\left(\frac{MN}{PQ}-3\right)^2$.

Solution. *Let CC_1, DD_1 be the altitudes of triangle FCD. Note that*

$$\frac{FD_1}{FD} = \cos\angle AFB = \frac{FC_1}{FC}.$$

Thus, it follows that

$$\triangle C_1FD_1 \sim \triangle CFD.$$

Therefore

$$C_1D_1 = CD\cos\angle AFB. \tag{7.19}$$

On the other hand, points F, D_1, P, C_1 are on the circle with diameter FP. Hence, according to the law of sines, we obtain that

$$FP = \frac{C_1D_1}{\sin\angle AFB}. \tag{7.20}$$

From (7.19) *and* (7.20)*, we deduce that*

$$FP = CD \cot \angle AFB.$$

In a similar way, we obtain that

$$FQ = AB \cot \angle AFB.$$

Now, let us consider a parallelogram $ADCK$*. Note that* $FP \perp AK$*,* $FQ \perp AB$*. Therefore* $\angle PFQ = \angle KAB$*. On the other hand,*

$$\frac{FP}{AK} = \frac{FQ}{AB}.$$

Thus, it follows that

$$\triangle PFQ \sim \triangle KAB.$$

Hence

$$PQ = KB \cot \angle AFB.$$

In a similar way, we obtain that

$$MN = KB \cot \angle BEC.$$

We deduce that

$$\frac{MN}{PQ} = \frac{\cot 15^\circ}{\cot 60^\circ} = \sqrt{3}(2+\sqrt{3}).$$

Therefore

$$\left(\frac{MN}{PQ} - 3\right)^2 = 12.$$

Problem 12. Let $ABCD$ be a parallelogram, such that $\angle A$ is an acute angle. Denote by M, N the midpoints of sides AD and BC. Let CE be the altitude of parallelogram $ABCD$, such that $E \in AD$ and K be a point on line MN, such that ray EB is the bisector of angle AEK. Given that $DE = 6$ and $EK = 15$. Find AK.

Solution. *Let* $\angle BAD = \alpha$*,* $\angle AEB = \phi$*,* $AM = a$*,* $DE = 2b$*. We have that*

$$\angle KME = \angle BAD = \alpha,$$

and

$$\angle MEK = 2\phi.$$

From triangle MKE, according to the law of sines, it follows that

$$MK = \frac{(a+2b)\sin 2\phi}{\sin(\alpha+2\phi)},$$

and

$$KE = \frac{(a+2b)\sin\alpha}{\sin(\alpha+2\phi)}.$$

From triangle AMK, according to the law of cosines, it follows that

$$AK^2 = a^2 + \left(\frac{(a+2b)\sin 2\phi}{\sin(\alpha+2\phi)}\right)^2 + 2a\cdot\frac{a+2b}{\sin(\alpha+2\phi)}\cdot\sin 2\phi\cos\alpha.$$

Let us prove that $AK = KE + ED$. Hence, we need to prove that

$$\left(\frac{(a+2b)\sin\alpha}{\sin(\alpha+2\phi)}+2b\right)^2 = a^2 + \left(\frac{(a+2b)\sin 2\phi}{\sin(\alpha+2\phi)}\right)^2 + 2a\frac{(a+2b)\sin 2\phi\cos\alpha}{\sin(\alpha+2\phi)},$$

or

$$\frac{(a+2b)\sin^2\alpha}{\sin^2(\alpha+2\phi)} + \frac{4b\sin\alpha}{\sin(\alpha+2\phi)} = a-2b+\frac{(a+2b)\sin^2 2\phi}{\sin^2(\alpha+2\phi)} + \frac{2a\sin 2\phi\cos\alpha}{\sin(\alpha+2\phi)}. \tag{7.21}$$

Note that, from triangle ABE and CED we have that

$$AB = \frac{(2a+2b)}{\sin(\alpha+\phi)}\cdot\sin\phi,$$

and

$$CD = \frac{2b}{\cos\alpha}.$$

Therefore, from the assumption $AB = CD$, we deduce that

$$a = b\tan\alpha\cot\phi.$$

Then, (7.21) can be rewritten as

$$(\tan\alpha\cot\phi+2)\sin^2\alpha + 4\sin\alpha\sin(\alpha+2\phi) = (\tan\alpha\cot\phi-2)\sin^2(\alpha+2\phi) +$$

$$+(\tan\alpha\cot\phi+2)\sin^2 2\phi + 4\sin\alpha\cos^2\phi\sin(\alpha+2\phi),$$

or

$$(\tan\alpha\cot\phi+2)\sin(\alpha-2\phi)+4\sin\alpha = (\tan\alpha\cot\phi-2)\sin(\alpha+2\phi)+4\sin\alpha\cos^2\phi,$$

$$-2\tan\alpha\cot\phi\sin 2\phi\cos\alpha + 4\sin\alpha\cos 2\phi + 4\sin\alpha\sin^2\phi = 0,$$

$$-4\sin\alpha\cos^2\phi + 4\sin\alpha\cos^2\phi = 0.$$

Hence, we obtain that $AK = KE + DE = 21$.

7.1.15 Problem Set 15

Problem 1. Let AB be the hypotenuse of a right triangle ABC. Given that $AB = 200$ and $\angle B = \frac{5\pi}{24}$. Consider the altitude CH and the median CM of triangle ABC. Find the distance from the point H to the line CM.

Solution. *Consider the altitude HE and the median HF of right triangle CHM. We have that*

$$CM = MB = MA = 100,$$

and

$$\angle HMC = 2\angle B = \frac{5\pi}{12}.$$

Thus, it follows that

$$\angle HCM = \frac{\pi}{12}.$$

On the other hand

$$CF = FM = HF.$$

Therefore

$$HF = 50,$$

and

$$\angle HFE = 2\angle HCM = \frac{\pi}{6}.$$

From triangle HFE, we obtain that

$$HE = \frac{HF}{2} = 25.$$

Problem 2. Find the greatest possible value of the expression

$$5\tan x + \frac{13}{\cos x} + 30,$$

where $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Solution. *We have that*

$$5\tan x + 12 + \frac{13}{\cos x} + 18 = \frac{5\sin x + 12\cos x}{\cos x} + \frac{13}{\cos x} + 18 =$$

$$= \frac{13\sin(x+\phi)}{\cos x} + \frac{13}{\cos x} + 18,$$

where $\phi = \arccos\frac{5}{13}$. *Note that* $\sin(x+\phi) \geq -1$ *and* $\cos x < 0$. *Thus, it follows that*

$$5\tan x + \frac{13}{\cos x} + 30 = \frac{13(\sin(x+\phi)+1)}{\cos x} + 18 \leq 18.$$

If $x = \pi + \arcsin\frac{5}{13}$, *then*

$$5\tan x + \frac{13}{\cos x} + 30 = 18.$$

Therefore, the greatest possible value of the given expression is equal to 18.

Problem 3. Let the diagonals of a convex quadrilateral $ABCD$ intersect at point M. Given that $(ABC) = 25$, $(BCD) = 16$, $(ACD) = 15$. Find (AMB).

Solution. *Let* $(AMB) = x$, *then*

$$(CMB) = 25 - x,$$

$$(CMD) = x - 9,$$

$$(AMD) = 24 - x.$$

Note that

$$\frac{(AMB)}{(CMB)} = \frac{AM}{CM} = \frac{(AMD)}{(CMD)}.$$

Therefore

$$x(x-9) = (25-x)(24-x).$$

Thus, it follows that $x = 15$. *Hence, we obtain that*

$$(AMB) = 15.$$

Problem 4. Let M, N be the intersection points of two circles with centres O_1, O_2 and radiuses 25, 60, respectively. Given that $O_1O_2 = 65$. Let line l passes through the point M, such that points O_1 and O_2 are on the same side of line l. Given also

that the distance from the point O_1 to the line l is equal to 15. Find the distance from the point O_2 to the line l.

Solution. *Note that*

$$65^2 = 25^2 + 60^2.$$

Thus, it follows that

$$\angle O_1MO_2 = 90^\circ.$$

Let $K, E \in l$ and $O_1K \perp l$, $O_1E \perp l$. Note that

$$\angle O_1MK + \angle O_2ME = 90^\circ.$$

Therefore

$$\triangle O_1KM \sim \triangle MEO_2.$$

We obtain that

$$\frac{O_2E}{KM} = \frac{O_2M}{O_1M}.$$

We have that

$$KM = \sqrt{O_1M^2 - O_1K^2} = \sqrt{25^2 - 15^2} = 20.$$

Thus, it follows that

$$O_2E = \frac{20 \cdot 12}{5} = 48.$$

Problem 5. Let ABC be an acute triangle, such that $\angle C \leq \arccos(\sqrt{5} - 2)$ and $AB = 5(\sqrt{5} + 1)$. Consider the altitudes AA_1 and BB_1. Given that AB_1A_1B is a circumscribed quadrilateral. Find the perimeter of quadrilateral AB_1A_1B.

Solution. *Let $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$. From right triangles ABB_1 and ABA_1, we have that*

$$AB_1 = AB\cos\alpha,$$

$$BA_1 = AB\cos\beta.$$

On the other hand, from the condition

$$\frac{CA_1}{AC} = \frac{CB_1}{CB} = \cos\gamma,$$

it follows that

$$\triangle A_1CB_1 \sim \triangle ACB.$$

Therefore

$$A_1B_1 = AB\cos\gamma.$$

According to the assumptions of the problem, we have that

$$AB_1 + BA_1 = AB + A_1B_1.$$

Thus, it follows that

$$\cos\alpha + \cos\beta = 1 + \cos\gamma \geq \sqrt{5} - 1.$$

On the other hand

$$\cos\alpha + \cos\beta \leq 2\cos\frac{\alpha+\beta}{2} = 2\sin\frac{\gamma}{2} \leq 2 - 2\sin^2\frac{\gamma}{2} = 1 + \cos\gamma.$$

We obtain that

$$\cos\alpha + \cos\beta = \sqrt{5} - 1.$$

Therefore

$$AB_1 + BA_1 = AB(\cos\alpha + \cos\beta) = 20.$$

Hence, using that AB_1A_1B is a circumscribed quadrilateral, we deduce that the perimeter of quadrilateral AB_1A_1B is equal to 40.

Problem 6. Let M be a chosen point on the incircle of an isosceles triangle ABC and N be the intersection point of that circle with the line segment BM. Given that $\angle B = 120^\circ$ and $\dfrac{BN}{BM} = \dfrac{2-\sqrt{3}}{2}$. Let $\angle AMC = n^\circ$. Find n.

Solution. *Let the incircle of triangle ABC is tangent to the sides AB and BC at the points E and F, respectively. Denote by I the incenter of triangle ABC.*

We have that $\angle IBE = 60^\circ$, $\angle IAB = 15^\circ$ and $IE \perp AB$. Thus, it follows that

$$\frac{BE}{EA} = \frac{IE\cot 60^\circ}{EA\cot 15^\circ} = \frac{\tan 15^\circ}{\tan 60^\circ} = \frac{2-\sqrt{3}}{\sqrt{3}}.$$

Therefore

$$\frac{BE}{EA} = \frac{2-\sqrt{3}}{\sqrt{3}} = \frac{BN}{NM}.$$

We deduce that $EN \parallel AM$. In a similar way, we obtain that $MC \parallel NF$. Hence

$$\angle AMC = \angle ENF = \frac{360^\circ - \angle EIF}{2} = \frac{360^\circ - 60^\circ}{2} = 150^\circ.$$

Thus, it follows that $n = 150$.

Problem 7. Let the greatest value of the expression $|\sin x(1-\cos y)| + |\sin y(1-\cos x)|$ be equal to M. Find $4M^2$.

Solution. *We have that*

$$|\sin x(1-\cos y)|+|\sin y(1-\cos x)|=\frac{1}{\sqrt{3}}|\sqrt{3}\sin x(1-\cos y)|+$$

$$+\frac{1}{\sqrt{3}}|\sqrt{3}\sin y(1-\cos x)|\leq\frac{1}{2\sqrt{3}}(3\sin^2 x+(1-\cos y)^2)+$$

$$+\frac{1}{2\sqrt{3}}(3\sin^2 y+(1-\cos x)^2)=\frac{1}{2\sqrt{3}}(8-2\cos^2 x-2\cos x-2\cos^2 y-2\cos y)=$$

$$=\frac{1}{2\sqrt{3}}\left(9-2\left(\cos x+\frac{1}{2}\right)^2-2\left(\cos y+\frac{1}{2}\right)^2\right)\leq\frac{3\sqrt{3}}{2}.$$

If $x=y=\dfrac{2\pi}{3}$, *then*

$$|\sin x(1-\cos y)|+|\sin y(1-\cos x)|=\frac{\sqrt{3}}{2}\cdot\frac{3}{2}+\frac{\sqrt{3}}{2}\cdot\frac{3}{2}=\frac{3\sqrt{3}}{2}.$$

Thus, it follows that $M=\dfrac{3\sqrt{3}}{2}$ *and* $4M^2=27$.

Problem 8. Let $ABCD$ be a cyclic quadrilateral and $AB=20$, $BC=18$, $CD=45$. Given points M, N on the diagonal AC, such that $\angle ABM=\angle CBN$ and $\angle ADM=\angle CDN$. Find AD.

Solution. *Let* $\angle ABM=\angle CBN=\phi$ *and* $\angle ABC=\alpha$. *We have that*

$$\frac{AM}{MC}=\frac{(ABM)}{(BMC)}=\frac{AB\sin\phi}{BC\sin(\alpha-\phi)}. \tag{7.22}$$

In a similar way, we obtain that

$$\frac{BC\sin\phi}{AB\sin(\alpha-\phi)}=\frac{CN}{NA}. \tag{7.23}$$

Multiplying the equations (7.22) *and* (7.23), *we deduce that*

$$\frac{AB^2}{BC^2}=\frac{AM\cdot AN}{CM\cdot CN}. \tag{7.24}$$

On the other hand, from the condition $\angle ADM=\angle CDN$, *in a similar way we obtain that*

$$\frac{AD^2}{CD^2}=\frac{AM\cdot AN}{CM\cdot CN}. \tag{7.25}$$

From the equations (7.24) *and* (7.25), *it follows that*

$$AB \cdot CD = AD \cdot BC.$$

Therefore $AD = 50$.

Problem 9. Evaluate the expression

$$\frac{\cos\frac{2\pi}{3^{100}+1}}{\sin\frac{3\pi}{3^{100}+1}} + \frac{\cos\frac{2\cdot 3\pi}{3^{100}+1}}{\sin\frac{3^2\pi}{3^{100}+1}} + \cdots + \frac{\cos\frac{2\cdot 3^{99}\pi}{3^{100}+1}}{\sin\frac{3^{100}\pi}{3^{100}+1}}.$$

Solution. *Note that*

$$\frac{1}{\sin\alpha} - \frac{1}{\sin 3\alpha} = \frac{2\sin\alpha\cos 2\alpha}{\sin\alpha\sin 3\alpha} = \frac{2\cos 2\alpha}{\sin 3\alpha}.$$

Thus, it follows that

$$\frac{\cos 2\alpha}{\sin 3\alpha} = \frac{1}{2}\left(\frac{1}{\sin\alpha} - \frac{1}{\sin 3\alpha}\right). \tag{7.26}$$

From the equation (7.26), *it follows that*

$$\frac{\cos\frac{2\pi}{3^{100}+1}}{\sin\frac{3\pi}{3^{100}+1}} + \frac{\cos\frac{2\cdot 3\pi}{3^{100}+1}}{\sin\frac{3^2\pi}{3^{100}+1}} + \cdots + \frac{\cos\frac{2\cdot 3^{99}\pi}{3^{100}+1}}{\sin\frac{3^{100}\pi}{3^{100}+1}} = \frac{1}{2}\left(\frac{1}{\sin\frac{\pi}{3^{100}+1}} - \right.$$

$$\left. - \frac{1}{\sin\frac{3\pi}{3^{100}+1}} + \frac{1}{\sin\frac{3\pi}{3^{100}+1}} - \frac{1}{\sin\frac{3^2\pi}{3^{100}+1}} + \cdots + \frac{1}{\sin\frac{3^{99}\pi}{3^{100}+1}} - \frac{1}{\sin\frac{3^{100}\pi}{3^{100}+1}}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{\sin\frac{\pi}{3^{100}+1}} - \frac{1}{\sin\frac{3^{100}\pi}{3^{100}+1}}\right) = 0.$$

Problem 10. Given that a tetrahedron $ABCD$ does not have any points outside a cube with a side length of $4\sqrt[4]{3}$. Find the greatest possible value of the total surface area of tetrahedron $ABCD$.

Solution. *At first, let us prove the following lemmas.*

Lemma 7.6. *Let XYZ be a triangle, then it holds true*

$$(XYZ) \leq \frac{XY^2 + YZ^2 + ZX^2}{4\sqrt{3}}.$$

Proof. *Let $XY = a$, $YZ = b$, $XZ = c$ and $p = \dfrac{a+b+c}{2}$. According to Heron's formula, we have that*

$$(XYZ) = \sqrt{p(p-a)(p-b)(p-c)}.$$

On the other hand, according to the AM-GM inequality, it follows that

$$(p-a)(p-b)(p-c) \leq \left(\frac{p-a+p-b+p-c}{3}\right)^3 = \frac{p^3}{27}.$$

Therefore

$$(XYZ) \leq \frac{p^2}{3\sqrt{3}} = \frac{(a+b+c)^2}{4 \cdot 3\sqrt{3}} \leq \frac{a^2+b^2+c^2}{4\sqrt{3}} = \frac{XY^2 + YZ^2 + ZX^2}{4\sqrt{3}}.$$

Hence, we obtain that

$$(XYZ) \leq \frac{XY^2 + YZ^2 + ZX^2}{4\sqrt{3}}.$$

This ends the proof of the lemma.

Lemma 7.7. *If a triangle XYZ does not have any points outside a cube with a side length of $4\sqrt[4]{3}$, then*

$$XY^2 + YZ^2 + ZX^2 \leq 96\sqrt{3}.$$

Proof. *Let us choose rectangular coordinate axes, such that the vertices of a cube with a side length of $a = 4\sqrt[4]{3}$ are $(0,0,0)$, $(a,0,0)$, $(0,a,0)$, $(a,a,0)$, $(0,0,a)$, $(a,0,a)$, $(0,a,a)$, (a,a,a). Let $X(x_1,y_1,z_1)$, $Y(x_2,y_2,z_2)$, $Z(x_3,y_3,z_3)$. We have that*

$$XY^2 + YZ^2 + ZX^2 = (x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_1)^2 + (y_1-y_2)^2 + (y_2-y_3)^2 +$$

$$+(y_3-y_1)^2 + (z_1-z_2)^2 + (z_2-z_3)^2 + (z_3-z_1)^2 \leq 2(\max(x_1,x_2,x_3) - \min(x_1,x_2,x_3))^2 +$$

$$+2(\max(y_1,y_2,y_3) - \min(y_1,y_2,y_3))^2 + 2(\max(z_1,z_2,z_3) - \min(z_1,z_2,z_3))^2 \leq$$

$$\leq 2a^2 + 2a^2 + 2a^2 = 6a^2 = 96\sqrt{3}.$$

Thus, it follows that

$$XY^2 + YZ^2 + ZX^2 \leq 96\sqrt{3}.$$

This ends the proof of the lemma.

According to Lemma 7.6 and Lemma 7.7, it follows that

$$S=(ABC)+(ACD)+(BCD)+(ABD)\leq\frac{AB^2+BC^2+AC^2}{4\sqrt{3}}+$$

$$+\frac{AC^2+AD^2+CD^2}{4\sqrt{3}}+\frac{BC^2+CD^2+BD^2}{4\sqrt{3}}+\frac{AB^2+BD^2+AD^2}{4\sqrt{3}}\leq\frac{96\sqrt{3}}{4\sqrt{3}}\cdot 4=96,$$

where we denote by S the total surface area of the tetrahedron. Therefore $S\leq 96$.

Note that, for the points $A(0,0,0), B(a,a,0), C(a,0,a), D(0,a,a)$, we obtain that

$$S=4\cdot\frac{\sqrt{4}}{4}\cdot(\sqrt{2}a)^2=2\sqrt{3}a^2=96.$$

Hence, the greatest possible value of the total surface area of tetrahedron ABCD is equal to 96.

Problem 11. Let $ABCDE$ be a convex pentagon, such that $AB=AE$, $\angle ABC+\angle AED=180^\circ$ and $\angle ACB=\angle ACD$. Given that $BC=12$ and $DE=13$. Find CD.

Solution. *Let us choose a point F on the ray CB, such that $BF=13$. Note that $\angle ABF=180^\circ-\angle ABC=\angle AED$, $AB=AE$ and $BF=DE$. Thus, it follows that $\triangle ABF=\triangle AED$. Hence, we deduce that $AF=AD$ and $\angle AFB=\angle ADE$.*

Now, let us choose a point K on the ray CB, such that $CK=CD$.

Let us prove that points K and F coincide. Proof by contradiction argument. Assume that points K and F do not coincide, then as $\triangle ACD=\triangle ACK$, thus it follows that $\angle AKC=\angle ADC$ and $AK=AD$.

Thus, it follows that $AK=AD=AF$. Therefore $\angle AFK=\angle AKF$. Hence $\angle AFC+\angle AKC=180^\circ$. On the other hand, we have that

$$\angle AFC+\angle AKC=\angle ADE+\angle ADC<180^\circ.$$

This leads to a contradiction.

Therefore $CD=CK=CF=CB+BF=25$. Hence $CD=25$.

Problem 12. Given the points M, N, P, Q on the diagonal AC of a cyclic quadrilateral $ABCD$, such that $\angle ABM=\angle CBN$, $\angle ADM=\angle CDN$, $\angle ABP=\angle CBQ=100^\circ$ and $\angle ADP=30^\circ$. Let $\angle PBQ+\angle PDQ=n^\circ$. Find n.

Solution. *At first, let us prove the following lemma.*

Lemma 7.8. *If given the points X, Y on the diagonal AC of a cyclic quadrilateral $ABCD$, then*

$$\frac{AB^2\cdot CD^2}{BC^2\cdot AD^2}=\frac{\sin\angle CBX\cdot\sin\angle CBY}{\sin\angle ABX\cdot\sin\angle ABY}\cdot\frac{\sin\angle ADX\cdot\sin\angle ADY}{\sin\angle CDX\cdot\sin\angle CDY}. \tag{7.27}$$

Proof. *We have that*

$$\frac{AX}{CX} = \frac{(ABX)}{(CBX)} = \frac{AB \sin \angle ABX}{BC \sin \angle CBX}.$$

In a similar way, we obtain that

$$\frac{AY}{CY} = \frac{AB \sin \angle ABY}{BC \sin \angle CBY}.$$

Multiplying these equations, we deduce that

$$\frac{AB^2}{BC^2} = \frac{AX \cdot AY}{CX \cdot CY} \cdot \frac{\sin \angle CBX \cdot \sin \angle CBY}{\sin \angle ABX \cdot \sin \angle ABY}. \tag{7.28}$$

In a similar way, we obtain that

$$\frac{AD^2}{CD^2} = \frac{AX \cdot AY}{CX \cdot CY} \cdot \frac{\sin \angle CDX \cdot \sin \angle CDY}{\sin \angle ADX \cdot \sin \angle ADY}. \tag{7.29}$$

From the equations (7.28) *and* (7.29)*, it follows that the equation* (7.27) *holds true.*

This ends the proof of the lemma

According to the lemma, for the points M, N *that are on the diagonal* AC *of a cyclic quadrilateral* $ABCD$*, we have that* $AB \cdot CD = BC \cdot AD$.

On the other hand, for the points P, Q *that are on the diagonal* AC*, from the lemma, it follows that*

$$1 = \frac{\sin \angle ADP \cdot \sin \angle ADQ}{\sin \angle CDP \cdot \sin \angle CDQ}. \tag{7.30}$$

Let $\angle ADC = \alpha$, $\angle ADP = \beta$, $\angle CDQ = \gamma$. *According to* (7.30)*, it follows that*

$$\frac{\sin(\alpha - \beta)}{\sin \beta} = \frac{\sin(\alpha - \gamma)}{\sin \gamma},$$

$$\sin \alpha \cot \beta - \cos \alpha = \sin \alpha \cot \gamma - \cos \alpha.$$

Therefore $\beta = \gamma$. *Hence*

$$\angle PBQ + \angle PDQ = \angle ABP + \angle CBQ + \angle ADP + \angle CDQ - \angle ABC - \angle ADC =$$

$$= 200^\circ + 60^\circ - 180^\circ = 80^\circ.$$

We obtain that $n = 80$.

7.2 Number Theory

7.2.1 Problem Set 1

Problem 1. Given that p, q, p^2+q^3, p^3+q^2 are prime numbers, where p and q are positive integers. Find the value of the sum $p+q+p^2+q^2+p^3+q^3$.

Solution. *Note that if p and q are simultaneously odd numbers, then p^2+q^3 is an even number greater than 2. Thus p^2+q^3 is a composite number, which is a contradiction. Therefore, $p=2$ or $q=2$. If $p=2$, then $8+q^2$ is a prime number, hence q is divisible by 3. Thus $q=3$. If $q=2$, in a similar way we deduce that $p=3$. We obtain that $p+q+p^2+q^2+p^3+q^3=53$.*

Problem 2. A positive integer is called "interesting", if the sum of its digits is a square of a positive integer. What is the possible maximum number of consecutive "interesting" positive integers?

Solution. *Note that 9 and 10 are (consecutive) "interesting" numbers. Let us prove that it does not exist consecutive $n, n+1, n+2$ "interesting" numbers. We proceed by a contradiction argument. Denote by $S(m)$ the sum of all the digits of the number m. Assume that $S(n)=a^2, S(n+1)=b^2, S(n+2)=c^2$, where $a,b,c\in\mathbb{N}$. Note that $S(n+1)-S(n)=1$ or $S(n+2)-S(n+1)=1$. We obtain a contradiction, as none of the following equalities may hold true $b^2-a^2=1$ and $c^2-b^2=1$.*

Problem 3. What is the maximal length of a geometric progression consisting of distinct prime numbers?

Solution. *Let us prove that a geometrical progression cannot include three distinct prime numbers. We proceed by a contradiction argument. Assume prime numbers p, r, s are respectively the m, n, k-th terms ($m<n<k$) of a geometrical progression with a common ratio q. We have $r=pq^{n-m}$, $s=pq^{k-m}$, thus $r^{k-m}=s^{n-m}p^{k-n}$, which is impossible, as $s^{n-m}p^{k-n}$ is not divisible by r. We get a contradiction.*

Problem 4. Find the total number of all positive integers n, such that $n(n+1)(n+4)(n+5)+4$ is the fourth power of a positive integer.

Solution. *Note that $n(n+1)(n+4)(n+5)+4=(n^2+5n+2)^2$ and $(n+1)^4<(n^2+5n+2)^2<(n+3)^4$, thus $n(n+1)(n+4)(n+5)+4$ is fourth power of a positive integer. Therefore $(n^2+5n+2)^2=(n+2)^4$, hence $n=2$.*

Problem 5. Let x, y, z be positive integers, such that $\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=1$. Find the value of the expression $\frac{1}{x^2}+\frac{1}{y^2}-\frac{1}{z^2}$.

Solution. *Note that $x=1$ or $y=1$. We proceed by a contradiction argument. Assume $x>1$ and $y>1$, then $\frac{1}{x}+\frac{1}{y}\leq 1$, which is not possible. If $x=1$, then $y=z$, thus $\frac{1}{x^2}+\frac{1}{y^2}-\frac{1}{z^2}=1$. If $y=1$, then $x=z$, thus $\frac{1}{x^2}+\frac{1}{y^2}-\frac{1}{z^2}=1$.*

Problem 6. Let a,b,c,d be positive integers, such that $a! = b! + c! + d! + 1$ (for a positive integer n, we define $n! = 1 \cdot 2 \cdot \dots \cdot n$). Find the value of the sum of a,b,c,d.

Solution. *Note that if one of the numbers b,c,d is not equal to* 1*, then the left side of the equation is even number and the right side of the equation is odd number, which is impossible. Suppose $d = 1$, if one of the numbers b,c is not equal to* 1 *or* 2*, then the left side of the equation is divisible by* 3 *and the right side is not divisible by* 3*, which is impossible. Therefore, $b = c = 2$, $a = 3$.*

Problem 7. Find the sum of all the solutions of the equation $\{2\{3\{4x\}\}\} = x$, where $\{a\}$ is a fractional part of a.

Solution. *Note that, if $a \in \mathbb{Z}$, then $\{a\{b\}\} = \{a(b-[b])\} = \{ab\}$, thus the given equality and the following equality $\{24x\} = x$ are equivalent. The solutions of the last equality are $0, \frac{1}{23}, \frac{2}{23}, \dots, \frac{22}{23}$, hence the sum of all the solutions is equal to* 11.

Problem 8. Let n and k be positive integers, $n \neq k$, such that $n+1 \vdots k+1$, $n+2 \vdots k+2, \dots, n+11 \vdots k+11$. Find the possible minimum value of $n - 27000$.

Solution. *Note that $n-k = n+1-(k+1) \vdots k+1$. In a similar way, we have $n-k \vdots k+2, \dots, n-k \vdots k+11$. Obviously, one of the numbers $k+1, k+2, \dots, k+11$ is divisible by* 11*, thus $n-k \vdots 11$. In a similar way, we deduce that $n-k \vdots 9, n-k \vdots 8, n-k \vdots 7, n-k \vdots 5$, thus $n-k \vdots 11 \cdot 9 \cdot 8 \cdot 7 \cdot 5$. We obtain that $n \geq 11 \cdot 9 \cdot 8 \cdot 7 \cdot 5 + k \geq 11 \cdot 9 \cdot 8 \cdot 7 \cdot 5 + 1 = 27721$. Therefore $n \geq 27721$. Hence, the minimal value of $n - 27000$ is 721 (when we take $k = 1$).*

Problem 9. The sum of three positive integers is equal to 2014. Denote by L the possible maximum value of the least common multiple of that numbers. Find the value of $\frac{L}{449570}$.

Solution. *Let $a+b+c = 2014$, $a,b,c \in \mathbb{N}$.*

We consider two different cases.

a) If a, b or c are equal. For example $a = b$, then

$$[a,b,c] = [a,c] \leq ac \leq \frac{1}{2}\left(\frac{2a+c}{2}\right)^2 = 507024,5.$$

b) If $a \neq b \neq c \neq a$, then

If a,b,c are even numbers, then

$$[a,b,c] = 2\left[\frac{a}{2},\frac{b}{2},\frac{c}{2}\right] \leq 2 \cdot \frac{abc}{8} \leq \frac{1}{4}\left(\frac{a+b+c}{3}\right)^3 = \frac{2}{27} \cdot 1007^3.$$

If two of the numbers a,b,c are odd and the other one is even, then we consider the following problem. Find the possible maximum value of abc, if their sum is equal

to 2014. *Without loss of generality, we may suppose that* a,b *are odd numbers and* $a<b$. *Let us prove that* $b-a=2$.

We proceed by a contradiction argument. Suppose $b-a>2$, *then*

$$abc<\Big(\frac{a+b}{2}-1\Big)\Big(\frac{a+b}{2}+1\Big)c,$$

which is impossible.

Let $a\neq b\neq c\neq a$ *are such that either* a *is even and* c *is odd, or* c *is even and* a *is odd and*

$$b\notin\Big\{\frac{a+c-1}{2},\frac{a+c+1}{2}\Big\},$$

and the sum of a,b,c *is equal to 2014 and* abc *has the possible maximum value.*

Let us prove that $|c-a|=1$. *We proceed by a contradiction argument. Assume* $|c-a|>1$, *then* $abc<\frac{a+c-1}{2}\cdot\frac{a+c+1}{2}b$, *which is impossible. We obtain a contradiction.*

Thus, the possible maximum value of abc *is* $670\cdot671\cdot673$. *Therefore, when two of* a,b,c *are odd and the other one is even, then* $[a,b,c]\leq abc\leq 670\cdot671\cdot673$.

Hence, $[a,b,c]\leq\max(670\cdot671\cdot673;507024,5;\frac{2}{27}\cdot1007^3)=670\cdot671\cdot673$ *and when* $a=670,b=671,c=673$, *then* $[a,b,c]=670\cdot671\cdot673$. *Therefore, the possible maximum value of the least common multiple of* a,b,c *is equal to* $670\cdot671\cdot673$.

7.2.2 Problem Set 2

Problem 1. The entries of 3×3 table are integers from 1 to 9. Consider the row and column sums of numbers in the table. How many of those six sums at maximum can be prime numbers?

Solution. *All the six numbers can be primes, for example*

2	1	8
9	3	5
6	7	4

Problem 2. Let x,y,z be positive integers, such that $\sqrt{x+2\sqrt{2015}}=\sqrt{y}+\sqrt{z}$. Find the possible smallest value of x.

Solution. *From the given equation, it follows that*

$$x-y-z=2(\sqrt{yz}-\sqrt{2015}).$$

Hence

$$8\sqrt{2015yz} = 4(yz + 2015) - (x - y - z)^2.$$

Therefore, $yz = 2015k^2, k \in \mathbb{N}$ *and* $x - y - z = 2\sqrt{2015}(k-1)$. *Thus* $x = y + z, k = 1$. *Hence, the smallest possible value of* x *is* $5 \cdot 13 + 31 = 96$.

Problem 3. Given that infinite arithmetic progression with a positive common difference includes finitely many prime numbers (at least one term). Find the number of those primes.

Solution. *Let the arithmetic progression* $an + b$ *has more than one term, which are primes. We have* $a \in \mathbb{N}$, $a + b \in \mathbb{N}, am + b = p, ak + b = q$, *where* $m, k \in \mathbb{N}$ *and* p, q *are prime numbers* ($p \neq q$). *If* $(a, b) = d$, *then* $d \mid q$ *thus* $d \mid q$, *thus* $d = 1$. *By Dirichlet's theorem, that arithmetic progression has infinitely many prime terms, which leads to a contradiction. Thus, the number of such prime numbers is* 1. *For example* $5, 10, 15, 20, \ldots$.

Problem 4. Given that the sum of the squares of four positive integers is equal to 9×2^{2015}. Find the ratio of the greatest and smallest numbers among them.

Solution. *Note that the square of an odd number is divisible by 8 with the remainder equal to 1. Thus, we obtain that we can represent that four positive integers in the following way* $2^{1007}a_i$, *where* $i = 1, 2, 3, 4$, $a_i \in \mathbb{N}$ *and* $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 18$. *Therefore, two numbers among* a_1, a_2, a_3, a_4 *are equal to* 2, *and the others are equal to* 1 *and* 3. *Hence, the ratio of the greatest and smallest numbers among those numbers is equal to 3.*

Problem 5. Find the number of all positive integers n, such that for each of them $n(n+1)(n+2)$ is a product of six consecutive positive integers.

Solution. *Let* m *and* n *be positive integers, such that* $n(n+1)(n+2) = m(m+1)(m+2)(m+3)(m+4)(m+5)$. *We have that* $n(n+1)(n+2) = a(a+4)(a+6)$, *where* $a = m^2 + 5m$. *Thus* $n \geq a + 1$ *and* $n + 2 \leq a + 5$*; hence* $n \in \{a+1, a+2, a+3\}$. *Therefore* $a = 6, m = 1, n = 8$.

Problem 6. Given that p, q, r are prime numbers, such that $p(p-5) + q(q-5) = r(r+5)$. Find the value of the product pqr.

Solution. *If none of the numbers* p, q, r *is divisible by* 5, *then the left side is divisible by 5 with the remainder equal to 2 or 3 and the right side with the remainder 1 or 4, which leads to a contradiction.*

If $p = 5$, *then* $q - r = 5$. *Hence* $r = 2$, $q = 7$.

If $q = 5$, *then we obtain that* $r = 2, p = 7$.

If $r = 5$, *then* $p(p-5) + q(q-5) = 50$. *Note that the last equation has no solutions in the set of prime numbers.*

Problem 7. Let x, y be rational numbers. Find the number of (x, y) couples, such that $x + \dfrac{2}{y}$ and $y + \dfrac{2}{x}$ are positive integers.

Solution. *Let* $x+\dfrac{2}{y}=m$ *and* $y+\dfrac{2}{x}=n$*, where* $m,n\in\mathbb{N}$*. Hence, we have that*

$$\left(x+\frac{2}{y}\right)\left(y+\frac{2}{x}\right)=mn,$$

and $x,y>0$*. Therefore, the equation* $(xy)^2-(mn-4)xy+4=0$ *has a rational solution. Thus* $D=(mn-4)^2-16=k^2$*, where* k *is a non-negative integer. We deduce that* $(mn-k-4)(mn+k-4)=16$*, hence* $mn=8$ *or* $mn=9$*.*

If $mn=8$*, then we obtain the following couples* $\left(\frac{1}{2},4\right)$*,* $\left(4,\frac{1}{2}\right)$*,* $(1,2),(2,1)$*.*

If $mn=9$*, then we obtain the following couples* $\left(\frac{1}{3},3\right)$*,* $\left(3,\frac{1}{3}\right)$*,* $\left(\frac{2}{3},6\right)$*,* $\left(6,\frac{2}{3}\right)$*,* $(1,1)$*,* $(2,2)$*.*

Problem 8. Let a and b be positive integers. Find the number of all positive integers, such that any of them is not possible to represent in the unique way in the following form $\dfrac{a^2+b}{ab^2+1}$.

Solution. *Let* $\dfrac{a^2+b}{ab^2+1}=k$*,* $k\in\mathbb{N}$*, thus* $a(a-kb^2)=b-k$*. Consider the following cases.*

If $k=b$*, then* $a=b^3$*.*

If $k>b$*, then* $a>kb^2$*, thus* $a(a-kb^2)\geq a>kb^2>b-k$*, which leads to a contradiction.*

If $k<b$*, then* $a(kb^2-a)=b-k$*, thus* $b-k\geq 1\cdot(kb^2-1)$*. Therefore* $k=b=1$*, which leads to a contradiction.*

Thus, the equation $\dfrac{a^2+b}{ab^2+1}=k$*, for any* $k\in\mathbb{N}$*, has the unique solution* (k^3,k)*.*

Problem 9. Given that there exist n distinct positive integers, such that we cannot choose four numbers a,b,c,d among them in a way that $ab-cd$ is divisible by n. Find the greatest positive integer n satisfying this condition.

Solution. *If* $n=6$*, we give the following example of such numbers* $1,2,5,6,7,13$*.*

If $n\geq 7$*, let us prove that there does not exist such* n *numbers.*

Given positive integers $a_1,a_2,\dots,a_n$ *which are divisible by* n *with the remainders* $r_1,r_2,\dots,r_n$*. Without loss of generality, we may assume that* $r_1\leq r_2\leq\cdots\leq r_n$*. Note that, if* $r_i=r_{i+1},r_j=r_{j+1}$*,* $(j>i+1)$*, then* $a_{i+1}a_{j+1}-a_ia_j$ *is divisible by* n*. Therefore, we need to make the proof for the case when at least* $n-2$ *of the remainders* $r_1,r_2,\dots,r_n$ *are pairwise different.*

If $n\geq 9$*, then among the couples* $(1,n-1),(2,n-2),(3,n-3),(4,n-4)$ *there exist couples* $(i,n-i)$ *and* $(j,n-j)$*, such that* $1\leq i<j\leq 4$ *and the numbers* i,j*,* $n-i,n-j$ *are the terms of the sequence* $r_1,r_2,\dots,r_n$*. Let* a,b,c,d *are some terms of the sequence* $a_1,a_2,\dots,a_n$ *and are divisible by* n *with the remainders* $n-i,n-j,i,j$*, then* $n\mid ab-cd$*.*

If $n=8$, we need to make the proof for the case when at least $n-2$ of the remainders $r_1,r_2,\dots,r_n$ are pairwise different and among the couple $(1,7),(2,6),(3,5)$ there are no couples $(i,n-i)$, $(j,n-j)$, $1\le i<j\le 3$, such that the numbers $i,j,n-j,n-j$ are the terms of the sequence $r_1,r_2,\dots,r_n$. Therefore, $0,4$ and at least one of the numbers $2,6$ are the terms of the sequence $r_1,r_2,\dots,r_n$. Hence, the statement is proved for this case.

If n=7, it is left to prove the statement, when five among the numbers $a_1,a_2,\dots,a_n$ are divisible by 7 with the nonzero remainders, such that four of those remainders are different. Assume those numbers are b_1,b_2,b_3,b_4,b_5, such that the last four numbers are divisible by 7 with non-equal remainders and b_1,b_2 are divisible by 7 with the same remainder. Consider the following numbers b_1b_2, b_1b_3, b_1b_4, b_1b_5, b_3b_4, b_3b_5, b_4b_5. Two among those numbers are divisible by 7 with the same remainders; therefore, their subtraction is divisible by 7. This ends the solution.

7.2.3 Problem Set 3

Problem 1. Given that $16y(x^2+1)=25x(y^2+1)$, where x,y are positive integers. Find the possible minimum value of xy.

Solution. *We have that $16\mid x(y^2+1)$ and $4\nmid y^2+1$, thus $8\mid x$. Therefore $x\ge 8$.*
If $x=8$, then $5y^2-26y+5=0$, hence $y=5$.
If $x>8$, then $x+\dfrac{1}{x}>16$. Hence $25(y^2+1)=16y(x+\dfrac{1}{x})>256y$.
We obtain that $y>5$, then $xy>40$.
Therefore, the possible minimum value of xy is equal to 40.

Problem 2. Let x,y,z,t be positive integers. Given that $68(xyzt+xy+zt+xt+1)=157(yzt+y+t)$. Find the value of the product $xyzt$.

Solution. *The given equation can be rewritten in the following form*

$$x+\cfrac{1}{y+\cfrac{1}{z+\cfrac{1}{t}}}=2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5}}}.$$

Therefore $x=2,y=3,z=4,t=5$. Hence $xyzt=120$.

Problem 3. The entries of 3×3 table are integers from 1 to 9. Consider the row, column and diagonal sums of numbers in the table. At most, how many of those eight sums can be prime numbers?

Solution. *Let us at first consider the following problem.*

The entries of 3×3 table are integers, such that three of those nine numbers are divisible by 3, three of them are divisible by 3 with a remainder of 1 and three of

them are divisible by 3 *with a remainder of* 2. *Let us prove that there exists a row, a column or a diagonal such that the sum of its three numbers is divisible by* 3.

Without loss of generality, one can assume that in the central square of 3×3 *table is written number* 0. *Consider four rectangles which include the central square. Now, let us consider two of those rectangles, which do not include a number divisible by* 3. *In two squares of any of these two rectangles are written numbers, which are divisible by* 3 *with a same remainder (otherwise, the statement is proved). By this argument, we obtain the proof of the statement.*

Therefore, we deduce that at most seven among these eight numbers can be prime numbers.

We give the following example, where seven of those eight sums are prime numbers.

2	1	8
9	3	5
6	7	4

Problem 4. Let n be a positive integer and p be a prime number. Find the number of solutions of the following equation $n^5 - p = 5p^2 - n^2$.

Solution. *We have that* $n^2(n^3+1) = p(5p+1)$, *thus either* $p \mid n$ *or* $p \mid n+1$ *or* $p \mid n^2-n+1$. *Hence, if* $n > 1$, *we deduce that* $n^2-n+1 \geq p$. *Therefore* $n^2 > p$. *Thus* $p(5p+1) > p(np+1)$. *We obtain that* $n \in \{2,3,4\}$. *By a simple verification, one may deduce that the unique solution of given equation is* $n=3$ *and* $p=7$.

Problem 5. Let a,b be positive integers. Given that b is an odd number. Find the number of solutions (in the set of integer numbers) of the following equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{x-y} = \frac{a}{b}.$$

Solution. *The given equation is equivalent to the following equation*

$$q(x^2 - y^2 + xy) = pxy(x-y). \tag{7.31}$$

Note that, if at least one of the numbers x,y *is odd, then the left-hand side of the equation* (7.31) *is odd and the right-hand side is even. Hence, the given equation has no solutions. On the other hand, if* x *and* y *are even numbers, then* (7.31) *is equivalent to*

$$\frac{1}{x_1} + \frac{1}{y_1} + \frac{1}{x_1-y_1} = \frac{p_1}{q},$$

where $x = 2x_1, y = 2y_1, p_1 = 2p$. *Therefore, in a similar way we deduce that the given equation has no solutions.*

Problem 6. Consider three pairwise distinct positive integers. Given that the sum of the greatest common divisor and the least common multiple of any two numbers

among those three numbers is divisible by the third one. Denote by D the ratio of the greatest number to the smallest one. Find the possible values of D.

Solution. *Let a,b,c satisfy the conditions of the given problem. Without loss of generality, one can assume that a,b,c are relatively prime numbers. (If, for example, a and b are not relatively prime numbers, then a and b have a common positive factor. Let p be a common prime factor for a and b. Thus $(a,c)+[a,c]$ is divisible by b, then (a,c) is divisible by p. Hence, c is divisible by p. We have that $\left(\frac{a}{p},\frac{b}{p}\right)=\frac{(a,b)}{p}$ and $\left[\frac{a}{p},\frac{b}{p}\right]=\frac{[a,b]}{p}$. Therefore, we deduce that $\frac{a}{p},\frac{b}{p},\frac{c}{p}$ satisfy the conditions of the given problem. Hence, after several similar steps, we will obtain relatively prime numbers, which satisfy the conditions of the given problem.)*

Thus, a,b,c are relatively prime numbers. We have that $1+[b,c]$, $1+[a,c]$, $1+[a,b]$ are divisible by a, b, c, respectively. Hence, $1+[b,c]+[a,c]+[a,b]$ is divisible by a, b and c; therefore $1+[b,c]+[a,c]+[a,b]$ is divisible by abc. We obtain that $1+[b,c]+[a,c]+[a,b]\geq abc$. Hence, according to the property $xy\geq[x,y]$, we deduce that

$$1+bc+ac+ab\geq abc. \tag{7.32}$$

Without loss of generality, one can assume that $a>b>c$. Therefore, by (7.90) we obtain that $1+3ab>abc$. Hence $1>ab(c-3)$, thus $c\in\{1,2,3\}$.

If $c=1$, then according to the conditions of the problem we have that $1+b$ is divisible by a, thus $1+b=a$. We also have that $1+a$ is divisible by b, therefore $b=2$ and $a=3$. Obviously, the obtained triple satisfies the conditions of the problem.

If $c=2$, then according to the conditions of the problem we have that $1+2b$ is divisible by a, thus $1+2b=a$. We also have that $1+2a$ is divisible by b, therefore $b=3$ and $a=7$. Obviously, the obtained triple satisfies the conditions of the problem.

If $c=3$, then according to the conditions of the problem we have that $1+3b$ is divisible by a, thus $1+3b=a$ or $1+3b=2a$. We also have that $1+3a$ is divisible by b, therefore $b=4$ and $a=13$ or $b=5, a=8$. By simple verification, we obtain that none of these triple satisfies the conditions of the problem.

Therefore $D=5$.

Problem 7. Given that $xy^2-(3x^2-4x+1)y+x^3-2x^2+x=0$, where x,y are integer numbers. Find the possible maximum value of the sum $|x|+|y|$.

Solution. *If $x=0$, then $y=0$.*

If $x\neq 0$, then we have that $y=xy^2-x(3x-4)y+x^3-2x^2+x$. Therefore, we deduce that $y=xz$, where $z\in\mathbb{Z}$. We obtain that

$$x^2z^2-(3x^2-4x+1)z+(x-1)^2=0.$$

Hence

$$z=\frac{3x-1\pm\sqrt{(3x-1)^2-4x^2}}{x^2}\cdot(x-1).$$

If $x = 1$, *then* $y = 0$.

If $x(x-1) \neq 0$, *we have that* $x^2 \mid 3x - 1 + \sqrt{(x-1)(5x-1)}$ *or* $x^2 \mid 3x - 1 - \sqrt{(x-1)(5x-1)}$.

If $x \geq 2$, *then* $3x - 1 + \sqrt{(x-1)(5x-1)} \geq x^2$. *Thus* $x^2 \leq 3x - 1 + \frac{x-1+5x-1}{2} = 6x - 2$. *Hence, we obtain that* $x = 2$, $x = 3$ *or* $x = 4$.

If $x = 2$, *then* $y = 2$.

If $x = 3$, *then* y *is not an integer number.*

If $x = 4$, *then* y *is not an integer number.*

If $x \leq -1$, *then we have that* $-3x + 1 + \frac{1-x+1-5x}{2} \geq x^2$. *Thus* $x = -1$, $x = -2$, $x = -3$, $x = -4$, $x = -5$ *or* $x = -6$.

Therefore, we obtain the solution $(-3, -12)$. *Thus, the greatest value of the sum* $|x| + |y|$ *is equal to 15.*

Problem 8. Let n be a positive integer, which is a multiple of 2015. Given that any divisor of n (except itself) is equal to the difference of any two other divisors of n. Find the sum of the digits of the smallest value of n.

Solution. *Note that if* d *is a divisor of a positive integer* n, *then* $\frac{n}{d}$ *is a divisor of* n *too. Given that* $2015 \mid n$, *thus* $13 \mid n$ *and* $31 \mid n$. *We have that* $\frac{n}{31} = \frac{n}{a} - \frac{n}{b}$, *where* $a, b \in \mathbb{N}$ *and* $a \mid n$, $b \mid n$. *We have that* $(31-a)(31+b) = 31^2$, *hence* $a = 30$. *Thus* $30 \mid n$. *In a similar way, we deduce that* $\frac{n}{13} = \frac{n}{c} - \frac{n}{d}$, *therefore* $c = 12$. *Thus* $12 \mid n$. *We obtain that* $2^2 \cdot 3 \cdot 5 \cdot 13 \cdot 31 \mid n$, *hence* $n \geq 2^2 \cdot 3 \cdot 5 \cdot 13 \cdot 31 = 24180$.

Let us show that every divisor m of 24180, which is smaller than 24180, can be written as a difference of two divisors of 24180.

If $4 \nmid m$, *then* $m = 2m - m$.

If $4 \mid m$, *then* $5 \nmid m$. *Thus* $m = \frac{5m}{4} - \frac{m}{4}$.

If $20 \mid m$, *then* $3 \nmid m$. *Thus* $m = \frac{3m}{2} - \frac{m}{2}$.

If $60 \mid m$, *then* $13 \nmid m$. *Thus* $m = \frac{13m}{12} - \frac{m}{12}$.

If $60 \cdot 13 \mid m$, *then* $31 \nmid m$. *Thus* $m = \frac{31m}{30} - \frac{m}{30}$.

Therefore, the smallest value of n is equal to 24180. Thus, it follows that the sum of the digits of the smallest value of n is equal to $2+4+1+8+0 = 15$.

Problem 9. Let m, n be positive integers, such that $m < 301$ and $n < 301$. Find the number of (m,n)-couples, such as $\frac{m^3+1}{mn+1}$ is a positive integer.

Solution. *Let* $\frac{m^3+1}{mn+1} = k$, *where* $k \in \mathbb{N}$. *Consider the following two cases.*

Case 1. If $m < n$, *then* $k = \frac{m^3+1}{mn+1} < m$ *and* $k = m^3 - kmn + 1$. *Therefore,* k *is divisible by* m *with a remainder of 1. Hence* $k = 1, n = m^2$.

Case 2. If $m \geq n$*, then we have that* $kn^3 = \dfrac{m^3n^3 + n^3}{mn+1} = m^2n^2 - mn + 1 + \dfrac{n^3-1}{mn+1}$.
Therefore $\dfrac{n^3-1}{mn+1} \in \mathbb{Z}$.

If $n = 1$*, then we can choose* m *to be any positive integer.*

If $n > 1$*, then we have that* $l = \dfrac{n^3-1}{mn+1} \in \mathbb{N}$ *and* $l < n$*. Hence* $l = n^3 - lmn - 1$*, thus* l *is divisible by* n *with a remainder of* $n-1$*. Therefore* $l = n-1$*,* $m = n+1$*.*

Thus, the solutions of the given equation in the set of positive integers have the following forms:

(a, a^2)*, where* $a = 1, 2, \ldots$

$(a, 1)$*, where* $a = 2, 3, \ldots$

$(a, a-1)$*, where* $a = 3, 4, \ldots$

On the other hand, the number of the solutions smaller than 301 *is* $17 + 299 + 298 = 614$.

7.2.4 Problem Set 4

Problem 1. Consider k consequent positive integers, such that their sum is equal to 2015. Find the greatest value of k.

Solution. *Let the sum of* $n+1, n+2, \ldots, n+k$ *is equal to 2015. We have that*

$$k \cdot \frac{2n+k+1}{2} = 2015.$$

Note that either k *or* $\dfrac{k}{2}$ *is a divisor of* 2015 *and* $2015 \geq \dfrac{k(k+3)}{2}$*, thus* $k \leq 62$*. If* $k = 62, n = 1$*, then the greatest possible value of* k *is equal to 62.*

Problem 2. Let a and b be positive divisors of 720. Given that $720^2(b-a) = a^2(721+a)$. Find the product of a and b.

Solution. *Note that* $b - a > 0$*, then* $b - a \geq 1$*. Let* $720 = ak$*, where* $k \in \mathbb{N}$*. Thus* $k^2(b-a) = ka + a + 1$*, then* $(b, a) = 1$*. We have that* $b \mid ak$*, hence* $k = bm, m \in \mathbb{N}$*.*

Therefore $bm(bm(b-a)-a) = a+1$*. On the other hand,* $m \geq 1$ *and* $b \geq a+1$*, thus* $bm(bm(b-a)-a) \geq bm(bm-a) \geq (a+1)(a+1-a) = a+1$*. Hence* $m = 1$ *and* $b = a+1$*, therefore* $720 = ak = abm = ab$*.*

Problem 3. Find the number of integer solutions of the following equation

$$(x^2+y^2+z^2)(x^4+y^4+z^4+x^3y+y^3z+z^3x-xy^3-yz^3-zx^3) = 201520152015.$$

Solution. *We have that* $x^4+y^4+z^4+x^3y+y^3z+z^3x-xy^3-yz^3-zx^3-(x^2+y^2+z^2) = x^2(x^2-1)+y^2(y^2-1)+z^2(z^2-1)+xy(x^2-y^2)+yz(y^2-z^2)+zx(z^2-x^2)$.

Note that if a and b are positive integers, then $3 \mid a(a-1)a(a+1) = a^2(a^2-1)$ *and* $3 \mid ab(a^2-b^2)$.

Therefore, the difference of the following expressions $x^4+y^4+z^4+x^3y+y^3z+z^3x-xy^3-yz^3-zx^3$ *and* $x^2+y^2+z^2$ *is divisible by 3. From the equality, we have that their product is divisible by 3 too; therefore both expressions are divisible by 3. This is not possible, as 201520152015 is not divisible by 9. Hence, given equation has no solutions in the set of positive integers.*

Problem 4. A positive integer is called "interesting", if we can choose some of its digits and write a new number with those digits which is divisible by 7, otherwise it is called "uninteresting" number. Find the ratio of the greatest "uninteresting" number to 1001.

Solution. *Let us prove that any seven-digit number is "interesting". We proceed by a contradiction argument, assume that* $\overline{a_7a_6\dots a_1}$ *is a "uninteresting" number. Consider the numbers* a_1, $\overline{a_2a_1}$, $\overline{a_3a_2a_1}$, $\dots,\overline{a_7a_6\dots a_1}$. *Note that either any two among those numbers are divisible by 7 with the same remainder or* $\overline{a_7a_6\dots a_1}$ *is divisible by 7. In the first case, the difference of those two numbers is divisible by 7. Hence, we obtain that* $\overline{a_7a_6\dots a_1}$ *is an "interesting" number. This leads to a contradiction. In the second case, when* $\overline{a_7a_6\dots a_1}$ *is divisible by 7. Consider the following number* $\overline{a_6\dots a_1a_7}$. *Note that* $\overline{a_6\dots a_1a_7}$ *is a "uninteresting" number. Hence, we obtain that* $\overline{a_6\dots a_1a_7}$ *is divisible by 7. Therefore* $\overline{a_7a_6\dots a_1} - \overline{a_6\dots a_1a_7} = (10^6-1)a_7 - 9\cdot\overline{a_6\dots a_1}$ *is divisible by 7. Hence,* $\overline{a_6\dots a_1}$ *is divisible by 7, thus* $\overline{a_7a_6\dots a_1}$ *is an "interesting" number. Therefore, the greatest "uninteresting" number is 999999 and* $\dfrac{999999}{1001} = 999$.

Problem 5. Find the number of all positive integers n, such that for any n the number 2^n+5^n-65 is a square of some positive integer.

Solution. *Let n be an odd number, then* 2^n+5^n-65 *is divisible by* 5 *with remainder* 2 *or* 3. *Therefore, this number is not a square of a positive integer.*

If $n=2k, k\in\mathbb{N}$ *and* $k\geq 4$, *we have that* $25^k < 4^k+25^k-65 < (5^k+1)^2$, *thus* 2^n+5^n-65 *is not a square of a positive integer.*

By a technical, but straightforward verification, we obtain that only for the case $n=4$ *(from the possible values* $2,4,6$*) we have that* 2^n+5^n-65 *is a square of a positive integer.*

Problem 6. Let $p_1 < p_2 < \dots < p_{11}$ be prime numbers. Given that $210 \mid p_1^{12} + p_2^{12} + \dots + p_{11}^{12}$. Find $p_1p_2p_3p_4$.

Solution. *By Fermat's little theorem, if* $(a,p)=1$ *and* $p\in\{2,3,5,7\}$, *then* $p \mid a^{p-1}-1$. *Thus* $p \mid a^{12}-1$. *Hence* $p_1=2, p_2=3, p_3=5, p_4=7$ *and* $p_1p_2p_3p_4 = 210$.

Problem 7. Let (u_n) be a sequence of positive integers, such that $u_{n+1} = u_n^3 + 2015$, $n=1,2,\dots$ Let k be the number of terms, which are squares of some positive integers. Find the greatest value of k.

Solution. *Let u_n is divisible by 4 with a remainder of v_n. Consider sequence (v_n) and note that $4a+v_{n+1}=(4b+v_n)^3+2015$, where $a,b\in\mathbb{Z}$. Therefore $4\mid v_{n+1}-v_n^3-2015$.*

If $u_n=c^2$, where $c\in\mathbb{N}$, then $v_n=0$ or $v_n=1$.

In this case, when $v_n=0$, then $v_{n+1}=3$, $v_{n+2}=2$, $v_{n+3}=3,\ldots$

In the case, when $v_n=1$, then $v_{n+1}=0$, $v_{n+2}=3$, $v_{n+3}=2$, $v_{n+4}=3,\ldots$

Therefore, at most two terms of the sequence (u_n) can be a square of a positive integer. Let us prove that simultaneously two cannot be. We proceed by a contradiction argument. Assume that $u_n=m^2$ and $u_{n+1}=k^2$, where $m,k\in\mathbb{N}$. Then $(k-m^3)(k+m^3)=2015$, note that this equation has no solution in the set of positive integers. Therefore, only one term of the sequence (u_n) can be a square of a positive integer, for example (u_1).

Problem 8. Find the number of all positive integers n, such that for any of them $2^{2^{n^{2015}}-1}-1$ is divisible by n.

Solution. *If $n=1$, we have that $n\mid 2^{2^{n^{2015}}-1}-1$.*

If $n>1$, let us prove that $n\nmid 2^{2^{n^{2015}}-1}-1$.

Let $n>1$ and $n\mid 2^{2^{n^{2015}}-1}-1$. Let p be the least prime divisor of n. We have that $p\mid 2^{2^{n^{2015}}-1}-1$ and $p\mid 2^{p-1}-1$, thus $p\mid 2^{(2^{n^{2015}}-1,p-1)}-1$. Therefore $(2^{n^{2015}}-1,p-1)>1$.

Let q be a prime divisor of $(2^{n^{2015}}-1,p-1)$, then $q\mid p-1$, hence $p-1\geq q$. On the other hand, $q\mid 2^{n^{2015}}-1$ and $q\mid 2^{q-1}-1$, thus $q\mid 2^{(n^{2015},q-1)}-1$.

Therefore $(n^{2015},q-1)>1$, we obtain that n has a prime divisor not greater than $q-1$. This leads to a contradiction, as $p-2\geq q-1$.

Problem 9. Let m and n be relatively prime numbers, such that $m+n=1000$. Consider the couples (a,b) of positive integers a and b, such that for any of them $m\mid a+nb$ and $n\mid a+mb$. Denote by $f(s)=\dfrac{mn}{m+n}s-(m+n-1)\left\{\dfrac{m^2s}{m+n}\right\}$, where $\{x\}$ stands for the fractional part of a real number x. Find the difference of $\min(a+b)$ and $\min(f(1),f(2),\ldots,f(m+n))$.

Solution. *Without loss of generality, one can assume that $1<m<n$. We have that $a+mb=nk$ and $a+nb=ml$, where $k,l\in\mathbb{N}$ and $k<l$. Therefore, we deduce that $(n-m)b=m(l-k)-(n-m)k$, then $b=\dfrac{m}{n-m}(l-k)-k$ and $(m,n-m)=1$, thus $l-k=(n-m)s$, $s\in\mathbb{N}$.*

We obtain that $b=ms-k$, $a=(m+n)k-m^2s$ and $\dfrac{m^2s}{m+n}<k<ms$. Note that $ms-\dfrac{m^2s}{m+n}=\dfrac{mns}{m+n}\geq\dfrac{2ns}{m+n}>s$, thus $a+b=(m+n-1)k-(m^2-m)s$. Therefore, for the fixed s the expression $(m+n-1)k-(m^2-m)s$ will accept its minimal value for $k=\left[\dfrac{m^2s}{m+n}\right]+1$.

Hence, we need to find the minimal value of the following expression (for a positive integer s) $a+b=(m+n-1)k-(m^2-m)s=m+n-1+(m+n-1)\left[\frac{m^2s}{m+n}\right]-(m^2-m)s$.

Note that

$$m+n-1+(m+n-1)\left[\frac{m^2s}{m+n}\right]-(m^2-m)s=m+n-1+\frac{mn}{m+n}s-(m+n-1)\left\{\frac{m^2s}{m+n}\right\}.$$

For $s>m+n$, *we have that*

$$a+b=m+n-1+\frac{mn}{m+n}s-(m+n-1)\left\{\frac{m^2s}{m+n}\right\}>$$

$$>m+n-1+\frac{mn}{m+n}(s-m-n)-(m+n-1)\left\{\frac{m^2(s-m-n)}{m+n}\right\}.$$

Therefore, $\min(a+b)$ *is equal to* $m+n-1+\min(f(1),f(2),\dots,f(m+n))$, *where* $f(s)=\frac{mn}{m+n}s-(m+n-1)\frac{m^2s}{m+n}$. *Hence, we deduce that the value of difference we are looking for is equal to 999.*

7.2.5 Problem Set 5

Problem 1. Find the smallest possible value of a positive integer n, such that at least one of the digits of the number $25n+1$ is equal to 8 or 9.

Solution. *We have that* $25n+1=\frac{n}{4}\cdot 100+1$ *and* $\frac{1}{4}=0.25$, $\frac{2}{4}=0.5$ *and* $\frac{3}{4}=0.75$, *thus any of the numbers* $\frac{n}{4}\cdot 100$ *(where* $n\in\{1,2,\dots,31\}$*) does not have a digit greater than 7 and ends with 0 or 5. Therefore, any of the numbers* $25n+1$ *does not have a digit greater than 7.*

If $n=32$, *we have that* $25n+1=801$. *Hence, the smallest positive integer value of n is equal to 32.*

Problem 2. How many zeros does $(1^2+1)(2^2+1)\cdot\dots\cdot(100^2+1)$ end with?

Solution. *Note that the numbers* 2^2+1, 4^2+1, $\dots$, 100^2+1 *are odd numbers. On the other hand,* $V_2((2k-1)^2+1)=V_2(4k^2-4k+2)=1$, $k=1,\dots,50$. *Hence,* $V_2((1^2+1)(2^2+1)\cdot\dots\cdot(100^2+1))=50$. *We have that* $V_5((10k+r)^2+1)=0$, $k\in\mathbb{Z}$, $r\in\{0,1,4,5,6,9\}$.

$$V_5((10k+2)^2+1)=1,\ \ k=0,1,2,4,5,6,7,9$$

and

$$V_5(32^2+1)=2,\;\; V_5(82^2+1)=2.$$

$$V_5((10k+3)^2+1)=1,\;\; k=0,1,2,3,5,6,7,8$$

$$V_5(43^2+1)=2,\;\; V_5(93^2+1)=2.$$

$$V_5((10k+7)^2+1)=1,\;\; k=1,2,3,4,6,7,8,9$$

$$V_5(7^2+1)=2,\;\; V_5(57^2+1)=3.$$

$$V_5((10k+8)^2+1)=1,\;\; k=0,2,3,4,5,7,8,9.$$

$$V_5(18^2+1)=2,\;\; V_5(68^2+1)=3.$$

Therefore,

$$V_5((1^2+1)(2^2+1)\cdot\dots\cdot(100^2+1))=12+12+13+13=50.$$

Hence, we obtain that the given product ends with fifty zeros.

Problem 3. Let x,y be positive integers. Given that $y\leq 2015$ and

$$(x^2+x+1)(x^2+3x+3)=y^2+y+1.$$

Find the greatest possible value of $\frac{y}{2}$.

Solution. *Note that*

$$(x^2+x+1)(x^2+3x+3)=(x^2+2x+1)^2+x^2+2x+1+1.$$

Therefore $y=(x+1)^2$, thus the greatest value of $\frac{y}{2}$ is equal 968.

Problem 4. Let a,b be positive integers. Given that $(a,b)+[a,b]=a+b+1874$. Find the smallest possible value of the sum $a+b$.

Solution. *Let $(a,b)=d$, then $a=da_1$, $b=db_1$, $a_1,b_1\in\mathbb{N}$ and $(a_1,b_1)=1$. We have that*

$$d+\frac{ab}{d}-da_1-db_1=1874.$$

$$d+da_1b_1-da_1-db_1=1874.$$

$$d(a_1-1)(b_1-1)=2\cdot 937.$$

Hence $d=1$ or $d=937$.

Without loss of generality, one can assume that $a_1 \leq b_1$.
If $d = 1$, *then* $a_1 = 3$, $b_1 = 938$, $a+b = 941$ *or* $a_1 = 2$, $b_1 = 1875$, $a+b = 1877$.
If $d = 937$, $a_1 = 3$, $b_1 = 2$, $a+b = 4685$.
Hence, we obtain that the smallest value of $a+b$ *is equal to 941.*

Problem 5. Find the number of integer solutions of the following equation

$$x^3(y-z)^3 + y^3(z-x)^3 + z^3(x-y)^3 = 36.$$

Solution. *We have that*

$$x^3(y-z)^3 + y^3(z-x)^3 + z^3(x-y)^3 = (xy-xz)^3 + (yz-yx)^3 + (zx-zy)^3 =$$

$$= 3(xy-xz)(yz-yx)(zx-zy) = 3xyz(y-z)(z-x)(x-y) = 36.$$

Therefore,

$$xyz(y-z)(z-x)(x-y) = 12.$$

Let $d = \max(|x-y|, |y-z|, |z-x|)$. *Note that* $d \in \{2,3\}$.
If $d = 2$, $|xyz| = 6$, *thus the solutions are* $(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(-1,-2,-3)$, $(-2,-3,-1)$, $(-3,-1,-2)$.
If $d = 3$, $|xyz| = 2$, *thus the solutions are* $(-1,1,2)$, $(1,2,-1)$, $(2,-1,1)$, $(1,-2,-1)$, $(-2,-1,1)$, $(-1,1,-2)$.
Hence, the number of integer solutions of the given equation is equal to 12.

Problem 6. Let a,b be positive integers. Given that the value of the following expression

$$\frac{a^3-ab+1}{a^2+ab+2}$$

is an integer number. Find the greatest possible value of the sum $a+b$.

Solution. *If the value of the following expression*

$$\frac{a^3-ab+1}{a^2+ab+2}$$

is an integer number, then a *is an odd number. Therefore, the value of the following expression*

$$\frac{a^3-ab+1}{a^2+ab+2} - \frac{a+1}{2} = \frac{a(a^2-3b-ab-a-2)}{2(a^2+ab+2)}$$

is an integer number. Note that

$$(a, 2(a^2+ab+2)) = (a, 2(a^2+ab+2) - a(2a+2b)) = (a,4) = 1,$$

thus the value of the following expression

$$\frac{a^2 - 3b - ab - 2 - a}{2(a^2 + ab + 2)}$$

is also an integer number. Hence $a^2 - 3b - ab - 2 - a = 0$ *or* $a^2 - 3b - ab - 2 - a \geq 2(a^2 + ab + 2)$ *or* $3b + ab + a + 2 - a^2 \geq 2(a^2 + ab + 2)$.

If $a^2 - 3b - ab - 2 - a = 0$, *then* $(a+3)(4+b-a) = 10$. *Thus* $a = 7$, $b = 4$.

If $a^2 - 3b - ab - 2 - a \geq 2(a^2 + ab + 2)$, *then* $a^2 + 3ab + 6 + a + 3b \leq 0$. *This leads to a contradiction.*

If $3b + ab + a + 2 - a^2 \geq 2(a^2 + ab + 2)$, *then* $3a^2 + ab + 2 - a - 3b \leq 0$. *Thus* $ab - 3b < 0$, *hence we deduce that* $a = 1$ *and that the value of the expression* $\dfrac{2-b}{3+b}$ *is an integer number. On the other hand,*

$$\frac{2-b}{3+b} = \frac{5}{3+b} - 1.$$

Therefore, we obtain that $b = 2$.

Thus, the greatest value of the sum $a + b$ *is equal to 11.*

Problem 7. Let x, y be positive integers. Find the number of all (x, y) pairs, such that $x + y \leq 100$ and the value of the following expression

$$\frac{x^3 + y^3 - x^2y^2}{(x+y)^2}$$

is an integer number.

Solution. *We have that*

$$\frac{x^3 + y^3 - x^2y^2}{(x+y)^2} \in \mathbb{Z},$$

hence

$$(x+y)^2 \mid (x+y)^3 - 3xy(x+y) - x^2y^2.$$

Therefore

$$(x+y)^2 \mid 3xy(x+y) + x^2y^2.$$

Thus, we obtain that

$$(x+y)^2 \mid 9(x+y)^2 + 4(3xy(x+y) + x^2y^2)$$

$$(x+y)^2 \mid (3xy(x+y) + 2xy)^2$$

$$(x+y) \mid 3xy(x+y) + 2xy.$$

Hence $2xy = k(x+y)$, *where* $k \in \mathbb{N}$. *Using the following condition*

$$(x+y)^2 \mid (x+y)^2 \cdot \frac{6k+k^2}{4} = 3xy(x+y) + x^2y^2,$$

we obtain that k *is even, thus* $(x+y) \mid xy$.

(The last statement is possible to prove also in the following way: if p *is a prime number and* $p^\alpha \mid x+y$, $p^\alpha \nmid xy$, *then* $p^{2\alpha} \nmid xy(3(x+y)+xy)$.

On the other hand, if $(x+y) \mid xy$, *then* $(x+y)^2 \mid (x+y)^3 - 3xy(x+y) - x^2y^2 = x^3 + y^3 - x^2y^2$. *Therefore, we need to find all couples* (x,y), *such that* $xy = (x+y)z$, *where* $z \in \mathbb{N}$. *Note that* $xy > xz$, $xy > yz$ *and* $(x-z)(y-z) = z^2$. *Hence, if* $(x-z, y-z) = d$, *then* $x - z = da^2$, $y - z = db^2$, *where* $(a,b) = 1$. *We deduce that* $x = da(a+b)$, $y = db(a+b)$.

Note that $\frac{x}{y} = \frac{a}{b}$, $d = \frac{x+y}{(a+b)^2}$. *Thus, to different* (x,y), *couples correspond different* (d,a,b) *triples. Moreover, to different* (d,a,b), *triples correspond different* (x,y) *couples.*

We have that $x + y = d(a+b)^2 \leq 100$, *hence the number of* (d,a,b) *triples is equal to* $25 \cdot 1 + 11 \cdot 2 + 6 \cdot 2 + 4 \cdot 4 + 2 \cdot 2 + 2 \cdot 6 + 1 \cdot 4 + 1 \cdot 6 + 1 \cdot 4 = 105$.

Problem 8. Find the number of all positive integers n, such that for any of them the number $2014^{n^{2015}-1} + 1$ is divisible by n.

Solution. *Obviously, 1 satisfies to the conditions of the problem. Let us prove that there is no other number, which satifies the conditions of the problem.*

Let $n > 1$ *and* $2014^{n^{2015}-1} + 1$ *is divisible by* n. *Note that* n *is odd and* $n^{2015} - 1$ *is even.*

Let $n^{2015} - 1 = 2^s m$, *where* $s \in \mathbb{N}$, $m \in \mathbb{N}$ *and* m *is odd. Let* p *be any prime divisor of* n. *We have that* $a^{2^s} - 1$ *is not divisible by* p *and* $a^{2^{s+1}} - 1$ *is divisible by* p, *where* $a = 2014^m$. *On the other hand, according to Fermat's little theorem* $a^{p-1} - 1$ *is divisible by* p. *Thus,* $p - 1$ *is divisible by* 2^{s+1}. *Therefore,* $n - 1$ *is divisible by* 2^{s+1}. *Hence,* $n^{2015} - 1$ *is divisible by* 2^{s+1}. *We obtain that* m *is even, this leads to a contradiction.*

Problem 9. Let x, y, z be positive integers. Find the number of all triples (x,y,z), such that for any of them there exist three-digit numbers m and n with the following property

$$(x+y+z)^m = (x^2+y^2+z^2)^n.$$

(For different triples (x,y,z), the pairs (m,n) can be different.)

Solution. *Let* $(m,n) = d$, *then* $m = dm'$ *and* $n = dn'$, *where* $m', n' \in \mathbb{N}$, $(m',n') = 1$. *Thus, we deduce that* $(x+y+z)^{m'} = (x^2+y^2+z^2)^{n'}$. *Therefore* $x+y+z$ *and* $x^2 + y^2 + z^2$ *have the same number of prime divisors.*

Let $x+y+z = p_1^{\alpha_1}\cdot\dots\cdot p_k^{\alpha_k}$ *and* $x^2+y^2+z^2 = p_1^{\beta_1}\cdot\dots\cdot p_k^{\beta_k}$, *where* $\alpha_1,\beta_1,\dots,\alpha_k$, $\beta_k \in \mathbb{N}$ *and* $p_1,\dots,p_k$ *are distinct prime numbers. Hence, we obtain that* $m'\alpha_i = n'\beta_i$, $i = 1,\dots,k$. *Thus* $n' \mid \alpha_i$ *and* $m' \mid \beta_i$. *Therefore* $x+y+z = a^{n'}$ *and* $x^2+y^2+z^2 = a^{m'}$, *where* $a = p_1^{\frac{\alpha_1}{n'}}\cdot\dots\cdot p_k^{\frac{\alpha_k}{n'}} \in \mathbb{N}$. *We have that*

$$\frac{(x+y+z)^2}{3} \le x^2+y^2+z^2 < (x+y+z)^2.$$

Hence, if $a > 3$, *then* $a^{2n'-1} < a^{m'} < a^{2n'}$, *this leads to a contradiction. Therefore,* $a = 2$ *or* $a = 3$.

If $a = 2$, *then* $x^2+y^2+z^2 = 2^{m'}$. *Without loss of generality, we may assume that only one of the numbers* x,y,z *is even. Thus,* $x^2+y^2+z^2$ *has the following form* $4l+2$, *where* $l \in \mathbb{N}$. *On the other hand,* $2^{m'}$ *is not possible to represent in that form.*

If $a = 3$, *then* $3^{2n'-1} \le 3^{m'} < 3^{2n'}$. *Thus,* $m' = 2n'-1$, *therefore*

$$\frac{(x+y+z)^2}{3} = x^2+y^2+z^2.$$

Hence, we obtain that $x = y = z = 3^{n'-1}$.

Note that equality may hold true for $(3^s,3^s,3^s)$, *if we take* $m = (2s+1)d$, $n = (s+1)d$, *where* $d \in \mathbb{N}$, $s \in \mathbb{Z}$ *and* $s \ge 0$. *We have that* $\frac{100}{s+1} \le d \le \frac{999}{2s+1}$, *therefore* $s = 0,\dots,499$, *hence the number of* (x,y,z) *triples is equal to 500.*

7.2.6 Problem Set 6

Problem 1. Find the smallest positive integer n, such that $25 \mid n+3$ and $49 \mid n+8$.

Solution. *We have that* $n = 49m-8$, *where* $m \in \mathbb{N}$. *On the other hand,* $25 \mid 50m-(m+5) = n+3$. *Thus* $25 \mid m+5$. *Hence, the smallest value of n is equal to* $49\cdot 20 - 8 = 972$.

Problem 2. Find the possible smallest value of positive integer n, such that one of the digits of $125n+3$ is equal to 9.

Solution. *We have that* $125n+3 = \frac{n}{8}\cdot 1000+3$ *and* $\frac{1}{8} = 0.125$, $\frac{2}{8} = 0.25$, $\frac{3}{8} = 0.375$, $\frac{4}{8} = 0.5$, $\frac{5}{8} = 0.625$, $\frac{6}{8} = 0.75$, $\frac{7}{8} = 0.875$. *Therefore, any of the numbers* $\frac{n}{8}\cdot 1000$, *where* $n \in \{1,2,\dots,71\}$, *does not have a digit greater than 8 and ends with 0 or 5. Hence, any of the numbers* $125n+3$ *does not have a digit greater than 8.*

If $n = 72$, *then* $125n+3 = 9003$. *Thus, the smallest positive integer value of n is equal to 72.*

Problem 3. Find the number of integer solutions of the following equation

$$(100-x)^2 + (100-y)^2 = (x+y)^2.$$

Solution. *Let us rewrite the given equation in the following way:*

$$(x+100)(y+100) = 2^5 \cdot 5^4.$$

The number of integer solutions of the last equation is equal to $2(5+1)(4+1) = 60$.

Problem 4. Let p,q,r be prime numbers. Given that $p^3r + p^2 + p = 2qr + q^2 + q$. Find pqr.

Solution. *We have that* $2 \mid p(p+1)$ *and* $2 \mid q(q+1)$, *thus* $2 \mid p^3r$.

If $p = 2$, *then* $8r + 6 = 2qr + q^2 + q$. *Hence,* $q = 3, r = 3$.

If $r = 2$, *then*

$$2(p^3 - p) + p^2 + 3p = q^2 + 5q. \tag{7.33}$$

We have that $3 \mid p^3 - p$. *Therefore, if* $3 \nmid p$, *then the left-hand side of equation (7.33) is divisible by 3 with a remainder of 1. On the other hand, the right-hand side of equation (7.33) is divisible by 3 with a remainder of 0 or 2.*

Hence, $p = 3$. *Thus,* $q^2 + 5q = 66$. *Note that the last equation does not have solutions.*

Therefore, we obtain that $pqr = 18$.

Problem 5. Let x,y be positive integers. Find the number of all couples (x,y), such that $x \le 2015$ and $(x-y)^3 = x + 2y$.

Solution. *Let* $x - y = n$. *Note that,* $n \in \mathbb{N}$ *and*

$$y = \frac{(x-y)^3 - (x-y)}{3} = \frac{n^3 - n}{3}.$$

We have that if $n > 1$ *and* $n \in \mathbb{N}$, *then* $\frac{n^3-n}{3} \in \mathbb{N}$. *Hence,* $x = \frac{n^3+2n}{3}$ *and* $y = \frac{n^3-n}{3}$, *where* $n = 2,3,4,\dots$. *On the other hand, from the following condition,* $x \le 2015$, *we deduce that* $n = 2,3,4,\dots,18$.

Therefore, the total number of (x,y) *couples is equal to 17.*

Problem 6. Let a,b be positive integers. Given that $ab - \sqrt{a^2 - b^2} = 53$. Find a.

Solution. *Let* $(a-b, a+b) = d$, *we have that* $\sqrt{a^2 - b^2} \in \mathbb{N}$. *Hence,* $a - b = du^2$, $a + b = dv^2$, *where* $d,u,v \in \mathbb{N}$ *and* $(u,v) = 1$.

Therefore, $a = \frac{u^2+v^2}{2}d$ *and* $b = \frac{v^2-u^2}{2}d$. *Thus,*

$$\frac{v^4 - u^4}{4}d^2 - duv = 53. \tag{7.34}$$

If $v - u \ge 2$, *we have that*

$$53 = \frac{v^4 - u^4}{4}d^2 - duv \ge d\left(\frac{(v+u)(v^2+u^2)}{2} - uv\right) \ge \frac{d}{2}(v^3 + u^3).$$

Thus, we deduce that $v \leq 4$. *Note that, for any of the couples* $(4,2)$, $(4,1)$, $(3,1)$, d *is not a positive integer.*

Therefore, $v-u=1$. *Hence, we deduce that*

$$53 \geq \left(\frac{(v-u)(v+u)(v^2+u^2)}{4}-uv\right)d > u^3 d.$$

We obtain that $u \leq 3$. *One can easily verify and deduce that* $u=2$, $v=3$, $d=2$. *Hence,* $a=13$.

Problem 7. Let m,n be positive integers. Given that $\dfrac{m^2+n^2}{mn-5}$ is a positive integer. Find the possible greatest value of $\dfrac{m^2+n^2}{mn-5}$.

Solution. *Let* $m \geq n$. *If* $n=1$, *then*

$$\frac{m^2+1}{m-5}=m+5+\frac{26}{m-5} \in \mathbb{N}.$$

Therefore, $m \in \{6,7,18,31\}$ *and* $\frac{m^2+1}{m-5} \in \{25,37\}$.

Let us prove that positive integer $\frac{m^2+n^2}{mn-5}$ *cannot be greater than 37.*

We proceed by a contradiction argument. Assume that $\frac{m^2+n^2}{mn-5}=k \in \mathbb{N}$ *and* $k>37$. *Let* n_0 *be the smallest value of* n.

We have that $m_0 \geq n_0 > 1$ *and*

$$\frac{m_0^2+n_0^2}{m_0 n_0-5}=k.$$

Consider the following quadratic function $f(x)=x^2-kn_0x+n_0^2+5k$. *We have that* $f(m_0)=0$. *According to Vieat's theorem, if* $f(m_1)=0$, *then* $m_1=kn_0-m_0$ *and* $m_1=\frac{n_0^2+5k}{m_0}$. *Hence,* $m_1 \in \mathbb{N}$.

On the other hand, we have that, when $n_0 \geq 3$

$$f(n_0)=5k-(k-2)n_0^2 \leq 5k-9(k-2) < 0.$$

Therefore, $m_1 < n_0$. *This leads to a contradiction.*

If $n_0=2$, *then*

$$\frac{m^2+4}{2m-5}=\frac{1}{4}\left(2m+5+\frac{41}{2m-5}\right) \in \mathbb{N}.$$

Thus, $m \in \{3,23\}$ *and* $\frac{m^2+4}{2m-5}=13$. *This leads to a contradiction.*

Therefore, the required number is 37.

Problem 8. Let x, y be positive integers. Find the number of all couples (x, y), such that $\frac{x^5+y}{x^2+y^2}$ and $\frac{y^5+x}{x^2+y^2}$ are positive integers.

Solution. *We have that $x, y \in \mathbb{N}$, $x^2+y^2 \mid x^5+y$, $x^2+y^2 \mid y^5+x$. Let us prove the following properties:*

P1. (x,y) = 1. Indeed, if $d = (x,y) > 1$, then from the following condition $x^2+y^2 \mid x^5+y$ we deduce that $d^2 \mid y$. In a similar way, we obtain that $d^2 \mid x$. Thus, $d^2 \mid (x,y) = d$. This leads to a contradiction.

P2. $x^2+y^2 \mid xy^3+1$ and $x^2+y^2 \mid x^3y+1$.

We have that

$$x^2+y^2 \mid (x^2+y^2)(x^3-xy^2)+y(xy^3+1) = x^5+y.$$

Hence, $x^2+y^2 \mid y(xy^3+1)$. Therefore, $x^2+y^2 \mid xy^3+1$.

In a similar way, we obtain that $x^2+y^2 \mid x^3y+1$.

According to P2, we have that $x^2+y^2 \mid xy^3+x^3y+2$. Thus, $x^2+y^2 \mid 2$. Therefore, $x = y = 1$.

Hence, the number of required couples is equal to 1.

Problem 9. Find the smallest value of a positive integer n, if it is known that, for any division of numbers $1, 2, \ldots, n$ into two groups, in one of the groups there are three numbers that generate a geometric progression.

Solution. *Let us give an example of division of the set $\{1, 2, \ldots, 63\}$ into two subsets, such that in any of those two subsets there are no three numbers that are terms of the same geometric progression. Let us choose the following subsets*

$$A_1 = \{1,2,8,9,12,24,27,54,10,25,30,40,45,60,14,49,56,63,$$

$$11,22,33,13,26,51,38,57\},$$

and

$$A_2 = \{3,4,6,16,18,32,36,48,5,15,20,50,7,21,28,42,35,44,55,39,52,17,34,$$

$$19,23,46,29,58,31,62,37,41,43,47,53,59,61\}.$$

One can easily verify that in any of those two subsets there are no three numbers that are terms of the same geometric progression.

Let us now prove that if the set $\{1, 2, \ldots, 64\}$ is divided (in a random way) into subsets B and C, then there are three numbers in one of those subsets that are terms of the same geometric progression. We proceed by a contradiction argument. Assume that neither in B nor in C, there are no three numbers that are terms of the same geometric progression. Let us consider the following cases:

a) If $4,16 \in B$, *then* $8 \in C$, $(8^2 = 4 \cdot 16)$, $1 \in C$, $(4^2 = 1 \cdot 16)$, $64 \in B$, $(8^2 = 1 \cdot 64)$, *but* $16^2 = 4 \cdot 64$. *This leads to a contradiction.*

b) If $4 \in B$ *and* $12,16 \in C$, *then* $9 \in B$, $(12^2 = 9 \cdot 16)$, $6 \in C$, $(6^2 = 4 \cdot 9)$, $3 \in B$, $(6^2 = 3 \cdot 12)$, $1 \in C$, $(3^2 = 1 \cdot 9)$, $27 \in C$, $(9^2 = 3 \cdot 27)$, $18 \in B$, $(18^2 = 27 \cdot 12)$, $36 \in B$, $(6^2 = 1 \cdot 36)$.

We have obtained that $9 \in B, 18 \in B, 36 \in B$. *This leads to a contradiction, as* $18^2 = 9 \cdot 36$.

c) If $4,12 \in B$ *and* $16 \in C$, *then* $36 \in C$, $(12^2 = 4 \cdot 36)$, $24 \in B$, $(24^2 = 36 \cdot 16)$, $48 \in C$, $(24^2 = 48 \cdot 12)$, $64 \in B$, $(48^2 = 64 \cdot 36)$, $9 \in C$, $(24^2 = 9 \cdot 64)$, $18 \in B$, $(18^2 = 9 \cdot 36)$, $32 \in B$, $(16^2 = 8 \cdot 32)$. *Hence,* $18 \in B$, $24 \in B$, $32 \in B$. *This leads to a contradiction, as* $24^2 = 18 \cdot 32$.

This ends the proof of the statement.

7.2.7 Problem Set 7

Problem 1. Find the smallest positive integer n, such that n and $n + 2015$ are squares of some integers.

Solution. *Let* $n = a^2$, $n + 2015 = b^2$, *where* $a, b \in \mathbb{N}$. *We have that* $2015 = (b - a)(b + a)$, *thus* $b - a = k, b + a = \dfrac{2015}{k}$, $k\{1,5,13,31\}$ *and* $a = \dfrac{1}{2}\left(\dfrac{2015}{k} - k\right)$. *Therefore, the smallest possible value of a is equal to 17, and the smallest possible value of n is equal to 289.*

Problem 2. Find all positive integers n, such that $9^{2^{n-1}} + 3^{2^{n-1}} + 1$ is a prime number.

Solution. *If* $n \geq 2$, $2^{n-1} = 2k$, $k \in \mathbb{N}$, *then* $9^{2^{n-1}} + 3^{2^{n-1}} + 1 = (3^k)^4 + (3^k)^2 + 1 = (3^{2k} + 1)^2 - (3^k)^2 = (3^{2k} + 1 - 3^k)(3^{2k} + 1 + 3^k)$ *is not a prime number.*

If $n = 1$, *then* $9^{2^{n-1}} + 3^{2^{n-1}} + 1 = 13$ *is a prime number. Therefore, the number of all such positive integer n is equal to 1.*

Problem 3. Let n be a positive integer. Denote by S the sum of all the digits of $2n^2 + 44n + 443$. Find the possible minimal value of S.

Solution. *Note that* $2n^2 + 44n + 443 = (n+1)^2 + (n+21)^2 + 1$.

If $n = 59$, *then we have that* $2n^2 + 44n + 443 = 10001$. *Note that the sum of the digits is equal to 2.*

Let us show that the sum of the digits of $(n+1)^2 + (n+21)^2 + 1$ *cannot be equal to 1. We proceed by contradiction argument. Assume that* $(n+1)^2 + (n+21)^2 + 1 = 10^k$, *where* $k \in \mathbb{N}$, *then* $3 \mid (n+1)^2 + (n+21)^2$. *Therefore,* $3 \mid n+1$ *and* $3 \mid n+21$, *which leads to a contradiction, as* $3 \nmid 20$.

Problem 4. Find the number of all positive integers n, such that $3^{n-1} + 7^{n-1} \mid 3^{n+1} + 7^{n+1}$.

Solution. *Let $3^{n+1}+7^{n+1}=(3^{n-1}+7^{n-1})m$, where $m\in\mathbb{N}$. We have that $3^{n-1}(m-9)=7^{n-1}(49-m)$, thus $9<m<49$.*

If $n=1$, then $m=29$.

If $n=2$, then $m=37$.

If $n>2$, then $49\mid m-9$, which leads to a contradiction.

Problem 5. Let $n>1$ and $a_1,\dots,a_n$ be integer numbers. Given that one of the roots of the polynomial $p(x)=x^n+a_1x^{n-1}+\cdots+a_n$ and $p(2015)$ are prime numbers. Find the value of $p(2015)$.

Solution. *Let one of the roots of $p(x)$ is a prime number q, then we obtain that $p(x)=(x-q)p_1(x)$, where $p_1(x)$ is a polynomial with integer coefficients. We have that $p(2015)=(2015-q)p_1(2015)$ is a prime number and $p_1(2015)\in\mathbb{Z}$. Therefore, either $2015-q$ or $q-2015$ is a prime number. Hence, $q=2017$ and $p(2015)=2$.*

Problem 6. Let x,y,z be positive integers. Find all the possible triples (x,y,z), such that $x+y+z\le 1000$ and $xy^2+y^2z^4=5x+4z+x^2y^2z^2$.

Solution. *Let us consider the following cases:*

a) If $x=z$, then from the given equation we deduce that $y^2=9$. Hence, we obtain the following solutions $(n,3,n)$, where $n=1,2,\dots,498$.

b) If $x>z$, then we have that $y^2(x-z^2(x^2-z^2))=5x+4z$. Thus, $x>z^2(x^2-z^2)>z^2$. We obtain that $x\ge z^2+1$. Hence, $x^2y^2z^2\ge x(z^2+1)y^2z^2=xy^2z^4+xy^2z^2\ge z^4y^2+xy^2$. Therefore, the given equation does not have solution.

c) If $x<z$, then $z\ge x+1$. Thus, $y^2(x+z^2(z^2-x^2))\ge x+z^2(2x+1)$. We deduce that $9z>5x+4z=y^2(x+z^2(z^2-x^2))\ge x+z^2(2x+1)$. We obtain that $9>z$ $(2x+1)$. Hence $x=1$, $z=2$ and $y=1$. Therefore, the number of all solutions of the given equation is equal to 499.

Problem 7. Let x be a positive integer. Given that for any even value of x the quadratic expression $p(x)$ is a square of a positive integer. Find the value of $4p(2015)$, if $50<p(2015)<60$.

Solution. *Let $p(x)=ax^2+bx+c$ and $k\in\mathbb{N}$. According to the conditions of the problem, we have that $4k^2a+2bk+c=x_k^2$, where $x_k\ge 0$, $x_k\in\mathbb{Z}$, $k=1,2,3,\dots$*

We have that

$$y_k=x_{k+1}-x_k=\sqrt{4(k+1)^2a+2b(k+1)+c}-\sqrt{4^2a+2bk+c}=$$

$$=\frac{8a+\dfrac{2b+4a}{k}}{\sqrt{4a+\dfrac{8a+2b}{k}+\dfrac{4a+2b+c}{k^2}}+\sqrt{4a+\dfrac{2b}{k}+\dfrac{c}{k^2}}}.$$

Hence, $y_k\to 2\sqrt{a}$, when $k\to\infty$.

Note that all terms of sequence (y_n) are integer numbers, thus there exists a positive integer m, such that $y_k=2\sqrt{a}$, for $k=m,m+1,\dots$

Therefore, $x_k = x_m + 2\sqrt{a}(k-m)$, *for* $k = m, m+1, \dots$

We obtain that $4k^2a + 2bk + c = (2\sqrt{a}k + d)^2$, *where* $d = x_m - 2\sqrt{a}m \in \mathbb{Z}$, *for* $k = m, m+1, \dots$

Thus, the equation $ax^2 + bx + c - (\sqrt{a}x + d)^2 = 0$ *has infinitely many solutions. We deduce that, for any positive integer n, we have that* $an^2 + bn + c = (\sqrt{a}n + d)^2$ *and* $4(an^2 + bn + c) = (2\sqrt{a}n + 2d)^2$.

On the other hand, $200 < 4p(2015) < 240$. *Hence,* $4p(2015) = 225$.

Problem 8. Let x, y be positive integers. Find all couples (x, y), such that $x^2 + y^2 \mid x^{101} + y$ and $x^2 + y^2 \mid x + y^{101}$.

Solution. *We have that* $x^2 + y^2 \mid x^{101} + y$ *and* $x^2 + y^2 \mid x + y^{101}$, *where* $x, y \in \mathbb{N}$.

Let us prove the following two properties.

P1. (x,y) = 1.

We proceed the proof by contradiction argument. Assume that $d = (x, y) > 1$, *then from the conditions* $x^2 + y^2 \mid x^{101} + y$ *and* $x^2 + y^2 \mid x + y^{101}$, *we deduce that* $d^2 \mid y$ *and* $d^2 \mid x$. *Thus* $d^2 \mid (x, y) = d$. *This leads to a contradiction.*

P2. $x^2 + y^2 \mid xy^{99} + 1$ *and* $x^2 + y^2 \mid x^{99}y + 1$.

We have that

$$x^2 + y^2 \mid x^{101} + y = (x^2 + y^2)(x^{99} - x^{97}y^2 + x^{95}y^4 - \dots - xy^{98}) + xy^{100} + y.$$

We obtain that $x^2 + y^2 \mid y(xy^{99} + 1)$. *Hence, we deduce that* $x^2 + y^2 \mid xy^{99} + 1$.

In a similar way, we obtain that $x^2 + y^2 \mid x^{99}y + 1$.

According to P2, we have that $x^2 + y^2 \mid xy^{99} + x^{99}y + 2$. *Note that*

$$x^2 + y^2 \mid xy((x^2)^{49} + (y^2)^{49}) = xy^{99} + x^{99}y.$$

Thus $x^2 + y^2 \mid 2$. *Therefore,* $x = y = 1$.

Hence, the number of required couples is equal to 1.

Problem 9. Find the number of all triples (x, y, z), where x, y, z are odd positive integers, such that

$$x + y + z \leq 2015$$

and

$$3x^4 - 3x^2y^2 + y^4 = z^2.$$

Solution. *If* $x = y$, *then* $z = x^2$. *Hence, the triples* (a, a, a^2), *where a is an odd number, are solutions. Let us now prove that there are no other solutions. Proof by contradiction argument. Assume that* $(x, y) = d$, *thus* $x = dx_1$, $y = dy_1$, $(x_1, y_1) = 1$ *and* $d^4(3x_1^4 - 3x_1^2y_1^2 + y_1^4) = z^2$. *Hence,* $d^4 \mid z^2$. *Taking this into consideration, we deduce that* $d^2 \mid z$. *Therefore,* $3x_1^4 - 3x_1^2y_1^2 + y_1^4 = z_1^2$, *where* $z = d^2z_1$.

It follows that, if (x_0, y_0, z_0), *where* $x_0 \neq y_0$, *if the solution for that the smallest value of z is* z_0, *then* $(x_0, y_0) = 1$. *We have that*

$$(2z_0-2y_0^2+3x_0^2)(2z_0+2y_0^2-3x_0^2)=3x_0^4. \tag{7.35}$$

Note that $2z_0-2y_0^2+3x_0^2$ *and* $2z_0+2y_0^2-3x_0^2$ *are odd numbers. Let us prove that* $(2z_0-2y_0^2+3x_0^2, 2z_0+2y_0^2-3x_0^2)=1$. *Proof by contradiction argument. Assume that for some prime number* p, *we have that* $p \mid 2z_0-2y_0^2+3x_0^2$ *nad* $p \mid 2z_0+2y_0^2-3x_0^2$. *Then,* $p \mid 4y_0^2-6x_0^2$ *and* $p \mid 4z_0$. *Hence,* $p^2 \mid 3x_0^4$. *Therefore,* $p \mid x_0$ *and* $p \mid y_0$. *This leads to a contradiction.*

From (7.35), *we obtain that*

$$\begin{cases} 2z_0-2y_0^2+3x_0^2=3u^4, \\ 2z_0+2y_0^2-3x_0^2=v^4, \\ uv=x_0, \end{cases}$$

or

$$\begin{cases} 2z_0-2y_0^2+3x_0^2=u^4, \\ 2z_0+2y_0^2-3x_0^2=3v^4, \\ uv=x_0, \end{cases}$$

where $u>0$, $v>0$, $(u,v)=1$. *From the second system, we deduce that*

$$(2y_0)^2+(u^2)^2=6x_0^2+3v^4.$$

Hence, $3 \mid 2y_0$ *and* $3 \mid u^2$. *Thus,* $3 \mid x_0$ *and* $3 \mid y_0$. *This leads to a contradiction.*

From the first system, we deduce that

$$\left(\frac{v^2+3u^2}{2}-y_0\right)\cdot\left(\frac{v^2+3u^2}{2}+y_0\right)=3u^4. \tag{7.36}$$

Note that the numbers $\frac{v^2+3u^2}{2}-y_0$ *and* $\frac{v^2+3u^2}{2}+y_0$ *are odd and mutually prime. Proof by contradiction argument. Assume that they are not mutually prime, then there exists a prime and odd number* p, *such that* $p \mid \frac{v^2+3u^2}{2}-y_0$ *and* $p \mid \frac{v^2+3u^2}{2}+y_0$. *Hence,* $p \mid 2y_0$ *and* $p \mid v^2+3u^2$. *Thus,* $p^2 \mid 3u^4$. *Therefore,* $p \mid y_0$ *and* $p \mid x_0$. *This leads to a contradiction.*

From (7.36), *we deduce that*

$$\begin{cases} \dfrac{v^2+3u^2}{2}-y_0=s^4, \\ \dfrac{v^2+3u^2}{2}+y_0=3t^4, \\ st=u, \end{cases}$$

or

$$\begin{cases} \dfrac{v^2+3u^2}{2} - y_0 = 3s^4, \\ \dfrac{v^2+3u^2}{2} + y_0 = t^4, \\ st = u, \end{cases}$$

where s and t are odd numbers.

Hence, we obtain that $3t^4 - 3s^2t^2 + s^4 = v^2$ or $3s^4 - 3s^2t^2 + t^4 = v^2$.

According to the choice of z_0, we have that $z_0 \leq v$. It follows that $z_0^2 \leq v^2 \leq x_0^2$. Thus, $3x_0^4 - 3x_0^2y_0^2 + y_0^4 \leq x_0^2$. Or equivalently, $(3x_0^2 - 2y_0^2)^2 + x_0^2(3x_0^2 - 4) \leq 0$. We obtain that $x_0 = 1$, $y_0 = 1$. This leads to a contradiction. Therefore, the given equation has no other solutions, except the triples (a, a, a^2), where $a \in \mathbb{N}$ and a is odd.

Therefore, the number of triples satisfying the given assumptions is equal to 22.

7.2.8 Problem Set 8

Problem 1. Let p and q be prime numbers. Given that the equation $x^{2015} - px^{2014} + q = 0$ has an integer root. Find the sum $p+q$.

Solution. *Let a be an integer number and a root of the given equation. We have that $q = a^{2014}(p-a)$, thus $a = 1$ or $a = -1$.*

If $a = 1$, then $q = p - 1$. Hence, $p = 3$, $q = 2$.

If $a = -1$, then $q = p + 1$. Hence, $p = 2$ or $q = 3$.

Therefore, $p + q = 5$.

Problem 2. Find the number of all positive integer solutions of the equation

$$55x(x^3y^3 + x^2 + y) = 446(xy^3 + 1).$$

Solution. *Let us rewrite the given equation in the following way:*

$$x^3 + \cfrac{1}{y^2 + \cfrac{1}{xy}} = 8 + \cfrac{1}{9 + \cfrac{1}{6}}.$$

Therefore, $x^3 = 8$, $y^2 = 9$, $xy = 6$. Hence, we deduce that $x = 2$, $y = 3$.

Problem 3. Let x and y be positive integers, such that $x < 2^{10}$, $y < 2^{20}$ and the value of the expression

$$\frac{2^{10}}{x} + \frac{2^{20}}{y}$$

is a positive integer. Find the number of all possible values of $\dfrac{x}{y}$.

Solution. *Let* $x = 2^m \cdot x_1$, $y = 2^n \cdot y_1$, *where* $m, n \in \mathbb{Z}$, $m \geq 0$, $n \geq 0$ *and* $x_1, y_1 \in \mathbb{N}$. x_1, y_1 *are odd numbers.*

According to the assumption of the problem, we have that

$$\frac{2^{10}}{2^m x_1} + \frac{2^{20}}{2^n y_1} = \frac{2^{10-m}}{x_1} + \frac{2^{20-n}}{y_1} \in \mathbb{N},$$

$10 - m \in \mathbb{N}$, $20 - n \in \mathbb{N}$.

Therefore, the following number

$$2^{10-m} + \frac{2^{20-n} \cdot x_1}{y_1}$$

is a positive integer. Hence, it follows that $y_1 \mid x_1$. *In a similar way, one can prove that* $x_1 \mid y_1$. *Thus, we deduce that* $x_1 = y_1$.

Therefore,

$$\frac{x}{y} = 2^{m-n}, \ \ m \in \{0, 1, \ldots, 9\}, \ \ n \in \{0, 1, \ldots, 19\}.$$

We obtain that the number of possible values of $\frac{x}{y}$ *is equal to 29.*

Problem 4. Let n be a positive integer, such that the sum of the digits of n^2 is equal to the sum of the digits of $(n+2015)^2$. Find the smallest possible value of n.

Solution. *Note that the numbers* n^2 *and* $(n+2015)^2$ *are divisible by 9 with the same remainder. Thus, it follows that* $9 \mid (n+2015)^2 - n^2$. *This means that* $9 \mid 2n + 2015$.

Therefore, n is divisible by 9 with a remainder of 5. Straightforward but somewhere technical verification shows that the first number from the numbers 5, 14, 23, 32, 41, 50, 59, 68, 77, 86, 95,... that satisfies the assumptions of the problem is 95. We have that $2110^2 = 4452100$ *and* $95^2 = 9025$.

Problem 5. Find the sum of the last 24 digits of the number $4 \cdot 6 \cdot (4!)^2 + 5 \cdot 7 \cdot (5!)^2 + \cdots + 50 \cdot 52 \cdot (50!)^2$.

Solution. *Note that*

$$n(n+2)(n!)^2 = ((n+1)^2 - 1^2)(n!)^2 = ((n+1)!)^2 - (n!)^2,$$

where $n \in \mathbb{N}$.

Hence, it follows that

$$4 \cdot 6 \cdot (4!)^2 + 5 \cdot 7 \cdot (5!)^2 + \cdots + 50 \cdot 52 \cdot (50!)^2 = (5!)^2 - (4!)^2 + (6!)^2 - (5!)^2 + \ldots$$

$$+ (51!)^2 - (50!)^2 = (51!)^2 - (4!)^2 = (51!)^2 - 576.$$

51! ends with 12 zeros, thus $(51!)^2$ *ends with 24 zeros. Hence, the sum of the last 24 digits of the number* $4\cdot 6\cdot (4!)^2+5\cdot 7\cdot (5!)^2+\cdots+50\cdot 52\cdot (50!)^2$ *is equal to* $9\cdot 21+4+2+4=199$.

Problem 6. Given that the sum of five positive integers is equal to 595. At most, with how many zeros can end the product of those numbers?

Solution. *We have that*

$$595=5^3+5^3+2^5\cdot 5+2^5\cdot 5+5^2.$$

Thus, the product of five summands ends with 10 zeros.

Assume that the sum of positive integers a, b, c, d, e is equal to 595. Let us prove that the product abcde cannot end with 11 zeros. We proceed the proof by contradiction argument. Assume that $10^{11}\mid abcde$. *We can assume that* $5\mid a$, $5\mid b$, $5\mid c$, $5\mid d$, $5\mid e$. *Otherwise, two of those numbers are not divisible by 5. Thus, it follows that* $5^{11}\mid abc$, $(5\nmid d, 5\nmid e)$. *Therefore, either* $5^4\mid a$ *or* $5^4\mid b$ *or* $5^4\mid c$. *This leads to a contradiction.*

Let $a=5x$, $b=5y$, $c=5z$, $d=5t$, $e=5u$, *where* $x,y,z,t,u\in\mathbb{N}$, $x+y+z+t+u=119$, $5^6\cdot 2^{11}\mid xyztu$. *Note that two among the numbers x, y, z, t, u are divisible by 25. Assume that* $25\mid x$ *and* $25\mid y$. *Hence,* $25\mid x+y$.

Consider the following cases:

a) If $x+y=50$, *then* $x=y=25$ *and* $z+t+u=69$, $5^2\cdot 2^{11}\mid z\cdot t\cdot u$. *Thus, one of the numbers z, t, u is divisible by* 2^6. *This leads to a contradiction.*

b) If $x+y=75$, *then* $x=25$, $y=50$, $(x=50, y=25)$, $z+t+u=44$, $5^2\cdot 2^{10}\mid z\cdot t\cdot u$. *Thus,* $16\mid z$. *Therefore,* $2^5\cdot 25\mid t\cdot u$. *This leads to a contradiction.*

c) If $x+y=100$, *then* $z+t+u=19$, $5^2\cdot 2^9\mid z\cdot t\cdot u$. *This leads to a contradiction.*

Problem 7. Let x, y, z, t be positive integers. Find the number of all quadruples (x,y,z,t), such that $x+y+zt=xy+z+t$ and $x+y+z+t=245$.

Solution. *Let us rewrite the equation* $x+y+zt=xy+z+t$ *in the following way*

$$(x-1)(y-1)=(z-1)(t-1).$$

If $x>1$ *and* $y>1$, *then* $z>1$ *and* $x>1$.

Let $\dfrac{x-1}{z-1}=\dfrac{a}{b}$, *where* $a,b\in\mathbb{N}$ *and* $(a,b)=1$. *Hence,* $x-1=ac$, $z-1=bc$, *where* $c\in\mathbb{N}$. *We have that* $\dfrac{t-1}{y-1}=\dfrac{a}{b}$. *Thus, it follows that* $t-1=ad$, $y-1=bd$, *where* $d\in\mathbb{N}$.

Therefore, $x=ac+1$, $y=bd+1$, $z=bc+1$, $t=ad+1$. *From the condition* $x+y+z+t=245$, *we deduce that* $(a+b)(c+d)=241$. *This is impossible, as 241 is a prime number.*

We obtain that, $x=1$ *or* $y=1$.

The number of quadruples $(1,y,1,t)$ *is equal to 242.*

The number of quadruples $(1,y,z,1)$ *is equal to 242.*

The number of quadruples $(x,1,z,1)$ *is equal to 242.*
The number of quadruples $(x,1,1,t)$ *is equal to 242.*
Hence, the total number is equal to $4 \cdot 242 - 4 = 964$.

Problem 8. Find the maximal number of summands, such that any summand is a positive number greater than 1, any two summands are mutually prime and the sum of all summands is equal to 2015.

Solution. *Note that the sum of following 32 prime numbers is equal to*

$$2+3+7+11+13+17+19+23+29+31+37+41+43+47+53+59+61+67+$$

$$+71+73+79+83+89+97+101+103+107+113+127+131+137+139=2015.$$

Now, let us prove that 2015 is not possible to write as a sum of 33 summands satisfying the assumptions of the problem.

Proof by contradiction argument. Assume that $2015 = a_1 + a_2 + \cdots + a_{33}$, $1 < a_1 < a_2 < \cdots < a_{33}$ *and* $(a_1, a_2, \ldots, a_{33})$ *are pairwise mutually prime.*

Let $p_1, p_2, p_3, \ldots$ *be a sequence of prime numbers.*

We are going to prove that $a_k \geq p_{k+1}$, $k = 1, 2, \ldots, 33$.

Proof by contradiction argument. Assume that for some k we have that $a_k < p_{k+1}$. *Then, two of the odd numbers* $a_1, a_2, \ldots, a_k$ *have the same prime divisor. This leads to a contradiction.*

Therefore,

$$a_1 + a_2 + \cdots + a_{33} \geq p_2 + \cdots + p_{33} + p_{34} = 2125.$$

If the number of summands is not less than 34, then we obtain that

$$2015 \geq p_1 + p_2 + \cdots + p_{33} + p_{34}.$$

This leads to a contradiction.

Problem 9. Find the number of all positive integers n, such that $2^{n!} + 1$ has a divisor less than or equal to $2n + 1$.

Solution. *Let p be a prime number and* $p \mid 2^{n!} + 1$, $p \leq 2n + 1$.

According to Fermat's little theorem

$$p \mid 2^{p-1} - 1.$$

Hence,

$$p \mid 2^{2k} - 1,$$

where

$$k = \frac{p-1}{2} \leq n.$$

Note that if $n \geq 4$*, then at least two of the numbers* $1, 2, \ldots, n$ *are even. It follows that* $2k \mid n!$*. Thus,* $2^{2k} - 1 \mid 2^{n!} - 1$*. Therefore,* $p \mid 2^{n!} - 1$*. We obtain that*

$$p \mid (2^{n!} + 1) - (2^{n!} - 1) = 2.$$

This leads to a contradiction.

If $n \leq 3$*, then* $5 \mid 2^{3!} + 1$*,* $5 \mid 2^{2!} + 1$*,* $3 \mid 2^{1!} + 1$*.*

Hence, the number of such n numbers is equal to 3.

7.2.9 Problem Set 9

Problem 1. Let n be a positive integer, such that $(n, n+20) = 20$ and $(n, n+15) = 15$. Find $(n, n+60)$.

Solution. *From the condition* $(n, n+20) = 20$*, it follows that* $20 \mid n$*, and from the condition* $(n, n+15) = 15$*, it follows that* $15 \mid n$*. Thus, we deduce that* $60 \mid n$*.*

Let $(n, n+60) = d$*, we have that* $60 \mid n$*, thus* $60 \mid n+60$*. Therefore,* $d \geq 60$*. On the other hand,* $d \mid 60 = (n+60) - n$*. Hence,* $d \leq 60$*. We obtain that* $d = 60$*.*

Problem 2. Find the number of all couples (p, q) of prime numbers p, q, such that $p^4 + q$ and $q^4 + p$ are prime numbers.

Solution. *Note that if p and q are odd, then* $p^4 + q$ *is even number greater than 2. Therefore, it is not a prime number.*

Hence, either $p = 2$ *or* $q = 2$*.*

If $p = 2$ *and q is not divisible by 3, then* $16 + q$ *or* $q^4 + 2$ *are divisible by 3. This leads to a contradiction.*

Thus, it follows that $q = 3$*. In this case, we obtain that* $p^4 + q = 19$ *and* $q^4 + p = 83$ *are prime numbers.*

If $q = 2$*, then* $p = 3$*.*

Therefore, the number of such (p, q) *couples is equal to 2.*

Problem 3. Find the number of integer solutions of the equation

$$x(x-1) = y(y-2).$$

Solution. *Note that*

$$x^2 - x + 1 = (y-1)^2.$$

On the other hand, if $x > 1$*, then*

$$(x-1)^2 < x^2 - x + 1 < x^2.$$

Thus, it follows that

$$x-1<|y-1|<x.$$

This leads to a contradiction.

If $x<0$*, then*

$$x^2<x^2-x+1<(x-1)^2.$$

We deduce that

$$-x<|y-1|<-x+1.$$

This leads to a contradiction.

If $x=0$*, then we obtain the solutions* $(0,2)$ *and* $(0,0)$.

If $x=1$*, then we obtain the solutions* $(1,2)$ *and* $(1,0)$.

Problem 4. Let a, b be rational numbers, such that

$$a^2-47b^2-18ab-4a+4b+2=0.$$

Find $a-b$.

Solution. *Let us solve the given equation with respect to a.*

$$a^2-2(9b+2)a-47b^2+4b+2=0.$$

Hence,

$$D_1=(9b+2)^2+47b^2-4b-2=128b^2+32b+2=2(8b+1)^2.$$

Thus, it follows that

$$a=9b+2\pm\sqrt{2}(8b+1).$$

Therefore, $8b+1=0$ *and* $a=9b+2$*. We obtain that*

$$a-b=8b+2=1.$$

Problem 5. Find the smallest positive integer n, such that the following equation

$$(2x+3y-5z)^3+(2y+3z-5x)^3+(2z+3x-5y)^3=n,$$

has a solution in the set of integer numbers.

Solution. *Note that*

$$(2x+3y-5z)+(2y+3z-5x)+(2z+3x-5y)=0.$$

Thus, according to the following equality

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ac),$$

it follows that

$$(2x+3y-5z)^3+(2y+3z-5x)^3+(2z+3x-5y)^3=3(2x+3y-5z)(2y+3z-5x)(2z+3x-5y).$$

Hence, we obtain that

$$3(2x+3y-5z)(2y+3z-5x)(2z+3x-5y)=n.$$

Let $2x+3y-5z<0$ *and* $2y+3z-5x<0$. *Note that*

$$19 \mid 2(2x+3y-5z)-3(2y+3z-5x).$$

Therefore, we deduce that

$$n=3ab(a+b)\geq 3\cdot 3\cdot 2\cdot 5,$$

where $2x+3y-5z=-a$, $2y+3z-5x=-b$.

If $n=90$, *then the triple* $(1,0,1)$ *is a solution. Hence, the smallest possible value of* n *is equal to 90.*

Problem 6. Find the greatest positive integer n, such that the numbers n and $8n+15$ have the same prime divisors.

Solution. *Note that*

$$15=8n+15-8n.$$

Thus, 3 and 5 are the prime divisors of n. *Let us consider the following three cases.*

a) If $n=3^m$, $8n+15=3^k$, *where* $m,k\in\mathbb{N}$.

If $m=1$, *this leads to a contradiction.*

If $m>1$, *then* $3\mid 8n+15$ *and* $9\nmid 8n+15$, *this leads to a contradiction.*

b) If $n=5^m$, $8n+15=5^k$, *where* $m,k\in\mathbb{N}$.

This leads to a contradiction.

c) If $n=3^m\cdot 5^k$, $8n+15=3^s\cdot 5^p$, *where* m, k, s, $p\in\mathbb{N}$.

If $m>1$ *and* $k>1$, *we have that*

$$8n+15=15(8\cdot 3^{m-1}\cdot 5^{k-1}+1).$$

This leads to a contradiction.

Therefore, $n=3\cdot 5^k$ *or* $n=3^m\cdot 5$.

If $n=3\cdot 5^k$, *then*

$$8n+15=15(8\cdot 5^{k-1}+1).$$

If $k = 1$, then $n = 15$ satisfies the assumption of the problem.

Hence, if $k > 1$, then

$$8 \cdot 5^{k-1} + 1 = 3^l.$$

Note that $l = 2u$, $u \in \mathbb{N}$,

$$8 \cdot 5^{k-1} = (3^u - 1)(3^u + 1).$$

On the other hand, the numbers $3^u - 1$ and $3^u + 1$ are not simultaneously divisible by 5. Thus, this equation does not have a solution.

If $n = 3^m \cdot 5$, then

$$8n + 15 = 15(8 \cdot 3^{m-1} + 1).$$

We have that $m > 1$, thus

$$8 \cdot 3^{m-1} + 1 = 5^v.$$

Therefore, we deduce that

$$8 \cdot 3^{m-1} = (5^t - 1)(5^t + 1),$$

where $v = 2t$, $t \in \mathbb{N}$.

Note that the numbers $5^t - 1$ and $5^t + 1$ are not simultaneously divisible by 3. Hence, we deduce that $t = 1$, $m = 2$ and $n = 45$.

Problem 7. Consider a sequence (a_n) of positive integers, such that for any $i,j \in \mathbb{N}, i \neq j$ it holds true

$$(a_i, a_j) = (i^2 + 3i + 2, j^2 + 3j + 2).$$

Find the number of all possible values of the 2015th term of this sequence.

Solution. *Note that*

$$(a_n, a_{n^2+3n+1}) = (n^2 + 3n + 2, (n^2 + 3n + 2)(n^2 + 3n + 3)) = n^2 + 3n + 2.$$

Hence, we deduce that $n^2 + 3n + 2 \mid a_n$.

Let $a_n = (n^2 + 3n + 2)k$, where $k \in \mathbb{N}$.

If $k > 1$, then

$$(a_n, a_{(n^2+3n+2)k-1}) \leq n^2 + 3n + 2.$$

On the other hand, $m + 1 \mid a_m$. Therefore, $a_{(n^2+3n+2)k-1}$ is divisible by $(n^2 + 3n + 2)k = a_n$ and

$$(a_n, a_{(n^2+3n+2)k-1}) = a_n > n^2 + 3n + 2.$$

This leads to a contradiction.

Thus, $a_n = n^2 + 3n + 2$, $n = 1, 2, \dots$ *This sequence satisfies the assumptions of the problem. We obtain that*

$$a_{2015} = 2015^2 + 3 \cdot 2015 + 2.$$

Problem 8. Find the number of all positive integers, such that any of them is possible to represent, in more than one way, in the following form $\frac{x^2+y}{xy+1}$, where x, y are positive integers.

Solution. *Let positive integer n be presented in the following way*

$$n = \frac{x_0^2 + y_0}{x_0 y_0 + 1},$$

where $x_0, y_0 \in \mathbb{N}$.

Consider the following equation

$$n = \frac{x^2 + y_0}{x y_0 + 1}.$$

We have that one of the solutions of the equation

$$x^2 - n y_0 x + y_0 - n = 0,$$

is x_0. *Let the other solutions be* x_1.

If $n > 1$, *then*

$$f(1) = (1-n)(1+y_0) < 0,$$

where

$$f(x) = x^2 - n y_0 x + y_0 - n.$$

Therefore, $x_1 \leq 0$. *We have that* $x_1 x_0 = y_0 - n$. *Hence,* $y_0 \leq n$. *Thus,*

$$y_0 \leq \frac{x_0^2 + y_0}{x_0 y_0 + 1},$$

where $x_0 \geq y_0^2$. *On the other hand,*

$$x_0 y_0 + 1 \mid x_0(x_0 y_0 + 1) - y_0(x_0^2 + y_0) = x_0 - y_0^2.$$

Therefore, $x_0 = y_0^2$ *and* $n = y_0$.

If $n > 1$, *then the couple* (n^2, n) *is the unique solution of following equation*

$$n = \frac{x^2 + y}{xy + 1}. \tag{7.37}$$

If $n = 1$, then (7.50) has infinitely many solutions of the form $(1,k)$, where $k \in \mathbb{N}$.
Hence, the answer is 1.

Problem 9. Let x, y be rational numbers. Find the number of (x,y) couples, such that $x + \frac{2}{y}$ and $y + \frac{3}{x}$ are positive integers.

Solution. *Let us at first prove that $x > 0$ and $y > 0$. Let $x + \frac{2}{y} = m$ and $y + \frac{3}{x} = n$, where $m, n \in \mathbb{N}$.*

If $x < 0$, then

$$2 = \frac{2}{y} \cdot y = (m-x)\left(n - \frac{3}{x}\right) = mn + 3 - nx - \frac{3m}{x} > 3.$$

This leads to a contradiction.

If $y < 0$, then

$$3 = x \cdot \frac{2}{x} = \left(m - \frac{2}{y}\right)(n-y) = mn + 2 - \frac{2n}{y} - my > 3.$$

This leads to a contradiction.

Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where $a,b,c,d \in \mathbb{N}$ and $(a,b) = 1$, $(c,d) = 1$.

We have that

$$x + \frac{2}{y} = \frac{a}{b} + \frac{2d}{c} = \frac{ac + 2bd}{bc} \in \mathbb{N}.$$

Hence, $b \mid ac$. We deduce that $b \mid c$.

On the other hand, $c \mid 2bd$. Thus, $c \mid 2b$.

Therefore, $c = b$ or $c = 2b$.

In a similar way, we obtain that $a = d$ or $a = 3d$.

Consider the following four cases:

1) If $a = d$, $b = c$, then $y = \frac{1}{x}$ and $3x = m$, $\frac{4}{x} = n$. Therefore, $mn = 12$. Thus, $m = 1,2,3,4,6,12$. Then, the couples (x,y) will be equal to $\left(\frac{1}{3}, 3\right)$, $\left(\frac{2}{3}, \frac{3}{2}\right)$, $(1,1)$, $\left(\frac{4}{3}, \frac{3}{4}\right)$, $\left(2, \frac{1}{2}\right)$, $\left(4, \frac{1}{4}\right)$.

2) If $a = d$, $c = 2b$, then $y = \frac{2}{x}$ and $2x = m$, $\frac{5}{x} = n$. Therefore, $mn = 10$. Thus, $m = 1,2,5,10$. Then, the couples (x,y) will be equal to $\left(\frac{1}{2}, 4\right)$, $(1,2)$, $\left(\frac{5}{2}, \frac{4}{5}\right)$, $\left(5, \frac{2}{5}\right)$.

3) If $a = 3d$, $c = b$, then $y = \frac{3}{x}$, $\frac{5x}{3} = m$, $\frac{6}{x} = n$. Thus, $m = 1,2,5,10$. Then, the couples (x,y) will be equal to $\left(\frac{3}{5}, 5\right)$, $\left(\frac{6}{5}, \frac{5}{2}\right)$, $(3,1)$, $\left(6, \frac{1}{2}\right)$.

4) If $a = 3d$, $c = 2b$, then $y = \frac{6}{x}$. Therefore, $\frac{4x}{3} = m$, $\frac{9}{x} = n$. Thus, $mn = 12$. Then, the couples (x,y) will be equal to $\left(\frac{3}{4},8\right)$, $\left(\frac{3}{2},4\right)$, $\left(\frac{9}{4},\frac{8}{3}\right)$, $(3,2)$, $\left(\frac{9}{2},\frac{4}{3}\right)$, $\left(\frac{9}{2},\frac{2}{3}\right)$.

Hence, the number of (x,y) couples is equal to 20.

7.2.10 Problem Set 10

Problem 1. Find the smallest positive integer, such that the sum of its digits is equal to 19 and it is divisible by 11.

Solution. *As the sum of the digits is equal to 19, hence this number cannot be 1-digit or 2-digits number. Let this number be $\overline{abc}$. We have that $a+b+c=19$ and $11 \mid a-b+c$. Thus, it follows that $11 \mid 19-2b$. Hence, $b=4$ and $a+c=15$. Therefore, $a=6$, $c=9$ and the answer is 649.*

Problem 2. Find the number of all positive integers, such that any of them is impossible to represent as $\frac{1+xy}{x+y}$, where x, y are positive integers.

Solution. *Note that*

$$n = \frac{1+(2n+1)(2n-1)}{2n+1+2n-1}.$$

On the other hand, if $n \in \mathbb{N}$, then $2n+1$ and $2n-1$ are positive integers.

Therefore, any positive integer is possible to represent in the given form. Hence, the answer is 0.

Problem 3. Let p, q be prime numbers, such that the following equation

$$x^4 - (q+3)x^3 + (3q+2)x^2 - 2qx - 225 - p = 0,$$

has an integer root. Find $p+q$.

Solution. *We have that*

$$p = x_0(x_0-1)(x_0-2)(x_0-q) - 225,$$

where $x_0 \in \mathbb{Z}$ and is a root of the equation

$$x^4 - (q+3)x^3 + (3q+2)x^2 - 2qx - 225 - p = 0.$$

Note that the product of x_0-2, x_0-1, x_0 is divisible by 3. Thus, it follows that $3 \mid p$. Therefore, $p = 3$.

We obtain that

$$228 = x_0(x_0-1)(x_0-2)(x_0-q).$$

On the other hand, either $228 = 1\cdot 2\cdot 3\cdot 38$ *or* $228 = -3\cdot(-2)\cdot(-1)\cdot(-38)$. *Hence,* $x_0 = 3$, $q = -35$, *this leads to a contradiction. Otherwise,* $x_0 = -1$, $q = 37$. *Therefore,* $p+q = 40$.

Problem 4. Let a, b, c be positive integers greater than 1, such that $c+1 \mid a+b$, $a+1 \mid b+c$, $b+1 \mid c+a$. Find the greatest possible value of $a+b+c$.

Solution. *Let* $a \geq b \geq c$, *we have that* $a+1 \mid b+c$ *and* $b+c \leq 2a < 2(a+1)$. *Thus, it follows that* $b+c = a+1$. *Hence,* $a = b+c-1$ *and* $c+1 \mid 2b+c-1$, $b+1 \mid 2c+b-1$. *On the other hand,* $2c+b-1 \leq 3b-1 < 3(b+1)$.

Therefore, either $2c+b-1 = b+1$ *or* $2c+b-1 = 2b+2$. *We have that* $a = b+c-1$, *if* $c = 1$, $a = b$. *This leads to a contradiction.*

If $b = 2c-3$, $a = 3c-4$ *and* $c+1 \mid 5c-7$, *then* $c+1 \mid 12$. *On the other hand,* $a+b+c = 6c-7$. *We obtain that the greatest possible value of* c *is equal to 11 and the greatest possible value of* $a+b+c$ *is equal to 59.*

Problem 5. Find the number of integer solutions of the equation

$$x^5 + y^5 = 2^{2016}.$$

Solution. *Note that, if* $n > 1$, $n \in \mathbb{N}$, *then the equation* $x^5 + y^5 = 2^n$ *does not have odd solutions.*

Indeed,

$$x^5 + y^5 = (x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4),$$

if x *and* y *are odd numbers, then* $x^4 - x^3y + x^2y^2 - xy^3 + y^4$ *is also an odd number and* $x+y > 2$.

Therefore,

$$4(x^4 - x^3y + x^2y^2 - xy^3 + y^4) = 3x^4 + 3y^4 - 2x^2y^2 + (x-y)^4 =$$

$$= 2x^4 + 2y^4 + (x^2-y^2)^2 + (x-y)^4 > 32.$$

Hence,

$$x^4 - x^3y + x^2y^2 - xy^3 + y^4 > 8.$$

This leads to a contradiction.

From the given equation, we deduce that $x = 2^{403}x_1$, $y = 2^{403}y_1$, *where* $x_1, y_1 \in \mathbb{Z}$. *Thus,* $x_1^5 + y_1^5 = 2$. *On the other hand,*

$$4(x_1^4 - x_1^3y_1 + x_1^2y_1^2 - x_1y_1^3 + y_1^4) \geq 2x_1^4 + 2y_1^4 \geq 4.$$

Therefore, $x_1 = y_1 = 1$.

We obtain that the unique solution of the given equation is the couple $x = 2^{403}$, $y = 2^{403}$.

Problem 6. Let x, y, z be positive rational numbers, such that $x+\frac{1}{y}$, $y+\frac{2}{z}$, $z+\frac{3}{x}$ are integer numbers. Find the sum of all possible values of xyz.

Solution. *Let* $x=\frac{a}{b}$, $a,b\in\mathbb{N}$ *and* $(a,b)=1$; $y=\frac{c}{d}$, $c,d\in\mathbb{N}$ *and* $(c,d)=1$; $z=\frac{e}{f}$, $e,f\in\mathbb{N}$ *and* $(e,f)=1$.

From the following condition

$$x+\frac{1}{y}=\frac{a}{b}+\frac{d}{c}=\frac{ac+bd}{bc}\in\mathbb{Z},$$

$b\mid ac$ *and* $c\mid bd$. *Thus,* $b\mid c$ *and* $c\mid b$. *Hence* $b=c$.

On the other hand, from the condition

$$y+\frac{2}{z}=\frac{c}{d}+\frac{2f}{e}=\frac{ce+2fd}{de}\in\mathbb{Z},$$

we obtain that $d\mid ce$ *and* $e\mid 2fd$. *Thus, it follows that* $d\mid e$ *and* $e\mid 2d$. *Hence, either* $e=d$ *or* $e=2d$.

In a similar way, from the condition

$$z+\frac{3}{x}=\frac{e}{f}+\frac{3b}{a}\in\mathbb{Z},$$

we deduce that $f\mid a$ *and* $a\mid 3f$. *Thus, either* $a=f$ *or* $a=3f$.

Therefore, possible values of

$$xyz=\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f}=\frac{c}{b}\cdot\frac{e}{d}\cdot\frac{a}{f}.$$

are equal to 1,2,3,6. Hence, the sum of the possible values of xyz *is equal to 12.*

If $x=y=z=1$, *then* $xyz=1$.

If $x=1, y=1, z=2$, *then* $xyz=2$.

If $x=3, z=1, y=1$, *then* $xyz=3$.

If $x=3, z=2, y=1$, *then* $xyz=6$.

Problem 7. Find the number of integer solutions of the equation

$$x^{10}+y^{10}=x^6y^5.$$

Solution. *Let us consider the following cases:*

a) If $x,y\in\mathbb{N}$.

Let p *be a prime number and* $V_p(x)\neq V_p(y)$.

We have that
$$V_p(x^{10}+y^{10}) = 10\min(V_p(x),V_p(y)),$$
and
$$V_p(x^6y^5) \geq 11\min(V_p(x),V_p(y)) > V_p(x^{10}+y^{10}).$$

Hence, for any prime number p we have that $V_p(x)=V_p(y)$. Therefore, $x=y=2$.

b) If $xy=0$, then $x^{10}+y^{10}=0$. Thus, it follows that $x=0$, $y=0$.

c) If $xy \neq 0$, then $y \in \mathbb{N}$. Hence, if $x<0$, then the couple $(-x,y)$ is a solution. Therefore, according to case a), we obtain the solution $(-2,2)$.

Thus, the number of solutions of the given equation is equal to 3.

Problem 8. Let a, b be positive integers, such that
$$80 \mid 144a^3-32a^2b+92ab^2+b^3.$$
Find the possible smallest value of $16a^2+b^2$.

Solution. *Note that*
$$144a^3-32a^2b+92ab^2+b^3 = (4a+b)^3+80a^3-80a^2b+80ab.$$

Thus, $80 \mid (4a+b)^3$. It follows that $20 \mid 4a+b$. Hence, $b=4c$, $c \in \mathbb{N}$.

We have that
$$16a^2+b^2 = 16a^2+16c^2 = 16(a^2+c^2)$$
and $5 \mid a+c$. Therefore, the smallest value of $16a^2+b^2$ is equal to $16\cdot 13 = 208$.

Problem 9. Find the number of integer solutions of the equation
$$2x^2-y^6=1.$$

Solution. *Let us prove that the given equation is a unique solution $(1,1)$ in the set of positive integers.*

Let $x,y \in \mathbb{N}$ and $2x^2=1+y^6$, $2x^2=(1+y^2)(y^4-y^2+1)$.

If $y>1$, then $(y^2-1)^2 < y^4-y^2+1 < y^4$. Hence, y^4-y^2+1 is not a square number. We have that
$$(1+y^2,y^4-y^2+1) = (1+y^2,(1+y^2)^2-3y^2) = (1+y^2,3y^2) = (1+y^2,y^2) = 1.$$

Therefore, $1+y^2=u^2$ and $y^4-y^2+1=2v^2$, where $u,v \in \mathbb{N}$. On the other hand, the equation $1=(u-y)(u+y)$ does not have a solution in the set of positive integers.

Note that if the couple (x,y) is a solution of the given equation, then the couples $(-x,y)$, $(x,-y)$, $(-x,-y)$ are also solutions. Hence, the integer solutions of the given equation are $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$.

7.2.11 Problem Set 11

Problem 1. Find the number of all positive integers, such that any of them is not possible to represent as $\frac{1+xy}{x-y}$, where x and y are positive integers.

Solution. *Note that*

$$n = \frac{1+(n-1)(n^2-n+1)}{n^2-n+1-(n-1)}.$$

Therefore, any positive integer greater than 1 is possible to represent as $\frac{1+xy}{x-y}$. *For example,* $x = n^2-n+1$, $y = n-1$, $x, y \in \mathbb{N}$. *On the other hand, 1 is not to represent in the given form, as* $1+xy > x > x-y$. *Hence,* $1+xy > x-y$. *Thus, the number of required numbers is equal to 1.*

Problem 2. How many integer solutions does the following equation have?

$$xy^2 - 2xy + 3y^2 + 2y - 8 = 0.$$

Solution. *We have that*

$$(x+3)y^2 - 2(x-1)y - 8 = 0.$$

If $x = -3$, *then* $y = 1$. *Hence, the couple* $(-3, 1)$ *is a solution.*

If $x \neq -3$, *then*

$$\frac{D}{4} = (x-1)^2 + 8(x+3) = (x+3)^2 + 4^2.$$

Thus, it follows that

$$(x+3)^2 + 4^2 = z^2,$$

where $z \in \mathbb{Z}$. *Hence,* $x+3 = 3$ *or* $x+3 = -3$.

If $x = 0$, $y = -2$ *or* $x = -6$, $y = 4$.

Therefore, the integer solutions of the given equation are $(-3, 1)$, $(0, -2)$, $(-6, 4)$.

Problem 3. Find the smallest positive integer n, such that

$$0 < \{\sqrt{n}\} < 0.05,$$

where we denote by $\{x\}$ the fractional part of a real number x.

Solution. *We have that*

$$0 < \sqrt{n} - [\sqrt{n}] < \frac{1}{20}.$$

Thus, it follows that

$$0 < \frac{n - [\sqrt{n}]^2}{\sqrt{n} + [\sqrt{n}]} < \frac{1}{20}.$$

We deduce that

$$\frac{1}{20} > \frac{n - [\sqrt{n}]^2}{\sqrt{n} + [\sqrt{n}]} \geq \frac{1}{\sqrt{n} + [\sqrt{n}]} > \frac{1}{2\sqrt{n}}.$$

Hence, we obtain that $\sqrt{n} > 10$. *Therefore,* $n \geq 101$. *On the other hand,*

$$\{\sqrt{101}\} = \sqrt{101} - 10 < \frac{1}{20},$$

as

$$\left(10 + \frac{1}{20}\right)^2 = 101 + \frac{1}{400}.$$

Thus, the smallest possible value of n is equal to 101.

Problem 4. How many positive integer solutions does the following equation have?

$$(x^y - 1)(z^t - 1) = 2^{200}.$$

Solution. *Note that only one of the numbers x, z can be an even number. Moreover, that even number should be equal to 2.*

If $x = 2$, $y = 1$ *and* $z^t - 1 = 2^{200}$. *We have that* $z > 1$ *and* z *is an odd number. On the other hand, t is also an odd number, otherwise*

$$(z^{\frac{t}{2}} - 2^{100})(z^{\frac{t}{2}} - 2^{100}) = 1.$$

This leads to a contradiction.

If $t \geq 3$, *then* $z^t - 1 = (z-1)(z^{t-1} + \cdots + 1)$ *and* $z^{t-1} + \cdots + 1$ *is an odd number. This leads to a contradiction.*

Therefore, $t = 1$ *and* $z = 2^{200} + 1$.

If $z = 2$, *then* $t = 1$, $y = 1$ *and* $x = 2^{200} + 1$.

If x, z *are odd numbers, then* $x^y - 1 = 2^n$ *and* $z^t - 1 = 2^{200-n}$, *where* $n \in \{1, 2, \ldots, 199\}$.

If y, t *are odd numbers, then* $y = 1$, $t = 1$. *Hence, we have again obtained 199 solutions.*

If y *is an even number,* $y = 2l$, $l \in \mathbb{N}$, *then*

$$(x^l - 1)(x^l + 1) = 2^n.$$

We deduce that $x^l - 1 = 2^m$, $x^l + 1 = 2^k$, *where* $m < k$, $m, k \in \mathbb{N}$.

Thus, it follows that

$$2 = 2^k - 2^m.$$

Hence, $k = 2$, $m = 1$, $n = 3$, $x = 3$, $l = 1$.

Therefore, $y = 2$, $x = 3$, $t = 1$ *and* $z = 2^{197} + 1$.

In a similar way, if t *is an even number, then* $t = 2$, $z = 3$, $y = 1$ *and* $x = 2^{197} + 1$.

We obtain that the given equation has 203 positive integer solutions.

Problem 5. Find the smallest positive integer, such that it is possible to represent as $\frac{m^3 - n^3}{2016}$, where m and n are positive integers.

Solution. *Note that*

$$2 = \frac{16^3 - 4^3}{2016}.$$

Let $m, n, k \in \mathbb{N}$ *and*

$$k = \frac{m^3 - n^3}{2016}.$$

Let us prove that $k \geq 2$.

If m, n *are odd numbers, then we have that* $32 \mid m - n$. *Thus, it follows that*

$$k \geq \frac{m^2 + mn + n^2}{63} > \frac{33^2}{63} > 2.$$

Therefore, $k > 2$.

If $m = 2m_1$, $n = 2n_1$, *where* $m_1 \in \mathbb{N}$, $n_1 \in \mathbb{N}$ *and*

$$k = \frac{m_1^3 - n_1^3}{4 \cdot 63}.$$

If m_1, n_1 *are odd numbers, then* $4 \mid m_1 - n_1$. *Therefore,* $m_1 - n_1 = 4l$, *where* $l \in \mathbb{N}$ *and*

$$k = \frac{l(16l^2 + 12ln_1 + 3n_1^2)}{63}.$$

Hence, we obtain that $3 \mid l$ *and*

$$k \geq \frac{16 \cdot 9 + 36 + 3}{21} > 2.$$

If $m_1 = 2m_2$, $n_1 = 2n_2$, *where* $m_2 \in \mathbb{N}$, $n_2 \in \mathbb{N}$, *then*

$$k = 2 \cdot \frac{m_2^3 - n_2^3}{63} \geq 2.$$

Thus, the answer is 2.

Problem 6. Consider 8×8 grid square. One needs to write in its each square a positive integer, such that all written numbers are different and any two numbers written in the squares with a common side are not mutually prime numbers. Find the smallest possible value of the greatest number among those numbers.

Solution. *Let us consider an example of* 8×8 *grid square, such that any two numbers written in the squares with a common side are not mutually prime numbers and the greatest number is 77.*

19	57	27	3	9	12	39	13
38	76	24	72	18	8	26	65
56	70	54	44	6	32	64	52
28	49	63	33	48	16	4	58
7	21	77	11	22	36	2	62
14	35	55	66	40	75	60	74
46	42	10	15	20	50	68	34
23	69	45	5	25	30	51	17

Assume that there exists a 8×8 *grid square satisfying the assumptions of the problem, such that its greatest number is smaller than 77. Note that, in this case, the numbers 1, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73 are not written in that grid square. Therefore, the total number of numbers written in this* 8×8 *grid square is not more than* $76 - 13 = 63$. *This leads to a contradiction.*

Thus, the answer is 77.

Problem 7. Given that x, y and $\dfrac{x^{2016} + y^{2016}}{x^{1009} \cdot y^{1008}}$ are integer numbers. Find the greatest possible value of $\dfrac{x^{2016} + y^{2016}}{x^{1009} \cdot y^{1008}}$.

Solution. *Let* $(x,y) = d$, *then* $x = dx_1$, $y = dy_1$, $x_1 \in \mathbb{Z}$, $y_1 \in \mathbb{Z}$ *and* $(x_1, y_1) = 1$.

According to the assumption of the problem,

$$\frac{x_1^{2016} + y_1^{2016}}{d \cdot x_1^{1009} \cdot y_1^{1008}},$$

is an integer number.

Therefore,

$$x_1^{1009} \mid x_1^{2016} + y_1^{2016},$$

and

$$y_1^{1008} \mid x_1^{2016} + y_1^{2016}.$$

Hence, $|x_1| = 1$ *and* $|y_1| = 1$. *Thus, it follows that*

$$\left| \frac{x^{2016} + y^{2016}}{x^{1009} \cdot y^{1008}} \right| = \frac{2}{d}.$$

Therefore, the greatest possible value of $\frac{x^{2016}+y^{2016}}{x^{1009}\cdot y^{1008}}$ *is equal to 2. It is equal to 2, if, for example,* $x=1$ *and* $y=1$.

Problem 8. Let a, b, c, d be positive integers. Find the number of quadruples (a,b,c,d), such that for any of them it holds true $ad+bc<bd$, $b+d\leq 132$ and

$$(9ac+bd)(ad+bc)=a^2d^2+10abcd+b^2c^2.$$

Solution. *Note that* $(b-a)d>bc$. *Thus,* $b>a$ *and* $d>\frac{bc}{b-a}$. *We have that*

$$a(b-a)d^2+c(b-a)(b-9a)d+bc^2(9a-b)=0.$$

Therefore,

$$D=c^2(b-9a)(b-a)(b-3a)^2\geq 0.$$

Hence, either $b=3a$ *or* $b\geq 9a$.

If $b\geq 9a$, *then*

$$a(b-a)d^2+c(b-a)(b-9a)d+bc^2(9a-b)>a(b-a)\cdot\frac{b^2c^2}{(b-a)^2}+$$

$$+c(b-a)(b-9a)\cdot\frac{bc}{b-a}+bc^2(9a-b)=\frac{b^2c^2a}{b-a}>0.$$

This leads to a contradiction.

If $b=3a$, *then we obtain that* $d=3c$.

One can easily verify that any of the quadruples $(a,3a,c,3c)$, *where* $a+c\leq 44$ *and a, c are any positive integers, satisfy the given assumptions.*

Therefore, the number of quadruples satisfying the assumptions of the problem is equal to $1+2+\cdots+43=946$.

Problem 9. Given that the numbers 1,2, ..., 100 are written in a line (in a random way), such that we obtain the numbers a_1, $a_2, \ldots, a_{100}$. Let $S_1=a_1$, $S_2=a_1+a_2, \ldots, S_{100}=a_1+a_2+\cdots+a_{100}$. At most, how many square numbers can be among the numbers $S_1, S_2, \ldots, S_{100}$?

Solution. *Let us consider the following sequence*

$$1,3,\ldots,99,2,100,4,98,10,96,12,94,\ldots,58,72,60,70,30,40,64,66,68,6,8,$$

$$14,16,32,38,46,54,22,24,48,56.$$

We have that $S_i=i^2$, *for* $i=1,2,\ldots,50$, *and*

$$S_{54}=50^2+2\cdot 102=52^2,$$

$$S_{58} = 52^2 + 2 \cdot 106 = 54^2,$$

$$\dots$$

$$S_{82} = 64^2 + 2 \cdot 130 = 66^2,$$

$$S_{87} = 66^2 + 2 \cdot 134 = 68^2,$$

$$S_{96} = 70^2.$$

Now, let us prove that among the numbers S_1, $S_2, \dots,$ S_{100} cannot be 61 square numbers.

We proceed the proof by contradiction argument. Assume that among the numbers S_1, $S_2, \dots,$ S_{100} there are 61 square numbers. Consider two cases:

a) If $71^2 \notin \{S_1, S_2, \dots, S_{100}\}$, then 61 among the numbers 1^2, $2^2, \dots,$ 70^2 belong to the set $\{S_1, S_2, \dots, S_{100}\}$. Let the set A_i, $i=1,2,\dots,10$ be a subset of $\{S_1, S_2, \dots, S_{100}\}$ and consists of the squares of consequent numbers.

Note that if S_k and S_m, ($k < m$) have opposite parity, then among the numbers $a_{k+1}, \dots,$ a_m there is an odd number. Hence, among the numbers $a_1, \dots,$ a_{100}, at least $|A_1| - 1 + \dots + |A_{10}| - 1 = 51$ are odd. This leads to a contradiction.

b) If $71^2 \in \{S_1, S_2, \dots, S_{100}\}$, then 61 among the numbers 1^2, $2^2, \dots,$ 71^2 belong to the set $\{S_1, S_2, \dots, S_{100}\}$. Let the set A_i, $i=1,2,\dots,11$ be a subset of $\{S_1, S_2, \dots, S_{100}\}$ and consists of the squares of consequent numbers. Hence, among the numbers $a_1, \dots,$ a_l, at least $|A_1| - 1 + \dots + |A_{11}| - 1 = 50$ are odd, where $S_l = 71^2$. We have that $S_{100} = 5050$; thus, it follows that at least one of the numbers $a_{l+1}, \dots,$ a_{100} is an odd number.

Therefore, among the numbers $a_1, \dots,$ a_{100}, at least 51 are odd numbers. This leads to a contradiction.

7.2.12 Problem Set 12

Problem 1. Given that the age difference between father and son is not more than 40. At most, how many times can the age of the father be greater than the age of the son?

Solution. *Let the age difference between father and son is equal to m. Therefore, $m \leq 40$. We are interested in the number of positive integers n, such that $n \mid m+n$. Thus, it follows that $n \mid m$. Hence, the number of such positive integers n is equal to the number of divisors of m. Therefore, the answer is 9 (for $m = 36$).*

Problem 2. Let the entries of 3×3 grid square be the numbers $1, 2, \dots, 9$, such that in any square is written only one number. Consider row and column sums. At most, how many numbers among those six numbers (sums) can be a square number?

Solution. *Let us provide an example, where four among those six sums are square numbers (see the figure below).*

1	4	8
5	9	2
7	3	6

Now, note that five among those six sums cannot be square numbers. Otherwise, the equation $a^2+b^2+c^2=45$ *must have (positive) integer solutions. This leads to a contradiction, as* $a,b,c \geq 3$.

Problem 3. Find the greatest three-digit number that is divisible by the product of its digits.

Solution. *Let* $abc \mid \overline{abc}$.

If $a=9$*, then* $9 \mid \overline{9bc}$*. Thus, it follows that* $b+c=9$*. Obviously, the numbers 918, 936, 954, 972 do not satisfy the assumptions of the problem.*

If $a=8$*, then* $8 \mid \overline{bc}$*. Note that among the numbers 816, 824, 832, 848, 856, 864, 872, 888, 896, only the number 816 satisfies the assumptions of the problem.*

Therefore, the greatest three-digit number that is divisible by the product of its digits is 816.

Problem 4. Find the number of all positive integers a, such as for each of them there exists a positive integer b, such that $b>a$ and the sum of the remainders after dividing 2016 by a and dividing 2016 by b is equal to 2016.

Solution. *According to the assumption of the problem* $2016=aq+r$ *and* $2016=bq_1+2016-r$*, where* $q,q_1 \geq 0$, $0 \leq r < a$, $0 \leq 2016-r < b$, $q,q_1,r \in \mathbb{Z}$.

Note that $2016=aq+bq_1$.

If $q_1 \neq 0$*, then from the condition* $b>a$*, it follows that* $q \neq 0$ *and the sum of the remainders is less than* $a+b$*. On the other hand,* $a+b \leq aq+bq_1=2016$*. This leads to a contradiction.*

Therefore, $q_1=0$*. Hence,* $r=0$*. We obtain that* $a \mid 2016$*. Thus, it follows that the number of such positive integers is equal to 36 (for any such a, one can choose* $b=2017$*).*

Problem 5. Find the number of integer solutions of the equation

$$x^3-3x^2y+4y^3=2016.$$

Solution. *Note that*

$$x^3-3x^2y+4y^3=(x+y)(x-2y)^2,$$

and

$$3 \mid 3y=x+y-(x-2y).$$

On the other hand, $9 \mid 2016$ *and* $27 \nmid 2016$*. Therefore, 2016 is not possible to represent as* ab^2*, where* $a,b \in \mathbb{Z}$ *and* $3 \mid a-b$.

Hence, the given equation does not have integer solutions.

Problem 6. Find the number of all positive integers n, such that

$$n^8+n^6+n^4+n^2+1,$$

is a prime number.

Solution. *If* $n=1$, *then*

$$n^8+n^6+n^4+n^2+1=5,$$

is a prime number.

If $n>1$, *then according to the formula of the sum of geometric progression*

$$n^8+n^6+n^4+n^2+1=\frac{n^{10}-1}{n^2-1}=\frac{n^5+1}{n+1}\cdot\frac{n^5-1}{n-1}=$$

$$=(n^4-n^3+n^2-n+1)(n^4+n^3+n^2+n+1).$$

Note that

$$n^4+n^3+n^2+n+1>n^4-n^3+n^2-n+1=n^3(n-1)+n(n-1)+1>1.$$

Thus, we deduce that $n^8+n^6+n^4+n^2+1$ *is a composite number.*

Therefore, the answer is 1.

Problem 7. Find the number of couples (m,n), where m, n are positive integers, such that it holds true the following equation

$$m!+136=n^2.$$

Solution. *Note that if* $m\geq 7$, *then* $m!+136$ *is divisible by 7 with a remainder of 3. Therefore, it cannot be a square number. One can easily verify that only* $m=5$ *satisfies the assumptions of the problem and in this case* $n=16$. *Thus, it follows that the number of couples* (m,n) *(where m, n are positive integers, such that the given equation holds true) is equal to 1.*

Problem 8. Find the number of all three-digit numbers n, such that the following system

$$\begin{cases} x+y+z=2^n, \\ (x-y)^2+(y-z)^2+(z-x)^2=2^n, \end{cases}$$

has an integer solution.

Solution. *Let us prove the following lemma.*

Lemma 7.10. *Consider all integers x, y, z, such that it holds true*

$$(x-y)^2+(y-z)^2+(z-x)^2 = x+y+z.$$

Prove that the set of all possible values of $x+y+z$ is a set of numbers having the form $2(a^2+3b^2)$, where $a, b \in \mathbb{Z}$.

Proof. *Note that two among the numbers x, y, z have the same parity. Assume that $y-z=2b$, where $b \in \mathbb{Z}$.*

We have that

$$2x^2-(2y+2z+1)x+y^2+z^2+4b^2-y-z=0.$$

Thus, it follows that

$$D=(2y+2z+1)^2-8(y^2+z^2+4b^2-y-z)=n^2,$$

where $n \in \mathbb{Z}$.

Hence, we obtain that

$$y+z=\frac{n^2-1}{12}+4b^2.$$

Therefore

$$y=\frac{n^2-1}{24}+2b^2+b,$$

$$z=\frac{n^2-1}{24}+2b^2-b,$$

and

$$x=\frac{n^2+5}{24}\pm\frac{n}{4}+2b^2.$$

We deduce that

$$x+y+z=\frac{(n\pm 1)^2}{8}+6b^2.$$

Note that either $n=12c\pm 1$ or $n=12c\pm 5$, where $c \in \mathbb{Z}$.

Thus, it follows that either

$$x+y+z=2((3c)^2+3b^2),$$

or

$$x+y+z=2((3c\pm 1)^2+3b^2).$$

This ends the proof of the lemma.

Now, we need to find out how many numbers among the numbers having the form $2(a^2+3b^2)$ is possible to represent as 2^n, where n is a three-digit number.

Note that if $n = 2k+1$, then

$$2((2^{k-1})^2+3(2^{k-1})^2) = 2^{2k+1} = 2^n.$$

For $n = 2k$, where $k \in \mathbb{N}$, $k > 50$, if

$$2(a^2+3b^2) = 2^{2k},$$

then a and b are even numbers.

Continuing in a similar way, we obtain that

$$e^2+3f^2 = 2.$$

This leads to a contradiction.

Therefore, the given system has an integer solution only for odd n. Hence, the answer is 450.

Problem 9. A positive integer is called "interesting", if there are two divisors among its divisors, such that their difference is equal to 2. Find the number of all "interesting" numbers of the form $n^{12}+n^{11}+\cdots+n+1$, where n is a positive integer.

Solution. *Note that if $n^{12}+n^{11}+\cdots+n+1$ is divisible by prime number p, then either $p = 13$ or $p = 13k+1$, where $k \in \mathbb{Z}$.*

Indeed, we have that $p \mid n^{13}-1$ and according to the Fermat's little theorem $p \mid n^{p-1}-1$. Thus, it follows that

$$p \mid (n^{13}-1, n^{p-1}-1).$$

On the other hand,

$$p \mid n^{(13,p-1)}-1 = (n^{13}-1, n^{p-1}-1).$$

The following cases are possible:

a) If $(13, p-1) = 13$, then $p = 13k+1$, where $k \in \mathbb{Z}$.

b) If $(13, p-1) = 1$, then $p \mid n-1$. From the condition $p \mid n^{12}+n^{11}+\cdots+n+1$, we obtain that $p \mid 13$. Therefore, $p = 13$.

Hence, any divisor of $n^{12}+n^{11}+\cdots+n+1$ is divisible by 13 either without remainder or with remainder of 1. Thus, it follows that $n^{12}+n^{11}+\cdots+n+1$ cannot be an "interesting" number.

Problem 10. Find the greatest value of C, such that the following inequality

$$(a-b)^3 > Cab,$$

holds true for any positive integers a, b satisfying the conditions $a > b$ and $a^2 + ab + b^2 \mid ab(a+b)$.

Solution. *At first, let us prove that*

$$(a-b)^3 > 3ab.$$

Note that

$$a^2 + ab + b^2 \mid a(a^2 + ab + b^2) - ab(a+b) = a^3.$$

Hence, if $(a,b) = d$, *then* $a = da_1$, $b = db_1$, $a_1^2 + a_1b_1 + b_1^2 \mid da_1^3$. *We have that* $(a_1, b_1) = 1$. *Thus, it follows that*

$$(a_1^2 + a_1b_1 + b_1^2, a_1^3) = 1.$$

Therefore,

$$a_1^2 + a_1b_1 + b_1^2 \mid d,$$

and

$$(a-b)^3 \geq d^3 \geq d^2(a_1^2 + a_1b_1 + b_1^2) = a^2 + ab + b^2 > 3ab.$$

We deduce that

$$(a-b)^3 > 3ab.$$

Let $a^2 + ab + b^2 \mid ab(a+b)$ *and* C *be such number that*

$$(a-b)^3 > Cab.$$

Now, let us take $a = (3n(n+1)+1)(n+1)$ *and* $b = (3n(n+1)+1)n$, *where* $n \in \mathbb{N}$. *We obtain that*

$$a^2 + ab + b^2 \mid ab(a+b).$$

Therefore, $3n(n+1)+1 > Cn(n+1)$. *Thus, it follows that for any* $n \in \mathbb{N}$

$$3 + \frac{1}{n(n+1)} > C.$$

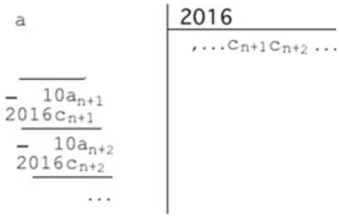
We deduce that $C \leq 3$. *Hence, the answer is 3.*

Problem 11. Given that positive integer n is not divisible by 2016 and one divides n by 2016 using the long division method. At most, how many numbers 9 can there be after the decimal point?

Solution. *Let us divide 2015 by 2016 (using the long division method).*

$$
\begin{array}{r|l}
2015 & 2016 \\ \cline{2-2}
-\ \ 0 & 0.9995 \\ \cline{1-1}
20150 & \\
-\ 18144 & \\ \cline{1-1}
20060 & \\
-\ 18144 & \\ \cline{1-1}
19160 & \\
-\ 18144 & \\ \cline{1-1}
10160 & \\
-\ 10080 & \\ \cline{1-1}
80 &
\end{array}
$$

Therefore, we have obtained three 9 after the decimal point. Now, let us prove that there cannot be four 9 after the decimal point.



Let $c_{n+1} = c_{n+2} = \cdots = c_{n+k} = 9$.
We have that

$$10a_{n+1} - 9 \cdot 2016 = a_{n+2},$$

$$10a_{n+2} - 9 \cdot 2016 = a_{n+3},$$

$$\cdots$$

$$10a_{n+k} - 9 \cdot 2016 = a_{n+k+1},$$

where $0 < a_{n+i} < 2016$, $i = 1, 2, \ldots, k$ *and* $a_{n+k+1} \geq 0$.

Let us denote $a_{n+i} = 2016 - b_{n+i}$, $i = 1, 2, \ldots, k+1$. *Then* $1 \leq b_{n+1}, b_{n+2}, \ldots, b_{n+k} < 2016$ *and* $1 \leq b_{n+k+1} \leq 2016$.

Note that $b_{n+2} = 10b_{n+1}$, $b_{n+3} = 10b_{n+2}, \ldots$, $b_{n+k+1} = 10b_{n+k}$. *Thus, it follows that*

$$b_{n+k+1} = 10^k b_{n+1}.$$

We deduce that

$$10^k \leq 10^k b_{n+1} \leq 2016.$$

Hence $k \leq 3$. *Therefore, the answer is 3.*

Problem 12. Find the number of all positive integers less than or equal to 2016, such that each of them is possible to represent as

$$\frac{m^3+n^2}{m^2n^2+1},$$

where m, n are positive integers.

Solution. *Let*

$$\frac{m^3+n^2}{m^2n^2+1}=k,$$

where $k \in \mathbb{N}$. Thus, it follows that

$$m^2(m-kn^2)=k-n^2.$$

Consider the following three cases.

a) If $k=n^2$, then $m=kn^2=n^4$.

b) If $k<n^2$, then $m<kn^2$. Therefore

$$m^2-k \leq m^2-1 \leq (m-1)\cdot kn^2 \leq (km-1)n^2.$$

We obtain that

$$m^2=\frac{n^2-k}{kn^2-m} \leq m.$$

Hence $m=1$, $k=1$ and $n>1$ is a random positive integer.

c) If $k>n^2$, then $m>kn^2$. Therefore

$$m^2 \geq m > kn^2 > k-n^2 \geq \frac{k-n^2}{m-kn^2}.$$

We deduce that

$$m^2 > \frac{k-n^2}{m-kn^2}.$$

This leads to a contradiction.

Hence, we have proven that only square numbers can be represented as

$$\frac{m^3+n^2}{m^2n^2+1},$$

where m, n are positive integers. On the other hand, the number of square numbers less than or equal to 2016 is equal to 44.

7.2.13 Problem Set 13

Problem 1. Let m, n be positive integers, such that

$$m(m,n)+n^2[m,n]=m^2+n^3-330.$$

Find $m+n$.

Solution. *Let $(m,n)=d$, therefore $d^2 \mid 2\cdot 3\cdot 5\cdot 11=330$. Thus, it follows that $d=1$. We deduce that $[m,n]=mn$. Hence*

$$m+mn^3=m^2+n^3-330.$$

This is equivalent to

$$(m-1)(m-n^3)=330.$$

We obtain that $m=23$ and $n=2$.

Thus, it follows that $m+n=25$.

Problem 2. Let m, n be positive integers, such that

$$m!+n!=(m+n+3)^2.$$

Find mn.

Solution. *Without less of generality, one can assume that $m \leq n$. Note that*

$$(2n+3)^2 \geq (m+n+3)^2 > n!.$$

On the other hand, if $n \geq 6$, then

$$(2n+3)^2 \leq 3\cdot 4(n-1)n < n!.$$

This leads to a contradiction.

Therefore, $n \leq 5$. One can easily verify and obtain that $m=4$, $n=5$.

Hence, there are only two couples satisfying the assumptions of the problem: $(4,5)$ and $(5,4)$.

We obtain that $mn=20$.

Problem 3. Find the number of all couples (m,n) of positive integers m, n, such that for any of them $n \mid 12m-1$ and $m \mid 12n-1$.

Solution. *If $m=n$, then $m \mid 12m-1$. Therefore $m=1$.*

Thus, we have obtained the couple $(1,1)$.

Note that if the couple (m,n) satisfies the assumptions of the problem, then the couple (n,m) also satisfies the assumptions of the problem.

Hence, it is sufficient to find all couples (m,n), where $m<n$.

Therefore $n \mid 12m-1$ *and* $12m-1 < 12n-1$*. Thus, we deduce that* $12m-1 = n$ *or* $12m-1 = 5n$ *or* $12m-1 = 7n$ *or* $12m-1 = 11n$*.*

If $n = 12m-1$*, then from the condition* $m \mid 12n-1$*, it follows that* $m \mid 144m-13$*. Thus, either* $m = 1$ *or* $m = 13$ *and we obtain the following couples* $(1,11)$*,* $(13,155)$*.*

If $5n = 12m-1$*, then* $m \mid 60n-5$*. Therefore* $m \mid 17$*. This leads to a contradiction.*

If $7n = 12m-1$*. In a similar way, we deduce that this case leads to a contradiction.*

If $12m-1 = 11n$*, we obtain that the couple* $(23,25)$*.*

Hence, the number of all couples satisfying the assumptions of the problem is equal to 7.

Problem 4. How many zeros does the expression

$$(4^3+1)(5^3+1)\cdot\cdots\cdot(2017^3+1)+(3^3-1)(4^3-1)\cdot\cdots\cdot(2016^3-1),$$

end with?

Solution. *Note that*

$$(k+1)^2-(k+1)+1 = k^2+k+1.$$

Thus, it follows that

$$A = (4^3+1)(5^3+1)\cdot\cdots\cdot(2017^3+1)+(3^3-1)(4^3-1)\cdot\cdots\cdot(2016^3-1) =$$

$$= 2015!\cdot(3^2+3+1)(4^2+4+1)\cdot\cdots\cdot(2016^2+2016+1)\cdot 341905705.$$

For $k \in \mathbb{N}$*, we have that*

$$5 \nmid (2k+1)^2+3 = 4(k^2+k+1).$$

Therefore

$$V_5(A) = V_5(2015!)+1 = 403+80+16+3+1 = 503.$$

On the other hand, $V_2(A) > 503$*.*

Hence, A ends with 503 zeros.

Problem 5. Find the number of integer solutions of the equation

$$(x+2015y)(x+2016y) = x.$$

Solution. *We have that*

$$2016(x+2015y)\cdot 2015(x+2016y) = 2015\cdot 2016x.$$

Denote by

$$u = 2016(x + 2015y),$$

$$v = 2015(x + 2016y).$$

Note that $u, v \in \mathbb{Z}$, $2016 \mid u$ *and* $2015 \mid v$. *On the other hand* $x = u - v$. *We have that*

$$uv = 2015 \cdot 2016(u - v).$$

This is equivalent to

$$(u + 2015 \cdot 2016)(v - 2015 \cdot 2016) = -2015^2 \cdot 2016^2.$$

Let $u + 2015 \cdot 2016 = 2016a$ *and* $v - 2015 \cdot 2016 = 2015b$, *where* $a, b \in \mathbb{Z}$.

We have that

$$x = 2016a - 2015b - 2015 \cdot 4032,$$

and

$$y = b - a + 4031.$$

Hence, we need to find the number of integer solutions of the equation

$$ab = -2015 \cdot 2016.$$

Note that $2015 \cdot 2016 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 31$ *and a is an integer divisor of this number, therefore, the number of all couples* (a, b) *is equal to* $2 \cdot 6 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 576$.

Problem 6. Find the number of all integers n, such that $n^5 - 2n^3 - n - 1$ is a prime number.

Solution. *We have that*

$$n^5 - 2n^3 - n - 1 = n^5 - 2n^3 - n^2 + n^2 - n - 1 = n^2(n^3 - 2n - 1) + n^2 - n - 1 =$$

$$= n^2(n^3 - n^2 - n + n^2 - n - 1) + n^2 - n - 1 = (n^2 - n - 1)(n^3 + n^2 + 1).$$

If $|n| \geq 3$, *then* $n^2 - n - 1 \geq 5$ *and* $|n^3 + n^2 + 1| \geq 17$. *Thus, it follows that* $n^5 - 2n^3 - n - 1$ *cannot be a prime number.*

One can easily verify that if $|n| \leq 2$, *then* $n^5 - 2n^3 - n - 1$ *is a prime number only for* $n = 2$.

Hence, the number of all integers n, such that $n^5 - 2n^3 - n - 1$ *is a prime number, is equal to 1.*

Problem 7. Let $p(x) = x^2 + x - 70$. A couple (m, n) of positive integers m, n is called “rare”, if $n > m$, $n \mid p(m)$ and $n + 1 \mid p(m + 1)$. Find the number of all “rare” couples.

Solution. *Let us prove that there do not exist "rare" couples.*

Let $n > m$, $n \mid p(m)$ and $n+1 \mid p(m+1)$. We have that

$$m^2+m-70 = nl.$$

Therefore, if $m \geq 8$, then $l > 0$ and

$$m^2+m > nl \geq (m+1)l.$$

Thus, it follows that $0 < l < m$.

Note that if $m \leq 7$, one can easily verify that there do not exist "rare" couples.

If $m \geq 8$, then we have obtained that $0 < l < m$, and

$$n+1 \mid l(n+1)+2m+2-l = nl+l+2m+2-l = m^2+m-70+2m+2 = p(m+1).$$

We deduce that

$$n+1 \mid 2m+2-l.$$

On the other hand,

$$m+2 < 2m+2-l < 2m+2 < 2n+2.$$

Therefore

$$2m+2-l = n+1.$$

We obtain that

$$m^2+m-70 = n(2m+1-n).$$

Thus, it follows that

$$(n-m)^2-(n-m)-70 = 0.$$

This leads to a contradiction.

Hence, the number of all "rare" couples is equal to 0.

Problem 8. At most, in how many ways the product of two distinct prime numbers can be represented as the sum of squares of two positive integers?

Solution. *Note that $5 \cdot 13 = 1^2+8^2 = 4^2+7^2$. Let us prove that if $p < q$ and p, q are prime numbers, then pq cannot be represented, in three ways, as the sum of squares of two positive integers. Proof by contradiction argument. Assume that*

$$pq = a^2+b^2 = c^2+d^2 = e^2+f^2,$$

where $a < c < e \leq f < d < b$, $a,b,c,d,e,f \in \mathbb{N}$. Note that

$$p^2q^2 = (ac+bd)^2+(ad-bc)^2 = (ad+bc)^2+(ac-bd)^2, \tag{7.38}$$

$$p^2q^2 = (ae+bf)^2 + (af-be)^2 = (af+be)^2 + (ae-bf)^2, \tag{7.39}$$

$$p^2q^2 = (ce+df)^2 + (cf-de)^2 = (cf+de)^2 + (ce-df)^2. \tag{7.40}$$

We have that

$$pq \mid (a^2+b^2)cd + ab(c^2+d^2) = (ac+bd)(ad+bc).$$

In a similar way, we obtain that

$$pq \mid (ae+bf)(af+be),$$

and

$$pq \mid (ce+df)(cf+de).$$

If $pq \mid ac+bd$, then by (7.38) we obtain that $ad = bc$. This leads to a contradiction.

In a similar way, we obtain that none of the numbers $ad+bc$, $ae+bf$, $af+be$, $ce+df$, $cf+de$ is divisible by pq.

Without loss of generality, one can assume that $p \mid ac+bd$ and $q \mid ad+bc$.

If $p \mid ae+bf$ and $q \mid af+be$, then

$$p \mid f(ac+bd) - d(ae+bf) = a(cf-ed),$$

and

$$q \mid b(cf-ed).$$

Note that if $p \mid a$, then $p \mid b$ and $p \mid q$. This leads to a contradiction. In a similar way, one can prove that $q \nmid b$.

Thus, it follows that $pq \mid cf-ed$. This leads to a contradiction (see (7.40)).

Therefore $p \mid ac+bd$, $q \mid ad+bc$ and $q \mid ae+bf$, $p \mid af+bc$.

Hence, from the condition $p \mid ac+bd$, $q \mid ad+bc$, we obtain that $q \mid ce+df$, $p \mid cf+de$. This leads to a contradiction, as $q \mid ae+bf$ and $p \mid af+bc$.

This ends the proof.

Problem 9. Find the smallest positive integer that can be represented at least in three different ways as the sum of squares of two positive integers.

Solution. *Note that*

$$325 = 1^2 + 18^2 = 6^2 + 17^2 = 10^2 + 15^2.$$

Let n be the smallest number, such that

$$n = a^2 + b^2 = c^2 + d^2 = e^2 + f^2, \tag{7.41}$$

where $a,b,c,d,e,f \in \mathbb{N}$ and $a < c < e \leq f < d < b$.

If $4 \mid n$, *then by* (7.41) *we obtain that* a, b, c, d, e, f *are even numbers. Thus, it follows that*

$$\frac{n}{4} = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = \left(\frac{c}{2}\right)^2 + \left(\frac{d}{2}\right)^2 = \left(\frac{e}{2}\right)^2 + \left(\frac{f}{2}\right)^2.$$

This leads to a contradiction.

We obtain that $n \nmid 4$. *In a similar way, we deduce that* $3 \nmid n$, $7 \nmid n$ *and* $11 \nmid n$.

If $2 \mid n$ *and* $4 \nmid n$, *then by* (7.41) *we obtain that* a, b, c, d, e, f *are odd numbers and*

$$\frac{n}{2} = \left(\frac{a+b}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2 = \left(\frac{c+d}{2}\right)^2 + \left(\frac{d-c}{2}\right)^2 = \left(\frac{e+f}{2}\right)^2 + \left(\frac{f-e}{2}\right)^2.$$

Therefore $f = e$ *and* $n = 2e^2$.

One can easily verify that (7.41) *does not hold true for numbers 2, 18, 50, 98, 162, 242.*

According to Problem 8, n is not a product of two distinct prime numbers, neither a prime number nor a square of a prime number (the proof is similar to the proof of Problem 8). Hence

$$n \in \{5^3, 5^2 \cdot 13 = 325\}.$$

One can easily verify that $n \neq 125$. *Thus, it follows that* $n = 325$.

Problem 10. A positive integer is called "extraordinary", if it is possible to represent as

$$\frac{a+1}{a} \cdot \frac{b+1}{b} \cdot \frac{c+1}{c} \cdot \frac{d+1}{d},$$

where $a, b, c, d \in \mathbb{N}$. Find the sum of all "extraordinary" numbers.

Solution. *Let* $k \in \mathbb{N}$, *denote by* M_k *the set of all positive integers, such that any of them is possible to represent as*

$$\frac{a_1+1}{a_1} \cdot \ldots \cdot \frac{a_k+1}{a_k},$$

where $a_1, \ldots, a_k \in \mathbb{N}$.

Hence, we need to find the sum of elements of set M_4. *Let us prove the following properties.*

P1. *If* $n \in \mathbb{M}_k$, *then* $1 < n \leq 2^k$.

Indeed, if $m \in \mathbb{N}$, *then*

$$1 < \frac{m+1}{m} \leq 2.$$

Thus, it follows that if $n \in \mathbb{M}_k$, *then there exist positive integers* $a_1, \ldots, a_k$, *such that*

$$n = \frac{a_1+1}{a_1} \cdot \dots \cdot \frac{a_k+1}{a_k}.$$

Therefore $1 < n \leq 2^k$.

P2. *If* $n \in \mathbb{M}_k$, *then* $2n \in \mathbb{M}_{k+1}$ *and* $n+1 \in \mathbb{M}_{k+1}$.

We have that

$$2n = \frac{1+1}{1} \cdot n \in \mathbb{M}_{k+1},$$

and

$$n+1 = \frac{n+1}{n} \cdot n \in \mathbb{M}_{k+1}.$$

P3. *Prove that*

$$\mathbb{M}_4 = \{2,3,4,5,6,7,8,9,10,12,16\}.$$

From the condition $2 \in \mathbb{M}_1$, *according to P1 and P2, we deduce that*

$$\mathbb{M}_2 = \{2,3,4\},$$

as $2 = \frac{4}{3} \cdot \frac{3}{2}$.

We have that

$$2 = \frac{8}{7} \cdot \frac{7}{6} \cdot \frac{3}{2} \in \mathbb{M}_3.$$

Therefore, if we prove that $7 \notin \mathbb{M}_3$, *then according to P1 and P2, we obtain that*

$$\mathbb{M}_3 = \{2,3,4,5,6,8\}.$$

Proof by contradiction argument. Assume that $7 \in \mathbb{M}_3$, *then there exist* $a_1, a_2, a_3 \in \mathbb{N}$, *such that*

$$7 = \frac{a_1+1}{a_1} \cdot \frac{a_2+1}{a_2} \cdot \frac{a_3+1}{a_3}. \tag{7.42}$$

We have that

$$7 \mid (a_1+1)(a_2+1)(a_3+1).$$

Without loss of generality, one can assume that $7 \mid a_1+1$. *Thus, it follows that*

$$\frac{a_1+1}{a_1} \leq \frac{7}{6},$$

and

$$\frac{a_1+1}{a_1} \cdot \frac{a_2+1}{a_2} \cdot \frac{a_3+1}{a_3} \leq \frac{7}{6} \cdot 2 \cdot 2 < 7.$$

This leads to a contradiction (see (7.42)).

In a similar way, we obtain that

$$11, 13, 14, 15 \notin \mathbb{M}_4.$$

We have that

$$2 = \frac{15}{14} \cdot \frac{7}{6} \cdot \frac{6}{5} \cdot \frac{4}{3} \in \mathbb{M}_4.$$

Therefore, according to P1 and P2, we obtain that

$$\mathbb{M}_4 = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\}.$$

Hence, the sum of all "extraordinary" numbers is equal to 82.

Problem 11. Let a, b, c, d be positive integers, such that
a) $a < b < c < d < 2016$,
b) $ad = bc$,
c) $a + d = 2^k$, $b + c = 2^m$, where $k, m \in \mathbb{N}$.
Find the number of all possible values of a.

Solution. *We have that*

$$\frac{a}{b} = \frac{c}{d} = \frac{u}{v},$$

where $u, v \in \mathbb{N}$ and $(u, v) = 1$.

Therefore, there exist positive integers S and t, such that $a = Su$, $b = Sv$, $c = tu$, $d = tv$.

We have that $Su < Sv < tu < tv$. Thus, it follows that

$$u < v < \frac{tu}{S}.$$

On the other hand

$$Su + tv = 2^k, \ Sv + tu = 2^m. \tag{7.43}$$

Note that

$$(a+d)^2 = (d-a)^2 + 4ad = (d-a)^2 + 4bc > (c-b)^2 + 4bc = (b+c)^2.$$

We obtain that $a + d > b + c$. Hence $k > m$.

If a is an odd number, then d is an odd number too. Therefore S, t, u, v are odd numbers.

By (7.43), we have that

$$(u+v)(S+t) = 2^m(2^{k-m} + 1),$$

$$(u-v)(S-t) = 2^m(2^{k-m} - 1). \tag{7.44}$$

Note that numbers $u+v$ *and* $u-v$ *simultaneously cannot be divisible by 4, as* $u+v+u-v=2u$. *In a similar way, numbers* $S+t$ *and* $S-t$ *simultaneously cannot be divisible by 4. Therefore, by* (7.44) *we obtain that* $u+v \geq 2^{m-1)}$ *and* $S+t \geq 2^{m-1)}$. *From the condition* $Sv+tu=2^m$, *we deduce that* $u=1$, $S=1$. *Thus, it follows that* $a=1$. *On the other hand,* $v \geq 3$, $t \geq 5$ *and* $d \geq 15$.

If a *is an even number, then* $a=2^{V_2(a)} \cdot a_1$, *where* a_1 *is an odd number.*

Hence, from the condition $a+d=2^k$, *we obtain that* $V_2(d)=V_2(a)$. *In a similar way, one can prove that* $V_2(b)=V_2(c)$. *From the condition* $ad=bc$, *we deduce that*

$$V_2(a)=V_2(b)=V_2(c)=V_2(d).$$

Thus, it follows that $a_1=1$. *Therefore*

$$a \in \{1,2,2^2,2^3,2^4,2^5,2^6,2^7\}.$$

We have that $a=2^l$, $l \in \{0,1,\dots,7\}$, *let us take* $b=3\cdot 2^l$, $c=5\cdot 2^l$, $d=15\cdot 2^l$. *Note that the obtained numbers satisfy the assumptions of the problem. Hence, the smallest possible value of* a *is equal to 8.*

Problem 12. For any positive integer n denote by M_n the set of remainders after division of 1^2-1, $2^2-1,\dots$, n^2-1, 1^2+1, $2^2+1,\dots$, n^2+1 by n. A positive integer n is called "nice", if the number of elements of set M_n is equal to n. Find the product of all "nice" even numbers.

Solution. *Let* n *be a "nice" even number. Consider the following two cases.*

a) If $n=4k$, *where* $k \in \mathbb{N}$.

Note that numbers i^2 *and* $(n-i)^2$ *are divisible by* n *with the same remainder. Therefore, numbers* 1^2-1, 2^2-1, $\dots$, $(2k)^2-1$, 1^2+1, $2^2+1,\dots$, $(2k)^2+1$ *are divisible by* n *with different remainders. This is not true if* $k>1$, *as numbers* 1^2+1 *and* $(2k-1)^2+1$ *are divisible by* n *with the same remainder.*

If $k=1$, *we obtain numbers 0, 3, 2, 5. Note that these numbers are divisible by 4 with different remainders.*

Hence, from numbers of form $4k$ *only 4 is a "nice" even number.*

b) If $n=4k+2$, *where* $k \in \mathbb{Z}$ *and* $k \geq 0$.

Note that the set of remainders after division of 1^2-1, $2^2-1,\dots$, $(2k+1)^2-1$, $(4k+2)^2-1$, 1^2+1, $2^2+1,\dots$, $(2k+1)^2-1$, $(4k+2)^2-1$ *by* n *is* M_n.

If $n=2$, *then we obtain numbers 0, 3, 2, 5. Therefore* $M_2=\{0,1\}$. *Thus, it follows that 2 is a "nice" even number.*

If $n=6$, *then we obtain numbers 0, 3, 8, 15, 24, 35, 2, 5, 10, 17, 26, 37. Therefore*

$$M_6=\{0,1,2,3,4,5\}.$$

Thus, it follows that 6 is a "nice" even number.

If $k \geq 2$, note that

$$(k+1)^2+1-((k-1)^2-1)=4k+2,$$

and

$$(k+2)^2-1-(k^2+1)=4k+2.$$

Hence, we obtain that the sums

$$1^2-1+2^2-1+\cdots+(2k+1)^2-1+(4k+2)^2-1+1^2+1+2^2+1+\cdots$$
$$+(2k+1)^2+1+(4k+2)^2+1-(k^2+1+(k-1)^2-1),$$

and

$$0+1+\cdots+4k+1,$$

are divisible by $n=4k+2$ with the same remainder. Therefore

$$4k+2 \mid 2\cdot\frac{(2k+1)(2k+2)(4k+3)}{6}-\frac{(4k+1)(4k+2)}{2}-2k^2+2k-1.$$

We deduce that

$$3(2k+1) \mid 8k^3+3k^2+7k. \tag{7.45}$$

Note that

$$12(8k^3+3k^2+7k)=(6k+3)(16k^2-2k+15)-45.$$

We deduce that $2k+1 \mid 15$. Thus, it follows that either $k=2$ or $k=7$.

If $k=7$, then note that (7.45) does not hold true.

If $k=2$, $n=10$, we obtain numbers 0, 3, 8, 15, 24, 35, 48, 63, 80, 99, 2, 5, 10, 26, 37, 50, 65, 82, 101. Therefore

$$M_{10}=\{0,1,2,3,4,5,6,7,8,9\}.$$

Thus, it follows that 10 is a "nice" even number.

Hence, the product of all "nice" even number is equal to $2\cdot 4\cdot 6\cdot 10=480$.

7.2.14 Problem Set 14

Problem 1. At most, how many composite numbers can one choose from the numbers $100, 101, \ldots, 500$, such that any two among the chosen numbers are mutually prime?

Solution. *Note that if a positive integer p is the smallest divisor of a composite number a, then p is a prime number and* $p \leq \sqrt{a}$*. The proof is straightforward; we have that* $a = pd$*, where* $d \geq p$*. Thus, it follows that* $a \geq p^2$*. Hence,* $p \leq \sqrt{a}$*. On the other hand, if p is not a prime number, then a has a divisor smaller than p.*

Note that any composite number among the chosen numbers is divisible by

$$2, 3, 5, 7, 11, 13, 17, 19.$$

Therefore, the number of the chosen numbers is not more than 8.

Let us provide an example of such eight numbers:

$$121, 125, 128, 169, 217, 243, 289, 361.$$

Problem 2. Let p, q, r be prime numbers, such that

$$\frac{1}{pq} + \frac{1}{qr} + \frac{1}{pr} = \frac{1}{839}.$$

Find the smallest possible value of $p + q + r$.

Solution. *We have that*

$$pqr = 839(p + q + r).$$

Note that 839 is a prime number, thus, one of the numbers p, q, r is equal to 839. Without loss of generality, one can assume that $p = 839$*, then it follows that*

$$(q-1)(r-1) = 840.$$

Assume that $q \leq r$*, then either* $q = 3$*,* $r = 421$ *or* $q = 13$*,* $r = 71$ *or* $q = 29$*,* $r = 31$.

Hence, we obtain that the smallest possible value of $p + q + r$ *is equal to 899.*

Problem 3. Find the smallest positive odd number that is not possible to represent as

$$\frac{2^m - 1}{2^n + 1},$$

where $m, n \in \mathbb{N}$.

Solution. *We have that*

$$1 = \frac{2^2 - 1}{2 + 1},\ 3 = \frac{2^4 - 1}{2^2 + 1},\ 5 = \frac{2^4 - 1}{2 + 1},\ 7 = \frac{2^6 - 1}{2^3 + 1}.$$

Note that the equation

$$9 = \frac{2^m - 1}{2^n + 1},$$

does not have a positive integer solution. We proceed the proof by contradiction argument. Assume that

$$9 \cdot 2^n + 9 = 2^m - 1.$$

Thus, it follows that

$$2^{m-1} - 9 \cdot 2^{n-1} = 5.$$

Hence, we obtain that either $n = 1$ or $m = 1$. This leads to a contradiction.

Therefore, the smallest odd number satisfying the assumptions of the problem is equal to 9.

Problem 4. Find the smallest number that is possible to represent at least in twenty different ways as $5m + 7n$, where m, n are non-negative integers.

Solution. *Assume that a is the required number, then $a = 5m_i + 7n_i$, where $i = 1, \ldots, 20$, $m_i, n_i \in \mathbb{Z}$, $m_i \geq 0$, $n_i \geq 0$. Let $m_1 = \min(m_1, \ldots, m_{20})$. We have that*

$$5(m_i - m_1) = 7(n_1 - n_i), i = 2, \ldots, 20,$$

thus it follows that $m_i - m_1 = 7l_{i-1}$ and $n_1 - n_i = 5l_{i-1}$, where $l_1, \ldots, l_{19}$ are pairwise distinct positive integers. Therefore

$$m_i = \max(m_1, \ldots, m_{20}) \geq 7 \cdot 19 = 133.$$

Hence, we obtain that

$$a = 5m_i + 7n_i \geq 5m_i = 665.$$

Note that

$$665 = 5 \cdot (7i) + 7 \cdot (5(19 - i)),$$

where $i = 0, 1, \ldots, 19$. Thus, the smallest possible number satisfying the assumptions of the problem is 665.

Problem 5. Let x, y be positive integers, such that

$$2201025(x^3y^3 + x^2 - y^2) = 6902413244(xy^3 + 1).$$

Find $x + y$.

Solution. *Let us rewrite the given equation in the following way*

$$x^2 - \frac{1}{xy + \frac{1}{y^2}} = 56^2 - \frac{1}{56 \cdot 34 + \frac{1}{34^2}}. \tag{7.46}$$

We have that $x, y \in \mathbb{N}$*; therefore, from* (7.46) *we deduce that* $x^2 = 56^2$ *and*

$$\frac{1}{xy + \frac{1}{y^2}} = \frac{1}{56 \cdot 34 + \frac{1}{34^2}}.$$

Thus, it follows that $y = 34$.
Hence, we obtain that $x + y = 90$.

Problem 6. Let m, n, k, s be positive integers, such that

$$(3^m - 3^n)^2 = 2^k + 2^s.$$

Find the greatest possible value of the product $mnks$.

Solution. *Without loss of generality, one can assume that* $m > n$ *and* $k > s$*. We have that*

$$3^{2n}(3^{m-n} - 1)^2 = 2^s(2^{k-s} + 1).$$

Thus, it follows that

$$3^{2n}a = 2^{k-s} + 1,$$

and

$$(3^{m-n} - 1)^2 = 2^s a,$$

where $a \in \mathbb{N}$*. From the first equation, we obtain that* a *is an odd number. On the other hand, from the second equation, we obtain that* $s = 2l$*, where* $l \in \mathbb{N}$ *and* $a = b^2$*, where* $b \in \mathbb{N}$*. Therefore*

$$3^{2n}b^2 = 2^{k-s} + 1,$$

and

$$3^{m-n} = 2^l b + 1.$$

Thus, it follows that

$$(3^n b - 1)(3^n b + 1) = 2^{k-s}.$$

Therefore

$$3^n b - 1 = 2^\alpha,$$

and

$$3^n b + 1 = 2^\beta,$$

where α*,* $\beta \in \mathbb{N}$*. We deduce that*

$$2^\beta - 2^\alpha = 2.$$

We obtain that $\alpha = 1$ *and* $\beta = 2$. *Hence* $b = 1$, $n = 1$ *and* $k - s = 3$. *Therefore*

$$3^{m-1} = 2^l + 1,$$

where $s = 2l$.

If $m - 1$ *is an even number, we have that* $m - 1 = 2$, $l = 3$, *then* $m = 3$, $n = 1$, $s = 6$, $k = 9$ *and* $mnks = 162$.

If $m - 1$ *is an odd number, we have that* 3^{m-1} *is divisible by 4 with the remainder of 3, then* $l = 1$, $m = 2$. *Hence* $s = 2$, $k = 5$. *Thus*

$$mnks = 2 \cdot 1 \cdot 5 \cdot 2 = 20.$$

Therefore, the greatest possible value of the product mnks is equal to 162.

Problem 7. Find the smallest positive integer that is not possible to represent as

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca},$$

where $a, b, c \in \mathbb{N}$.

Solution. *Note that*

$$1 = \frac{1^2 + 1^2 + 1^2}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1}, \; 2 = \frac{1^2 + 1^2 + 4^2}{1 \cdot 1 + 1 \cdot 4 + 1 \cdot 4}.$$

Let us prove that 3 is not possible to represent in this form.

We proceed the proof by contradiction argument. Assume that

$$\frac{a_0^2 + b_0^2 + c_0^2}{a_0 b_0 + b_0 c_0 + c_0 a_0} = 3,$$

where (a_0, b_0, c_0) *is such a solution of the equation*

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} = 3,$$

that the sum $a + b + c$ *is the smallest.*

We have that

$$(a_0 + b_0 + c_0)^2 = 5(a_0 b_0 + b_0 c_0 + c_0 a_0).$$

Therefore $5 \mid a_0 + b_0 + c_0$. *Thus, it follows that*

$$5 \mid a_0 b_0 + b_0 c_0 + c_0 a_0.$$

We obtain that

$$5 \mid a_0^2 + b_0^2 + c_0^2.$$

Hence, one of the numbers a_0, b_0, c_0 *is divisible by 5. Thus, from the condition* $5 \mid a_0b_0 + b_0c_0 + c_0a_0$, *we deduce that two among the numbers* a_0, b_0, c_0 *are divisible by 5. On the other hand, from the condition* $5 \mid a_0 + b_0 + c_0$, *it follows that any of the numbers* a_0, b_0, c_0 *is divisible by 5. Therefore*

$$3 = \frac{\left(\frac{a_0}{5}\right)^2 + \left(\frac{b_0}{5}\right)^2 + \left(\frac{c_0}{5}\right)^2}{\frac{a_0}{5} \cdot \frac{b_0}{5} + \frac{b_0}{5} \cdot \frac{c_0}{5} + \frac{c_0}{5} \cdot \frac{a_0}{5}}.$$

This leads to a contradiction, as

$$\frac{a_0}{5}, \frac{b_0}{5}, \frac{c_0}{5} \in \mathbb{N},$$

and

$$\frac{a_0}{5} + \frac{b_0}{5} + \frac{c_0}{5} < a_0 + b_0 + c_0.$$

Hence, the smallest positive integer satisfying the assumptions of the problem is equal to 3.

Problem 8. Let a, b, c be nonzero integers, such that

$$a^3 + b^3 + c^3 = 2.$$

Find the smallest possible value of $|a + b + c|$.

Solution. *Let us prove that* $6 \mid a + b + c - 2$.

We have that

$$6 \mid a - a^3 + b - b^3 + c - c^3 = a + b + c - 2,$$

as if $m \in \mathbb{Z}$, *then* $6 \mid (m-1)m(m+1) = m^3 - m$.

If we prove that $a + b + c \neq 2$, *then from the condition* $6 \mid a + b + c - 2$, *it follows that either* $a + b + c \leq -4$ *or* $a + b + c \geq 8$. *Hence* $|a + b + c| \geq 4$.

Indeed, if $a + b + c = 2$ *and* $a^3 + b^3 + c^3 = 2$, *then from the condition* $b + c \mid b^3 + c^3$, *we deduce that* $\frac{2 - a^3}{2 - a} \in \mathbb{Z}$. *On the other hand*

$$\frac{a^3 - 2}{a - 2} = a^2 + 2a + 4 + \frac{6}{a - 2}.$$

Therefore

$$a \in \{-4, -1, 1, 3, 4, 5, 8\}.$$

In a similar way, we obtain that

$$b, c \in \{-4, -1, 1, 3, 4, 5, 8\}.$$

By straightforward verification, one can see that from the numbers -4, -1, 1, 3, 4, 5, 8 is not possible to choose a triple (a, b, c), such that $a+b+c=2$ and $a^3+b^3+c^3=2$.

Hence $|a+b+c| \geq 4$. On the other hand, $7^3+(-5)^3+(-6)^3=2$. We obtain that the possible smallest value of $|a+b+c|$ is equal to 4.

Problem 9. Find the number of triples (x, y, z) of positive integers x, y, z, such that the following conditions hold true

$$x+y+z \leq 100,$$

and

$$x^2+y^2+z^2 = 2xy+2yz+2zx.$$

Solution. *At first, let us find all solutions (x, y, z) of the equation*

$$x^2+y^2+z^2 = 2xy+2yz+2zx, \tag{7.47}$$

such that $x \geq y \geq z$ and $x, y, z \in \mathbb{N}$.

Note that if (x, y, z) is a solution of (7.47), where $x, y, z \in \mathbb{N}$, then

$$(x+y-z)^2 = 4xy.$$

Thus, it follows that xy is a square of a positive integer. In a similar way, we obtain that yz and xz are squares of positive integers.

On the other hand, if p is a prime number, such that $p \mid x$, $p \mid y$, then $p \mid z$. Moreover, the triple

$$\left(\frac{x}{p}, \frac{y}{p}, \frac{z}{p}\right),$$

is also a solution of (7.47).

Therefore, any solution of (7.47) is of the form (da^2, db^2, dc^2), where $d, a, b, c \in \mathbb{N}$.

Let $x \geq y \geq z$, hence $a \geq b \geq c$. We have that

$$a^4+b^4+c^4-2a^2b^2-2b^2c^2-2a^2c^2 = 0.$$

We deduce that

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 0.$$

Thus, it follows that $a = b+c$.

Hence, solutions (x,y,z) *of* (7.47)*, where* $x \geq y \geq z$ *and* $x,y,z \in \mathbb{N}$*, are of the form* $(d(b+c)^2, db^2, dc^2)$*, where* $d,b,c \in \mathbb{N}$ *and* $(b,c)=1$.

Now, let us provide all such triples satisfying the condition

$$x+y+z \leq 100.$$

$(4d,d,d)$*, where* $d=1,2,\ldots,16$.
$(9d,4d,d)$*, where* $d=1,2,\ldots,7$.
$(16d,9d,d)$*, where* $d=1,2,3$.
$(25d,16d,d)$*, where* $d=1,2$.
$(36d,25d,d)$*, where* $d=1$.
$(49d,36d,d)$*, where* $d=1$.
$(25d,9d,4d)$*, where* $d=1,2$.
$(49d,25d,4d)$*, where* $d=1$.
$(49d,16d,9d)$*, where* $d=1$.
$(64d,25d,9d)$*, where* $d=1$.

Therefore, the total number of triples satisfying the assumptions of the problem is equal to

$$3\cdot 16+6\cdot 7+6\cdot 3+6\cdot 2+6\cdot 1+6\cdot 1+6\cdot 2+6\cdot 1+6\cdot 1+6\cdot 1=162.$$

Problem 10. A positive integer greater than one is called “uninteresting”, if it is not possible to represent as

$$\frac{a+1}{a}\cdot\frac{b+1}{b}\cdot\frac{c+1}{c}\cdot\frac{d+1}{d}\cdot\frac{e+1}{e}\cdot\frac{f+1}{f},$$

where a, b, c, d, $e,f \in \mathbb{N}$. Find the smallest “uninteresting” number.

Solution. *Let* $k \in \mathbb{N}$*, denote by* M_k *the set of all positive integers, such that any of them is possible to represent as*

$$\frac{a_1+1}{a_1}\cdot\frac{a_2+1}{a_2}\cdot\ldots\cdot\frac{a_k+1}{a_k},$$

where $a_1,\ldots,a_k \in \mathbb{N}$*. We need to find the smallest positive integer greater than one that does not belong to* M_6*. Let us prove the following properties.*

P1. $2 \in M_k$.

We have that $2=\dfrac{1+1}{1} \in M_1$ *and for* $k \geq 2$

$$2=\frac{2^k}{2^k-1}\cdot\frac{2^k-1}{2^k-2}\cdot\frac{2^{k-1}-1}{2^{k-1}-2}\cdot\ldots\cdot\frac{2^2-1}{2^2-2}.$$

P2. *If* $n \in M_k$*, then* $2n \in M_{k+1}$ *and* $n+1 \in M_{k+1}$.

We have that

$$2n = \frac{1+1}{1} \cdot n \in M_{k+1},$$

and

$$n+1 = \frac{n+1}{n} \cdot n \in M_{k+1}.$$

On the other hand, we have that $2 \in M_1$. *Therefore*

$$\{2,3,4\} \subseteq M_2,\ \{2,3,4,5,6,8\} \subseteq M_3,\ \{2,3,4,5,6,7,8,9,10,12,16\} \subseteq M_4,$$

$$\{2,3,4,5,6,7,8,9,10,11,12,13,14,16,17,18,20,24,32\} \subseteq M_5.$$

Thus, it follows that

$$\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22\} \subseteq M_6.$$

Let us prove that $23 \notin M_6$.

We proceed the proof by contradiction argument. Assume that $23 \in M_6$.

We have that

$$23 = \frac{a_1+1}{a_1} \cdot \frac{a_2+1}{a_2} \cdot \frac{a_3+1}{a_3} \cdot \frac{a_4+1}{a_4} \cdot \frac{a_5+1}{a_5} \cdot \frac{a_6+1}{a_6},$$

where $a_1, a_2, \ldots, a_6 \in \mathbb{N}$.

Let $23 \mid a_1+1$, *then* $a_1+1 = 23k$, $k \in \mathbb{N}$, $a_1 = 23k-1$.

If $23k-1$ *has a prime divisor greater than or equal to 5, then from the condition* $(k, 23k-1) = 1$, *it follows that*

$$23 \leq \frac{23k}{23k-1} \cdot \frac{5}{4} \cdot 2 \cdot 2 \cdot 2 \cdot 2.$$

Therefore $3k \leq 1$. *This leads to a contradiction.*

If 9 is a divisor of $23k-1$, *then*

$$23 \leq \frac{23k}{23k-1} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot 2 \cdot 2 \cdot 2.$$

This leads to a contradiction.

Hence $k \geq 5$. *Thus, it follows that* $2^6 \mid 23k-1$. *Therefore*

$$23 \leq \frac{23k}{23k-1} \cdot \frac{4}{3} \cdot 2 \cdot 2 \cdot 2 \cdot 2.$$

We deduce that $69k-3 \leq 64k$. *This leads to a contradiction.*

Thus, the statement does not hold true. We obtain that the smallest "uninteresting" number is equal to 23.

Problem 11. A positive integer n is called "amazing", if there exist pairwise distinct integers a, b, c, d, such that

$$n^2 = a+b+c+d,$$

and any among the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ is a square of an integer number. Find the smallest "amazing" number.

Solution. *At first, let us prove that a square of any "amazing" number n is possible to represent, at least in three different ways, as the sum of the squares of two non-negative integers. We have that*

$$n^2 = a+b+c+d,$$

and any among the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ is a square of an integer number, where a, b, c, $d \in \mathbb{Z}$.

Let $a < b < c < d$, then

$$n^2 = (a+b)+(c+d) = (a+c)+(b+d) = (b+c)+(a+d).$$

We have that $a+b < c+d$, $a+c < b+d$, $a+b < a+c$ and $a+c < \min(b+c, a+d)$.

This ends the proof of the statement.

Let us prove that if n is the smallest positive integer, such that n^2 is possible to represent, at least in three different ways, as the sum of the squares of two non-negative integers, then $n \geq 25$.

Note that $n^2 \geq 4+4 = 8$. Thus, it follows that $n > 2$.

If n is an even number, then $\left(\frac{n}{2}\right)^2$ is possible to represent, at least in three different ways, as the sum of two non-negative integers. Hence, without loss of generality, one can assume that $2 \nmid n$. In a similar way, we obtain that $3 \nmid n$.

One can easily prove that n is not a prime number; nevertheless, it is sufficient to check that $n \notin \{5, 7, 11, 13, 17, 19, 23\}$.

Therefore, n is either a prime number greater than 25 or a product of at least two prime numbers greater than 3. We deduce that $n \geq 25$.

On the other hand, $n = 25$ is an "amazing" number, as for the numbers $a = -88$, $b = 88$, $c = 137$, $d = 488$ the assumptions of the problem hold true.

Hence, the smallest "amazing" number is equal to 25.

Problem 12. Let $C(k)$ denotes the sum of all different prime divisors of a positive integer k. For example, $C(1) = 0$, $C(2) = 2$, $C(45) = 8$. Find all positive integers n such that $C(2^n + 1) = C(n)$.

Solution. *Let us prove that $n = 3$. For every positive integer t denote by $P(t)$ the greatest prime divisor of t. Let us separate the greatest odd divisor of n, then $n = 2^k m$. Therefore, $2^n + 1 = 2^{2^k m} + 1 = a^m + 1$, where $a = 2^{2^k}$. If $k > 0$, then n is odd and $C(n) = C(m) + 2$, $C(2^n + 1) = C(a^m + 1)$.*
Let us prove the following lemmas.

Lemma 7.11. *For every prime $p > 2$, it holds true $P\left(\frac{a^p+1}{a+1}\right) = p$ or $P\left(\frac{a^p+1}{a+1}\right) \geq 2p+1$.*

Proof. *Let $P\left(\frac{a^p+1}{a+1}\right) = q$. By Fermat's little theorem, q divides $a^{q-1} - 1$. Hence, we obtain that $(a^{2p} - 1, a^{q-1} - 1) = a^{(2p,q-1)-1}$. The greatest common divisor of $(2p, q-1)$ is even; thus, it is either equal to $2p$ or 2. In the first place, $q - 1$ is divisible by $2p$, hence $q \geq 2p + 1$. In the second case, $a^2 - 1$ is divisible by q, but $a - 1$ is not divisible by q (as $a^p + 1$ is divisible by q). We deduce that $a \equiv -1 (mod q)$. Thus, it follows that*

$$\frac{a^p+1}{a+1} = a^{p-1} - \cdots + 1 \equiv p(mod q).$$

Hence, we obtain that $p = q$.

Lemma 7.12. *If p_1, p_2 are distinct prime numbers, then $P\left(\frac{a^{p_1}+1}{a+1}\right)$ is not equal to $P\left(\frac{a^{p_2}+1}{a+1}\right)$.*

Proof. *We proceed the proof by contradiction argument. Assume that $P\left(\frac{a^{p_1}+1}{a+1}\right) = P\left(\frac{a^{p_1}+1}{a+1}\right) = q$, then both numbers $a^{2p_1} - 1$ and $a^{2p_2} - 1$ are divisible by q. Thus, it follows that $(a^{2p_1} - 1, a^{2p_2} - 1) = a^{(2p_1,2p_2)} - 1 = a^2 - 1$ is divisible by q and this means that $a + 1$ is also divisibly by q. In that case $p_1 = q$ and $p_2 = q$. This leads to a contradiction.*

Now, let us continue the solution of the problem. Let n has odd prime divisors $p_1, \ldots, p_s$. From the second lemma, it follows that

$$C(2^n+1) \geq P\left(\frac{a^{p_1}+1}{a+1}\right) + \cdots + P\left(\frac{a^{p_s}+1}{a+1}\right).$$

If $C(2^n+1) > P\left(\frac{a^{p_1}+1}{a+1}\right) + \cdots + P\left(\frac{a^{p_s}+1}{a+1}\right)$, then $2^n + 1$ has at least one other prime divisor (besides the ones on the right-hand side). Therefore,

$$C(2^n+1) \geq P\left(\frac{a^{p_1}+1}{a+1}\right) + \cdots + P\left(\frac{a^{p_s}+1}{a+1}\right) + \geq p_1 + \cdots + p_s + 3 \geq C(n).$$

Hence, it is enough to consider the following case

$$C(2^n+1) = P\left(\frac{a^{p_1}+1}{a+1}\right) + \cdots + P\left(\frac{a^{p_s}+1}{a+1}\right).$$

If in this case there exists i, such that $P\left(\frac{a^{p_i}+1}{a+1}\right) \neq p_i$, *then* $C(2^n+1) \geq p_1 + \cdots + p_s + p_i + 1 > C(n)$. *It is left to consider the case, when* $P\left(\frac{a^{p_i}+1}{a+1}\right) = p_i$ *for all i. In this case, we have that* $C(n) = C(2^n+1) = p_1 + \cdots + p_s$. *Therefore, n is odd and* $a = 2$. *On the other hand, we have that* $2^p \equiv 2 \pmod p$ *for all odd p. Hence, for* $p > 3$ *the number* $2^p + 1$ *is not divisible by p. We deduce that* $s = 1$, $p = 3$, $n = 3^r$, *where r is a positive integer. Then, the number* $2^n + 1 = 2^{3^r} + 1$ *must be some power of 3. But note that for example for* $r = 2$ *and in general for* $r \geq 2$ *this number is divisible by 19. Hence, only* $n = 3$ *is possible. Obviously* $n = 3$ *satisfies to the assumptions of the problem.*

7.2.15 Problem Set 15

Problem 1. Let m, n, k be such integers that

$$4^m \cdot 14^n \cdot 21^k = 2016.$$

Find $m^2 + n^2 + k^2$.

Solution. *We have that*

$$2^5 \cdot 3^2 \cdot 7 = 2^{2m+n} \cdot 7^{n+k} \cdot 3^k.$$

Thus, it follows that $k = 2$, $n = -1$, $m = 3$.

Hence, we obtain that

$$m^2 + n^2 + k^2 = 14.$$

Problem 2. Find the smallest odd positive integer that is not possible to represent as

$$\frac{2^m+1}{2^n-1},$$

where $m, n \in \mathbb{N}$.

Solution. *We have that*

$$1 = \frac{2+1}{2^2-1},\ 3 = \frac{2+1}{2-1},\ 5 = \frac{2^2+1}{2-1}.$$

Note that 7 is not possible to represent in this way. We proceed the proof by contradiction argument. Assume that

$$1 = \frac{2^m + 1}{2^n - 1},$$

where $m, n \in \mathbb{N}$. Thus, it follows that

$$7(2^n - 1) = 2^m + 1.$$

This leads to a contradiction, as 2^m is divisible by 7 with a remainder of 1, 2 or 4.

Therefore, the smallest odd positive integer that is not possible to represent in the given form is equal to 7.

Problem 3. Let p, q be prime numbers, such that

$$p^2 - 2q^2 = 2801.$$

Find $p + q$.

Solution. *We have that*

$$(p-1)(p+1) = 2q^2 + 2800.$$

Thus, it follows that $8 \mid 2q^2$. Hence, we obtain that $q = 2$ and $p = 53$. Therefore $p + q = 55$.

Problem 4. Find the number of all positive integers n, such that the equation

$$x^z + y^z = 2^n,$$

does not have any solution in the set of positive integers, where $z > 1$.

Solution. *If $n = 1$, then $x = y = 1$, $z = 2$ is a solution of the given equation.*

If $n = 2m + 1$, where $m \in \mathbb{N}$, then $x = 2$, $y = 2$, $z = 2m$ is a solution of the given equation.

If $n = 2m$, $m > 1$, then $x = 2$, $y = 2$, $z = 2m - 1$ is a solution of the given equation.

If $n = 2$, then the equation $x^z + y^z = 4$ does not have any solution in the set of positive integers.

Hence, the number of all positive integers n, such that the given equation does not have any solution in the set of positive integers, is equal to 1.

Problem 5. Find the number of all positive integers, such that any of them is not possible to represent as

$$\frac{xyz}{x + y + z},$$

where x, y, z are positive integers.

Solution. *Let us prove that any positive integer n is possible to represent in the given form. We have that*

$$n = \frac{n(n+2)\cdot(n+1)\cdot 1}{n(n+2)+(n+1)+1}.$$

This ends the proof.

Problem 6. Let x, y, z be integer numbers, such that

$$(x-y)^2+(y-z)^2+(z-x)^2 = 2016.$$

Find $|x-y|+|y-z|+|z-x|$.

Solution. *Let $x \geq y \geq z$. We denote $x-z=a$, $y-z=b$, $x-y=a-b$, where $a \geq b \geq 0$, $a,b \in \mathbb{Z}$. Therefore*

$$a^2+b^2+(a-b)^2 = 2016. \tag{7.48}$$

We have that $2016 = 32\cdot 63$. From (7.48), we deduce that $4 \mid a$, $4 \mid b$. Thus, it follows that $a = 4a_1$, $b = 4b_1$, where $a_1,b_1 \in \mathbb{Z}$, $a_1,b_1 \geq 0$, $a_1 \geq b_1$. Note that

$$a_1^2+b_1^2+(a_1-b_1)^2 = 126.$$

Hence, by straightforward verification, we obtain that $a_1 = 9$, $b_1 = 6$, $a_1-b_1 = 3$ or $a_1 = 9$, $b_1 = 3$, $a_1-b_1 = 6$. Therefore

$$|x-y|+|y-z|+|z-x| = 4\cdot 18 = 72.$$

Problem 7. Let a, b, c, d be positive integers, such that $a+b+c+d \leq 100$. Given that the sum of the squares of any two among the numbers a, b, c, d is divisible by the product of the other two numbers. Find the number of all possible values of the product $abcd$.

Solution. *Let m be the greatest common divisor of numbers a, b, c, d. We have that $a = ma_1$, $b = mb_1$, $c = mc_1$, $d = md_1$, where a_1, b_1, c_1, $d_1 \in \mathbb{N}$.*

Now, let us prove that numbers a_1, b_1, c_1, d_1 are of the form 2^k, where $k \geq 0$ and $k \in \mathbb{Z}$.

We proceed the proof by contradiction argument. Assume that $p \mid a_1$, where $p > 2$ is a prime number, then $p \mid b_1^2+c_1^2$, $p \mid c_1^2+d_1^2$, $p \mid b_1^2+d_1^2$. Thus, it follows that $p \mid b_1$, $p \mid c_1$, $p \mid d_1$. This leads to a contradiction.

Hence, we obtain that $a = 2^\alpha \cdot m$, $b = 2^\beta \cdot m$, $c = 2^\gamma \cdot m$, $d = 2^\delta \cdot m$, where α, β, γ, $\delta \geq 0$ and α, β, γ, $\delta \in \mathbb{Z}$.

Without loss of generality, one can assume that $\alpha \leq \beta \leq \gamma \leq \delta$. We have that

$$\gamma+\delta \leq V_2(a_1^2+b_1^2) \leq 2\beta+1.$$

Therefore $\gamma = \beta$.

If $\alpha < \beta$, then $\gamma + \delta \le V_2(a_1^2 + b_1^2) = 2\alpha$. This leads to a contradiction. We deduce that $\alpha = \beta = \gamma = 0$ and either $\delta = 0$ or $\delta = 1$.

Thus, it follows that either $abcd = m^4$, where $m = 1, 2, \ldots, 25$ or $abcd = 2m^4$, where $m = 1, 2, \ldots, 20$.

Hence, the greatest possible value of the product $abcd$ is equal to 45.

Problem 8. Find the number of all triples (a, b, c), where $a, b, c \in \mathbb{N}$, such that it holds true

$$a + b + c \le 100,$$

and

$$(a+b)(b+c)(c+a) = 2^n,$$

where $n \in \mathbb{N}$.

Solution. *If $a < b < c$, then $a + b < b + c < a + c$. Therefore $a + b = 2^r$, $b + c = 2^s$, $a + c = 2^t$, where $r, s, t \in \mathbb{N}$, $r < s < t$. Thus, it follows that*

$$b = \frac{2^r + 2^s - 2^t}{2} < 0,$$

as $2^t = 1 + 2 + \cdots + 2^{t-1} + 1$. Hence, we obtain that $b < 0$. This leads to a contradiction.

Therefore, two among those numbers are equal. Let $a = b$, then

$$2a \cdot (a+c)(a+c) = 2^n.$$

We deduce that $a = 2^m$, $a + c = 2^k$. Thus $a = 2^m$, $b = 2^m$, $c = 2^k - 2^m$, where $k, m \in \mathbb{Z}$, $m \ge 0$, $k > m$.

From the condition $a + b + c \le 100$, it follows that

$$2^k + 2^m \le 100.$$

Hence, we obtain that $k \le 6$.

Therefore, the number of all triples (a, b, c), where $a, b, c \in \mathbb{N}$, such that the given assumptions hold true, is equal to

$$1 + 4 + 7 + 10 + 13 + 16 = 51.$$

Problem 9. Let a, b be distinct positive integers, such that

$$a^2 + b^3 \mid a^3 + b^2.$$

Find the smallest possible value of the sum $a + b$.

Solution. *Let* $(a,b) = d$. *We have that*

$$a^2 + b^3 \mid a(a^2 + b^3) - (a^3 + b^2).$$

Therefore

$$a^2 + b^3 \mid b^2(ab - 1). \tag{7.49}$$

Let $a = da_1$, $b = db_1$, *where* a_1, $b_1 \in \mathbb{N}$ *and* $a_1 > b_1$.

From (7.49), *it follows that*

$$a_1^2 + db_1^3 \mid b_1^2(d^2a_1b_1 - 1). \tag{7.50}$$

We have that $(a_1, b_1) = 1$. *Therefore*

$$(b_1^2, a_1^2 + db_1^3) = 1.$$

From (7.50), *we obtain that*

$$a_1^2 + db_1^3 \mid d^2a_1b_1 - 1. \tag{7.51}$$

From (7.51), *we deduce that* $d > 1$.

Note that if $a_1 = 2$, *then* $b_1 = 1$, *and from the condition* $d + 4 \mid 2d^2 - 1$, *we obtain that* $d = 27$. *Hence* $a + b = 81$.

Therefore $d \geq 2$, $a_1 \geq 3$, $b_1 \geq 1$. *Thus, it follows that* $a + b \geq 8$.

If $a = 6$, $b = 2$, *then* $6^2 + 2^3 \mid 6^3 + 2^2$.

Hence, the smallest possible value of the sum $a + b$ *is equal to 8.*

Problem 10. Find the number of all triples (a,b,c), where $a,b,c \in \mathbb{Z}$, such that it holds true

$$a + b + c = 1,$$

and

$$a^5 + b^5 + c^5 = 31.$$

Solution. *We have that*

$$b + c = 1 - a,$$

and

$$b^5 + c^5 = 31 - a^5.$$

On the other hand, $b + c \mid b^5 + c^5$. *Therefore*

$$1 - a \mid 31 - a^5. \tag{7.52}$$

Note that

$$x^5 - 31 = (x-1)q(x) - 30,$$

where $q(x)$ is a polynomial with integer coefficients. Hence, from (7.52) we deduce that $1-a \mid 30$. Thus, it follows that

$$a \in \{-29, -14, -9, -5, -4, -2, -1, 0, 2, 3, 4, 6, 7, 11, 16, 31\} = M.$$

In a similar way, we obtain that $b, c \in M$.

Note that among the triples (a,b,c) satisfying the condition $a+b+c=1$, only the following six triples satisfy the assumptions of the problem:

$$(-1,0,2),\ (-1,2,0),\ (0,-1,2),\ (0,2,-1),\ (2,-1,0),\ (2,0,-1).$$

Therefore, the number of triples (a,b,c), where $a,b,c \in \mathbb{Z}$, satisfying the given assumptions of the problem is equal to 6.

Problem 11. Let x, y, z be positive integers, such that

$$(x^2+y)(x+y^2) = 3^z.$$

Find the greatest possible value of the product xyz.

Solution. *We have that*

$$x^2 + y = 3^\alpha,$$

and

$$y^2 + x = 3^\beta,$$

where $\alpha, \beta \in \mathbb{N}$ and $\alpha + \beta = z$. Let $\alpha \geq \beta$. Note that $\alpha \neq \beta$, otherwise $x = y$ and $x^2 + y$ is an even number. Therefore $\alpha > \beta$.

Subtracting from the first equation the second equation, we obtain that

$$(x-y)(x+y-1) = 3^\beta(3^{\alpha-\beta} - 1). \tag{7.53}$$

Note that $x-y$ and $x+y-1$ cannot be simultaneously divisible by 9. Otherwise, x and y are divisible by 9 with a remainder of 5. Therefore, x^2+y is divisible by 9 with a remainder of 3. This leads to a contradiction.

On the other hand, $x+y-1$ is an even number and $3 \mid x-y$. We have that

$$3^\beta = x + y^2 \geq x + y.$$

From (7.53), we deduce that

$$\begin{cases} x + y - 1 = 2 \cdot 3^{\beta-1}, \\ x - y = \dfrac{3(3^{\alpha-\beta} - 1)}{2}, \end{cases}$$

or

$$\begin{cases} x - y = 3^{\beta-1}, \\ x + y - 1 = 3(3^{\alpha-\beta} - 1). \end{cases}$$

In the first case (see the first system), we obtain that

$$x = 3^{\beta-1} + \frac{3^{\alpha-\beta+1} - 1}{4},$$

$$y = 3^{\beta-1} - \frac{3^{\alpha-\beta+1} - 5}{4}.$$

We have that $y \geq 1$. *Therefore*

$$4 \cdot 3^{\beta-1} \geq 3^{\alpha-\beta+1} - 1.$$

Thus, it follows that

$$9 \cdot 3^{\beta-1} > 3^{\alpha-\beta+1}.$$

Hence, we obtain that $2\beta \geq \alpha + 1$.

On the other hand

$$3^{\alpha} > x^2 > 3^{2\beta-2}.$$

Therefore $\alpha \geq 2\beta - 1$. *We deduce that* $\alpha = 2\beta - 1$.

Thus

$$x = \frac{7 \cdot 3^{\beta-1} - 1}{4},$$

$$y = \frac{3^{\beta-1} + 5}{4}.$$

From the equation $x + y^2 = 3^{\beta}$, *we obtain that* $\beta = 2$, $x = 5$, $y = 2$ *and* $z = 5$.

In the second case (see the second system), we have that

$$x = y + 3^{\beta-1}.$$

Thus, it follows that

$$y(y+1) = 2 \cdot 3^{\beta-1}.$$

Hence, either $y = 1$ *or* $y = 2$, *as* $3 \nmid x$, $3 \nmid y$.

If $y = 1$, $\beta = 1$, $x = 2$, *then this leads to a contradiction.*

If $y = 2$, $\beta = 2$, $x = 5$, *then this case holds true.*

Therefore, the solutions of the given equation are $(2, 5, 5)$ *and* $(5, 2, 5)$.

Thus, the greatest possible value of the product xyz is equal to 50.

Problem 12. Positive integer n is called "wonderful", if there exist positive integers a, b, k, such that

$$3^n = a^k + b^k,$$

where $k > 1$. Find the sum of all "wonderful" numbers smaller than 75.

Solution. *Note that if n is a "wonderful" number, then k is an odd number. Indeed, if $a = 3^{\alpha}a_1$, $b = 3^{\beta}b_1$, where a_1, $b_1 \in \mathbb{N}$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha, \beta \in \mathbb{R}$ and $(a_1, 3) = 1$, $(b_1, 3) = 1$. We have that*

$$3^{n-k\alpha} = a_1^k + (3^{\beta-\alpha}b_1)^k,$$

where $\beta \geq \alpha$. If k is an even number, then $3 \mid a_1^{\frac{k}{2}}$. This leads to a contradiction.

Note that $3m+2$ are "wonderful" numbers, where $m \in \mathbb{Z}$ and $m \geq 0$. Indeed, it holds true

$$3^{3m+2} = (2 \cdot 3^m)^3 + (3^m)^3.$$

Now, let us prove that all the other positive integers are not "wonderful" numbers.

We have that

$$3^n = (a+b)(a^{k-1} - a^{k-2}b + \cdots - ab^{k-2} + b^{k-1}).$$

Thus, it follows that

$$a + b = 3^{\alpha},$$

and

$$a^{k-1} - a^{k-2}b + \cdots - ab^{k-2} + b^{k-1} = 3^{\beta},$$

where α, $\beta \in \mathbb{N}$. We have that

$$a^{k-1} - a^{k-1}(3^{\alpha} - a) + \cdots - (3^{\alpha} - a)^{k-1} = 3^{\beta},$$

$$3^{\alpha} \cdot M + ka^{k-1} = 3^{\beta},$$

where $M \in \mathbb{Z}$.

Hence, we obtain that either $3 \mid k$ or $3 \mid a$.

If $3 \nmid k$, then $a = 3^u \cdot a_1$, $b = 3^v b_1$, where $u, v, a_1, b_1 \in \mathbb{N}$, $(a_1, 3) = (b_1, 3) = 1$.

Let $u \leq v$, then

$$3^{n-uk} = a_1^k + (3^{v-u}b_1)^k.$$

We deduce that $3 \mid a_1$. This leads to a contradiction.

Therefore $3 \mid k$.

$$3^n = (a^{\frac{k}{3}} + b^{\frac{k}{3}})((a^{\frac{k}{3}})^2 - a^{\frac{k}{3}}b^{\frac{k}{3}} + (b^{\frac{k}{3}})^2).$$

Thus, it follows that

$$a^{\frac{k}{3}}+b^{\frac{k}{3}}=3^s,$$

$$(a^{\frac{k}{3}})^2-a^{\frac{k}{3}}b^{\frac{k}{3}}+(b^{\frac{k}{3}})^2=3^t,$$

where $s,t\in\mathbb{N}$ *and* $n=s+t$. *Therefore*

$$(a^{\frac{k}{3}}+b^{\frac{k}{3}})^2-3a^{\frac{k}{3}}b^{\frac{k}{3}}=3^t,$$

$$l=V_3(a^{\frac{k}{3}})=V_3(b^{\frac{k}{3}})\le s-1.$$

On the other hand

$$V_3((a^{\frac{k}{3}}+b^{\frac{k}{3}})^2-3a^{\frac{k}{3}}b^{\frac{k}{3}})^2=2l+1,\ t=2l+1.$$

We have that $a^{\frac{k}{3}}=3^l\cdot a_1^{\frac{k}{3}}$, $b^{\frac{k}{3}}=3^l\cdot b_1^{\frac{k}{3}}$, $a_1,b_1\in\mathbb{N}$ *and* $(a_1,3)=1$, $(b_1,3)=1$.

Therefore

$$(a_1^{\frac{k}{3}}+b_1^{\frac{k}{3}})^2-3a_1^{\frac{k}{3}}b_1^{\frac{k}{3}}=3.$$

Thus, it follows that

$$(a_1^{\frac{k}{3}})^2+(a_1^{\frac{k}{3}}-b_1^{\frac{k}{3}})^2+(b_1^{\frac{k}{3}})^2=6.$$

We deduce that

$$a_1^{\frac{k}{3}}+b_1^{\frac{k}{3}}=3.$$

Hence $s=l+1$. *We obtain that* $n=s+t=3l+2$.

Thus, the sum of all "wonderful" numbers smaller than 75 is equal to

$$2+5+8+\cdots+74=38\cdot 25=950.$$

7.3 Algebra

7.3.1 Problem Set 1

Problem 1. Solve the equation $4^x+9^x+49^x=6^x+14^x+21^x$.

Solution. *We have that* $(2^x-3^x)^2+(7^x-3^x)^2+(2^x-7^x)^2=0$. *Thus, the solution is* $x=0$.

Problem 2. Let (b_n) be a geometric progression, such that $b_1+b_{10}=9$ and $b_2+b_3+\cdots+b_9=10$. Find the value of the expression $\frac{10b_2+b_1b_2+b_1^2}{b_1}$.

Solution. *Let us Note that* $S_{10} = 19$, *thus*

$$\frac{b_{10}q - b_1}{q-1} = 19.$$

Hence, $(9-b_1)q - b_1 = 19q - 19$. *Therefore*

$$\frac{10b_2 + b_1b_2 + b_1^2}{b_1} = 19.$$

Problem 3. Find the sum of the greatest and smallest solutions of the equation $4^x + 64 = 2^{x^2-5x}$.

Solution. *We have* $2^x + 2^{6-x} = 2^{(x-3)^2-9}$, *thus if* x_0 *is a solution, then* $6 - x_0$ *is also a solution. Therefore, the sum of the greatest and the smallest solutions is equal to* 6.

Problem 4. Find the sum of all the coefficients of the polynomial $(3x^{2014} - 2x^3 + 1)^9$.

Solution. *The sum of all the coefficients of the polynomial* $p(x) = (3x^{2014} - 2x^3 + 1)^9$ *is equal to* $p(1) = 2^9 = 512$.

Problem 5. Find the greatest value of the expression $\frac{3\sqrt{x^2+8x+15}+4}{x+4}$.

Solution. *We have that if* $x \geq -3$, *then*

$$\frac{3\sqrt{x^2+8x+15}+4}{x+4} = \frac{\sqrt{(9x+27)(x+5)}+4}{x+4} \leq \frac{\frac{1}{2}(9x+27+x+5)+4}{x+4} = 5.$$

Note that for $x = -2,75$ *we obtain*

$$\frac{3\sqrt{x^2+8x+15}+4}{x+4} = 5.$$

On the other hand, for $x \leq -5$ *we deduce that*

$$\frac{3\sqrt{x^2+8x+15}+4}{x+4} < 0,$$

thus the greatest value of the expression is equal to 5.

Problem 6. Find the product of the greatest and the smallest solutions of the inequality $\sqrt{6x-2} \leq 2 + \sqrt[3]{x+5}$.

Solution. *The given inequality is equivalent to the following inequality*

$$\sqrt{6-\frac{2}{x}}-\frac{2}{\sqrt{x}}-\sqrt[3]{\frac{1}{\sqrt{x}}+\frac{5}{x\sqrt{x}}}\leq 0,$$

the solution set of the last inequality is the following interval $\left[\frac{1}{3},3\right]$, *as the function* $f(x)=\sqrt{6-\frac{2}{x}}-\frac{2}{\sqrt{x}}-\sqrt[3]{\frac{1}{\sqrt{x}}+\frac{5}{x\sqrt{x}}}$ *is increasing on the interval* $\left[\frac{1}{3},\infty\right)$ *and* $f(3)=0$.

Problem 7. Find the product of the solutions of the equation $\log_3 x - 1 = \log_2(\sqrt{x}-1)$.

Solution. *Let* $\log_3 x = \log_2(\sqrt{x}-1)+1 = y$, *thus* $3^{\frac{y}{2}} - 2^{y-1} = 1$, *the solutions of the last equality are* 2 *and* 4 *(cf Calculus 8), thus the solutions of the initial equation are* 9 *and* 81.

Problem 8. Consider $n\times n$ chess board, $n>1$. In each square is written a positive integer. Given that the sum of the numbers of each column and the sum of the numbers of each row make a geometric progression (with $2n$ terms). Find the total number of all possible values of the common ratio of that geometric progression.

Solution. *The sum of all the numbers written on the board is equal to the sum of the sums corresponding to all the columns, and on the other hand, it is equal to the sum of the sums corresponding to all the rows. Let us denote by q the common ratio of that geometric progression. Obviously, q is a positive rational number and is a solution of* $x^k+a_1x^{k-1}+\cdots+a_k=0$, *where* $k=2n-1$, $a_i\in\{-1,1\}$, $i=1,2,\ldots,k$. *Let* $q=\frac{u}{p}$, *where* $u,p\in\mathbb{N}$ *and* $(u,p)=1$. *We have that* $u^k+a_1u^{k-1}p+\cdots+a_kp^k=0$, *thus* u^k *is divisible by* p *and* p^k *is divisible by* u, *therefore* $u=p=1$. *Thus, the common ration of that geometric progression is equal to* 1.

Problem 9. Let a,b,c be complex numbers different from 0, such that $a+b+c=a^2+b^2+c^2=a^3+b^3+c^3=a^4+b^4+c^4$. Find the sum of a,b,c.

Solution. *Let* $a+b+c=A$. *Consider*

$$p(x)=(x-a)(x-b)(x-c)(x-1)=x^4+mx^3+nx^2+kx+l.$$

We have that

$$0=p(a)+p(b)+p(c)=a^4+b^4+c^4+m(a^3+b^3+c^3)+n(a^2+b^2+c^2)+$$

$$+k(a+b+c)+3l=A(1+m+n+k)+3l=A(p(1)-l)+3l=-Al+3l=l(A-3),$$

thus $A=3$, *as* $l=abc\neq 0$.

7.3.2 Problem Set 2

Problem 1. Find the absolute value of the smallest integer solution of the following inequality

$$x \geq \frac{2015}{x}.$$

Solution. *The given inequality is equivalent to the following inequality*

$$\frac{(x-\sqrt{2015})(x+\sqrt{2015})}{x} \geq 0.$$

The solution set of this inequality is $[-\sqrt{2015},0)\cup[\sqrt{2015},\infty)$. *Thus, the smallest solution of the given inequality is* -44, *and the absolute value is equal to* 44.

Problem 2. Let $x \geq 0, y>0, z>0$ and $\frac{4x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}=2$. Find $\frac{x+z}{y}$.

Solution. *Let* $y+z=a, x+z=b, x+y=c$, *then we obtain that*

$$\frac{2(b+c-a)}{a}+\frac{a+c-b}{2b}+\frac{a+b-c}{2c}=2.$$

Hence

$$\frac{(2b-a)^2}{2ab}+\frac{(2c-a)^2}{2ac}+\frac{(c-b)^2}{2bc}=0.$$

Therefore $a=2b, b=c$, *thus* $y=z, x=0$ *and* $\frac{x+z}{y}=1$.

Problem 3. Find the greatest integer solution of the following inequality $4^x - 3^x + 2^x < 2015$.

Solution. *Note that the function*

$$f(x)=3^x\left(\left(\frac{4}{3}\right)^x-1\right)+2^x$$

is increasing function on $[0,\infty)$ *interval (as the functions* 3^x, $\left(\frac{4}{3}\right)^x - 1$ *and* 2^x *are non-negative and increasing functions on that interval). We also have that* $f(5)=813<2015<3431=f(6)$. *Thus, the greatest integer solution of the given inequality is* 5.

Problem 4. Let distinct complex numbers a and b be the roots of x^3+x^2-1. Find the value of $(a+b)^3+2(a+b)^2+a+b+1000$.

Solution. *Let the third root of the given polynomial be c. According to Vieta's theorem $c=-1-a-b$, thus $(-1-a-b)^3+(1-a-b)^2-1=0$. Hence*

$$(a+b)^3+2(a+b)+a+b+1000=999.$$

Problem 5. Given that $\{a\}^2+[b]=15-6\sqrt{5}$ and $\{b\}^2+[a]=11-4\sqrt{5}$, where $\{x\}$ and $[x]$ are the fractional and integer parts of a real number x, respectively. Find the value of $a+b$.

Solution. *Note that $1<\sqrt{15}-6\sqrt{5}<2$ and $2<11-4\sqrt{5}<3$, thus $[b]=1$, $\{a\}^2=14-6\sqrt{5}$ and $[a]=2$, $\{b\}^2=9-4\sqrt{5}$. Hence $\{a\}=3-\sqrt{5}$ and $\{b\}=\sqrt{5}-2$. We deduce that $a=[a]+\{a\}=5-\sqrt{5}$ and $b=[b]+\{b\}=\sqrt{5}-1$. Thus $a+b=4$.*

Problem 6. Let a,b,c,d be real numbers, such that $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}=0$ and $a^5+b^5+c^5+d^5=0$. Find the number of all the possible values of the sum $a^{2015}+b^{2015}+c^{2015}+d^{2015}$.

Solution. *Without loss of generality, we may assume that $a\le b\le c\le d$. Let us show that $d=-a$ and $c=-b$. Note that from the following condition*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=0$$

we deduce that $a<0<d$.

If $c<0$, we have that

$$\frac{d}{-a}+\frac{d}{-b}+\frac{d}{-c}=1.$$

Thus $\left(\frac{d}{-a}\right)^5<1$. Hence $a^5+d^5<0$ and $a^5+b^5+c^5+d^5<0$, which leads to a contradiction.

If $c>0$ and $b>0$, then

$$\frac{-a}{b}+\frac{-a}{c}+\frac{-a}{d}=1.$$

Thus $\left(\frac{-a}{d}\right)^5<1$. Hence $a^5+d^5>0$ and $a^5+b^5+c^5+d^5>0$, which leads to a contradiction.

Therefore $a\le b<0<c\le d$. Let $a=-m$, $b=-n$, then $m>0,n>0$ and $m\ge n$.

If $d\ne m$, we may assume that $d>m\ge n>c$, as $\frac{d-m}{md}=\frac{n-c}{nc}$ and $c^5+d^5=m^5+n^5$. Thus

$$1=\frac{d^5-m^5}{n^5-c^5}=\frac{md}{nc}\cdot\frac{d^4+d^3m+\cdots+m^4}{n^5+n^3c+\cdots+c^4}>1.$$

This leads to a contradiction.

Hence $d = m$ and the unique possible value of the expression $a^{2015} + b^{2015} + c^{2015} + d^{2015}$ is 0*.*

Problem 7. Find the greatest value of a positive integer n, for which there exist n numbers, such that their pairwise sums are $\dfrac{n(n-1)}{2}$ consecutive integer numbers.

Solution. *Let the pairwise sums of $a_1 \le a_2 \le \cdots \le a_n$ be $\frac{n(n-1)}{2}$ consequative integer numbers.*

If $a_i = a_j$, $(i \neq j)$, then $a_i + a_k = a_j + a_k$, $(k \neq i, k \neq j)$. Therefore, the number of the possible values of all the pairwise sums of that n numbers will be less than $\frac{n(n-1)}{2}$.

Hence, we deduce that $a_1 < a_2 < \cdots < a_{n-1} < a_n$. If $n \ge 6$, we have that $a_1 + a_3 = a_1 + a_2 + 1$ and $a_n + a_{n-1} = a_n + a_{n-2} + 1$. Thus $a_3 + a_{n-2} = a_2 + a_{n-1}$, which leads to a contradiction.

If $n = 5$, we have that $a_3 = a_2 + 1$, $a_4 = a_3 + 1$, $a_4 + a_5 = a_1 + a_2 + 9$ and $4(a_1 + a_2 + a_3 + a_4 + a_5) = 10\frac{a_1+a_2+a_4+a_5}{2}$. Therefore $a_3 = a_2 + 1$, $a_4 = a_2 + 2$ and $a_5 = a_1 + 7$. We obtain that $4(2a_1 + 3a_2 + 10) = 5(2a_1 + 2a_2 + 9)$, thus $a_2 = a_1 + 2{,}5$, which leads to a contradiction, as $a_1 + a_5 = a_2 + a_4$.

As the numbers $0, 1, 2, 4$ satisfy the problem conditions, thus the greatest possible values of n are equal to 4.

Problem 8. Find the value of the expression

$$21\sqrt[5]{6} - \sqrt{\frac{2205}{\sqrt[5]{1296} + 2\sqrt[5]{216} + 3\sqrt[5]{36} + 4\sqrt[5]{6} + 5}}.$$

Solution. *Let $\sqrt[5]{6} = a$, then $a^5 = 6$. We have*

$$21\sqrt[5]{6} - \sqrt{\frac{2205}{\sqrt[5]{1296} + 2\sqrt[5]{216} + 3\sqrt[5]{36} + 4\sqrt[5]{6} + 5}} = 21a - 21\sqrt{\frac{5}{a^4 + 2a^3 + 3a^2 + 4a + 5}} =$$

$$21a - 21\sqrt{\frac{5(a-1)}{(a-1)(a^4 + 2a^3 + 3a^2 + 4a + 5)}} = 21a - 21\sqrt{\frac{5(a-1)}{a^5 + a^4 + a^3 + a^2 + a - 5}} =$$

$$21a - 21\sqrt{\frac{5(a-1)}{a^4 + a^3 + a^2 + a + 1}} = 21a - 21\sqrt{\frac{5(a-1)^2}{a^5 - 1}} = 21a - 21\sqrt{(a-1)^2} = 21.$$

Problem 9. Find the smallest value of the expression

$$(1 + x^2)\sqrt[3]{\frac{1024}{1 + 6x^2 + 9x^4}}.$$

Solution. *Note that*

$$f(x) = (1+x^2)\sqrt[3]{\frac{1024}{1+6x^2+9x^4}} = 8\left(\left(\frac{1+x}{\sqrt[3]{2(1+3x^2)}}\right)^2 + \left(\frac{1-x}{\sqrt[3]{2(1+3x^2)}}\right)^2\right).$$

and

$$\left(\frac{1+x}{\sqrt[3]{2(1+3x^2)}}\right)^3 + \left(\frac{1-x}{\sqrt[3]{2(1+3x^2)}}\right)^3 = 1.$$

Therefore

$$\left(\frac{1+x}{\sqrt[3]{2(1+3x^2)}}\right)^2 + \left(\frac{1-x}{\sqrt[3]{2(1+3x^2)}}\right)^2 \geq 1.$$

Hence $f(x) \geq 8$ *and* $f(1) = 8$, *thus the minimum value of the function* $f(x)$ *is equal to* 8.

7.3.3 Problem Set 3

Problem 1. Evaluate the expression

$$4^{\sqrt[3]{\log_2^2 3}} - 9^{\sqrt[3]{\log_3 2}}.$$

Solution. *We have that*

$$4^{\sqrt[3]{\log_2^2 3}} = \left(9^{\log_9 4}\right)^{\sqrt[3]{\log_2^2 3}} = 9^{\log_3 2 \cdot \sqrt[3]{\log_2^2 3}} = 9^{\sqrt[3]{\log_3^3 2 \cdot \log_2^2 3}} = 9^{\sqrt[3]{\log_3 2}}.$$

Therefore

$$4^{\sqrt[3]{\log_2^2 3}} - 9^{\sqrt[3]{\log_3 2}} = 0.$$

Problem 2. Find the number of solutions (in the set of real numbers) of the following equation

$$x^6 - 2x^4 + 4x^3 - 3x^2 - 4x + 4 = 0.$$

Solution. *Let us rewrite the given equation in the following way*

$$x^4 + 4x + \frac{4}{x^2} - 2x^2 - \frac{4}{x} - 3 = 0.$$

Denote by $y = x^2 + \dfrac{2}{x}$. *Hence, we obtain*

$$y^2 - 2y - 3 = 0.$$

Thus $y = -1$ *or* $y = 3$. *We deduce that* $x^3 + x + 2 = 0$ *or* $x^3 - 3x + 2 = 0$. *Therefore* $(x+1)(x^2 - x + 2) = 0$ *or* $(x-1)(x^2 + x - 2) = 0$. *Thus, the given equation has 4 real roots.*

Problem 3. Given that in the geometric progression the sum of the first three terms with the odd indexes is equal to 10 and the sum of the first two terms with the even indexes is equal to 5. Find the sum of the squares of the first five terms.

Solution. *Consider a geometric progression* (b_n), *which satisfies the assumptions of the problem. Therefore, we have that* $b_1 + b_3 + b_5 = 10$ *and* $b_2 + b_4 = 5$. *Hence* $b_1 + b_3 + b_5 + b_2 + b_4 = 15$ *and* $b_1 + b_3 + b_5 - b_2 - b_4 = 5$. *Thus* $75 = (b_1 + b_3 + b_5 + b_2 + b_4)(b_1 + b_3 + b_5 - b_2 - b_4) = (b_1 + b_3 + b_5)^2 - (b_2 + b_4)^2 = b_1^2 + b_3^2 + b_5^2 + 2b_1b_3 + 2b_1b_5 + 2b_3b_5 - b_2^2 - 2b_2b_4 - b_4^2 = b_1^2 + b_3^2 + b_5^2 + 2b_2^2 + 2b_3^2 + 2b_4^2 - b_2^2 - 2b_3^2 - b_4^2 = b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2$.

Hence $b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 = 75$.

Problem 4. Find the product of the real-valued solutions of the following equation

$$x^2 + 7x - 5 = 5\sqrt{x^3 - 1}.$$

Solution. *Let us rewrite the given equation in the following way*

$$x^2 + x + 1 + 6(x-1) = 5\sqrt{(x-1)(x^2 + x + 1)}.$$

Therefore, we deduce that

$$\frac{x^2 + x + 1}{x - 1} - 5\sqrt{\frac{x^2 + x + 1}{x - 1}} + 6 = 0.$$

Hence, we obtain the following equations $x^2 - 3x + 5 = 0$ *and* $x^2 - 8x + 10 = 0$. *Note that first equation does not have real solutions. On the other hand, second equation has two real solutions and their product is equal to 10.*

Problem 5. Let x, y, z be nonzero real numbers. Given that $x^2 - yz = 1.5y^2$, $y^2 - xz = 1.5z^2$ and $z^2 - xy = 1.5x^2$. Find the absolute value of the following expression

$$\frac{24xyz}{x^2z + y^2x + z^2y}.$$

Solution. *We have that* $x^2 - yz = 1.5y^2$, $y^2 - xz = 1.5z^2$ *and* $z^2 - xy = 1.5x^2$. *After summing up these equations, we obtain that* $(x+y+z)^2 = 0$, *thus* $x+y+z = 0$. *On the other hand, we have that*

$$x^2z + y^2x + z^2y = x(y^2 - 1.5z^2) + y(z^2 - 1.5x^2) + z(x^2 - 1.5y^2) =$$

$$= xy^2 + yz^2 + zx^2 - 1.5(xz^2 + yx^2 + zy^2).$$

Thus

$$xz^2 + yx^2 + zy^2 = 0.$$

Therefore

$$xyz = -(y+z)(z+x)(x+y) = -(yx^2 + xz^2 + zy^2 + x^2z + y^2x + z^2y + 2xyz) =$$

$$= -x^2z - y^2x - z^2y - 2xyz.$$

We obtain that

$$x^2z + y^2x + z^2y = -3xyz.$$

Hence

$$\left|\frac{24xyz}{x^2z + y^2x + z^2y}\right| = 8.$$

Problem 6. At most, how many numbers one can choose from the sequence $1, 2, \ldots, 17$, such that among the chosen numbers there are no three numbers which create an arithmetic progression?

Solution. *Let us consider a sequence* (a_n), *where* $a_1 < a_2 < \cdots < a_n$, *such that there are no three numbers among them which create an arithmetic progression. We have that* $a_{k+2} - a_k = a_{k+2} - a_{k+1} + a_{k+1} - a_k \geq 3$. *In a similar way, one can obtain that* $a_{k+4} - a_k = a_{k+4} - a_{k+2} + a_{k+2} - a_k \geq 7$. *Moreover, if* $a_{k+4} - a_k = 7$, *then either* $a_{k+4} - a_{k+2} = 3$ *and* $a_{k+2} - a_k = 4$ *or* $a_{k+4} - a_{k+2} = 4$ *and* $a_{k+2} - a_k = 3$. *One can easily verify that none of those cases holds true, therefore* $a_{k+4} - a_k \geq 8$. *Hence, we deduce that* $a_{k+8} - a_k = a_{k+8} - a_{k+4} + a_{k+4} - a_k \geq 17$. *Therefore, it is not possible to choose 9 such numbers. We give the following example for 8 numbers: 1,2,4,5,10,11,13,14.*

Problem 7. Let x, y, z be real numbers. Given that $2x(y^2 - 1) + 2y(x^2 - 1) = (1 + x^2)(1 + y^2)$ and $4z(1 - y^2) + 4y(1 - z^2) = (1 + z^2)(1 + y^2)$. Find the value of the following expression

$$\left(\frac{2x}{1+x^2} - \frac{2z}{1+z^2}\right)^2 + \left(\frac{1-z^2}{1+z^2} - \frac{1-x^2}{1+x^2}\right)^2.$$

Solution. *Consider the following points*

$$A\Big(\frac{2x}{1+x^2},\frac{1-x^2}{1+x^2}\Big),B\Big(\frac{1-y^2}{1+y^2},\frac{2y}{1+y^2}\Big),C\Big(\frac{2z}{1+z^2},\frac{1-z^2}{1+z^2}\Big).$$

Note that these points are on a unit circle, as for any real number a it holds true

$$\Big(\frac{2a}{1+a^2}\Big)^2+\Big(\frac{1-a^2}{1+a^2}\Big)^2=1.$$

According to the assumptions of the problem, we have that

$$AB^2=\Big(\frac{2x}{1+x^2}-\frac{1-y^2}{1+y^2}\Big)^2+\Big(\frac{1-x^2}{1+x^2}-\frac{2y}{1+y^2}\Big)^2=2-2\frac{2x(1-y^2)+2y(1-x^2)}{(1+x^2)(1+y^2)}=4.$$

Therefore $AB=2$. On the other hand, we have that

$$BC^2=\Big(\frac{2y}{1+y^2}-\frac{1-z^2}{1+z^2}\Big)^2+\Big(\frac{1-y^2}{1+y^2}-\frac{2z}{1+z^2}\Big)^2=2-2\frac{2z(1-y^2)+2y(1-z^2)}{(1+z^2)(1+y^2)}=1.$$

Hence, we obtain that $BC=1$. From the condition $AB=2$, it follows that AB is a diameter, thus $\angle C=90^\circ$ and $AC^2=3$. Hence

$$\Big(\frac{2x}{1+x^2}-\frac{2z}{1+z^2}\Big)^2+\Big(\frac{1-z^2}{1+z^2}-\frac{1-x^2}{1+x^2}\Big)^2=AC^2=3.$$

Problem 8. Let f,g be a quadratic expression. Given that f, g has at least one root, $f+g$ has no roots and $f-g$ is a quadratic expression. Find the number of roots of $f-g$.

Solution. *Consider $f(x)=ax^2+bx+c$ and $g(x)=px^2+qx+r$. According to the assumptions of the problem, we have that $a\neq 0, p\neq 0$, $b^2-4ac\geq 0$ and $q^2-4pr\geq 0$. We have that $f(x)+g(x)=(a+p)x^2+(b+q)x+c+r$ has no roots. Let us consider two cases.*

Case 1. If $a+p\neq 0$, then $(b+q)^2-4(a+p)(c+r)<0$.

If $(b-q)^2-4(a-p)(c-r)\leq 0$, then by summing up the last two inequalities, we deduce that

$$2(b^2-4ac)+2(q^2-4pr)<0,$$

which leads to a contradiction, as $b^2-4ac\geq 0$ and $q^2-4pr\geq 0$.

Therefore, $(b-q)^2-4(a-p)(c-r)>0$. Thus $f-g$ has two roots.

Case 2. If $a+p=0$, then $b+q=0$ and $c+r\neq 0$. Hence $(b+q)^2-4(a+p)(c+r)=0$.

If $(b-q)^2-4(a-p)(c-r)\leq 0$, then by summing up the last inequality and the equality, we deduce that

$$0 \le 2(b^2 - 4ac) + 2(q^2 - 4pr) \le 0.$$

Therefore $b^2 = 4ac$ *and* $q^2 = 4pr$. *Hence, we obtain that* $4ac = 4pr$. *Thus* $c + r = 0$, *which leads to a contradiction.*

Therefore

$$(b-q)^2 - 4(a-p)(c-r) > 0.$$

Problem 9. Let $b_1, b_2, \ldots, b_{15}$ be a geometric progression with a common ratio q. Find the possible values of $\frac{b_1}{q}$, such as for any of the following system of equations has a solution in the set of real numbers

$$\begin{cases} x_1 + x_2 + \cdots + x_{15} = b_1, \\ x_1^2 + x_2^2 + \cdots + x_{15}^2 = b_2, \\ \ldots \\ x_1^{15} + x_2^{15} + \cdots + x_{15}^{15} = b_{15}. \end{cases}$$

Solution. *We prove that the given system of equations has a solution in the set of real numbers, iff* $\dfrac{b_1}{q} \in \{1, 2, \ldots, 15\}$.

We proceed by a contradiction argument. Suppose that the given system of equations has a solution in the set of real numbers. We have that $x_1^2 + x_2^2 + \cdots + x_{15}^2 = b_1 \cdot q$, $x_1^3 + x_2^3 + \cdots + x_{15}^3 = b_1 \cdot q^2$, $x_1^4 + x_2^4 + \cdots + x_{15}^4 = b_1 \cdot q^3$, *hence* $x_1^4 + x_2^4 + \cdots + x_{15}^4 - 2q(x_1^3 + x_2^3 + \cdots + x_{15}^3) + q^2(x_1^2 + x_2^2 + \cdots + x_{15}^2) = 0$. *Thus* $(x_1^2 - qx_1)^2 + \cdots + (x_{15}^2 - qx_{15})^2 = 0$. *Therefore* $x_i = 0$ *or* $x_i = q$, *for* $i = 1, 2, \ldots, 15$. *Suppose for some positive integer* k, *exactly* k *terms among* x_i, *for* $i = 1, 2, \ldots, 15$, *are equal to* q. *Hence* $k \in \{1, 2, \ldots, 15\}$ *and* $\dfrac{b_1}{q} = k$.

Thus, for $\dfrac{b_1}{q} = k$ *and* $k \in \{1, 2, \ldots, 15\}$ *we have that* $x_1 = q, x_2 = q, \ldots, x_k = q, x_{k+1} = 0, \ldots, x_n = 0$ *is a solution of a given system of equations.*

7.3.4 Problem Set 4

Problem 1. Evaluate the expression

$$\log_3^2 45 - \frac{\log_3 15}{\log_{135} 3}.$$

Solution. *We have that*

$$\log_3^2 45 - \frac{\log_3 15}{\log_{135} 3} = \log_3^2 45 - \log_3 15 \cdot \log_3 135 = \log_3^2 45 - (\log_3 45 - 1)(\log_3 45 + 1) =$$

$$= \log_3^2 45 - \log_3^2 45 + 1 = 1.$$

Problem 2. Let a,b,c be real numbers. Given that

$$\frac{a^{2014}}{b+c}+\frac{b^{2014}}{a+c}+\frac{c^{2014}}{a+b}=\frac{a^{2014}+b^{2014}+c^{2014}}{a+b+c}.$$

Find the value of the sum

$$\frac{a^{2015}}{b+c}+\frac{b^{2015}}{a+c}+\frac{c^{2015}}{a+b}.$$

Solution. *We have that*

$$(a+b+c)\left(\frac{a^{2014}}{b+c}+\frac{b^{2014}}{a+c}+\frac{c^{2014}}{a+b}\right)-a^{2014}-b^{2014}-c^{2014}=0.$$

Therefore

$$\frac{a^{2015}}{b+c}+a^{2014}+\frac{b^{2015}}{a+c}+b^{2014}+\frac{c^{2015}}{a+b}+c^{2014}-a^{2014}-b^{2014}-c^{2014}=0.$$

Hence

$$\frac{a^{2015}}{b+c}+\frac{b^{2015}}{a+c}+\frac{c^{2015}}{a+b}=0.$$

Problem 3. Find the greatest integer solution of inequality $x-\sin x<6$.

Solution. *Note that* $5-\sin 5<6$ *and* $\sin 6<0$*, thus* $6-\sin 6>6$*. On the other hand,* $x\geq 7$ *and* $x\in\mathbb{Z}$*,* $x-\sin x\geq 7-\sin x\geq 6$.

Therefore, the greatest solution of the given inequality is 5.

Problem 4. Evaluate the expression $\left(\sqrt[3]{22+10\sqrt{7}}\right)^2-2\sqrt[3]{22+10\sqrt{7}}$.

Solution. *Note that* $22+10\sqrt{7}=(1+\sqrt{7})^3$*, hence*

$$\left(\sqrt[3]{22+10\sqrt{7}}\right)^2-2\sqrt[3]{22+10\sqrt{7}}=(1+\sqrt{7})^2-2(1+\sqrt{7})=6.$$

Problem 5. Given that $[x]^3\cdot\{x\}\geq\frac{16}{3}$, where by $[x]$ and $\{x\}$ we denote, respectively, the integer and fractional parts of a real number x. Find the possible smallest value of $30x$.

Solution. *From the following condition* $[x]^3\cdot\{x\}\geq\frac{16}{3}$*, we obtain that* $[x]\geq 0$*. Therefore*

$$30x = 30([x]+\{x\}) = 30\left(\frac{[x]}{3}+\frac{[x]}{3}+\frac{[x]}{3}+\{x\}\right) \geq 120\sqrt[4]{\frac{[x]^3\cdot\{x\}}{27}} \geq 80.$$

If $x=\frac{8}{3}$*, then we have that* $[x]^3\cdot\{x\}=\frac{16}{3}$ *and* $30x=80$*. Therefore, the possible minimum value of* $30x$ *is equal to 80.*

Problem 6. Find the sum of all the solutions of the equation $5^{x^3-17x^2+20x-3} = 15\cdot 3^{-x}$.

Solution. *We have that* $(5^{x^2-16x+4}\cdot 3)^{x-1}=1$*, therefore* $(x-1)\ln(5^{x^2-16x+4}\cdot 3)=0$*. Thus* $x=1$ *or* $5^{x^2-16x+4}\cdot 3=1$*. Hence* $x^2-16x+4+\log_5 3=0$*. Note that the sum of the solutions of the last equation is equal to* 16*; therefore, the sum of all the solutions of the given equation is equal to* 17*.*

Problem 7. Evaluate the expression

$$\left[\frac{2}{5}+\frac{2}{5}\cdot\frac{4}{7}+\cdots+\frac{2}{5}\cdot\frac{4}{7}\cdot\cdots\cdot\frac{100}{103}\right],$$

where we denote by $[x]$ the integer part of a real number x.

Solution. *Let us prove that*

$$\frac{2}{5}+\frac{2}{5}\cdot\frac{4}{7}+\cdots+\frac{2}{5}\cdot\frac{4}{7}\cdot\cdots\cdot\frac{100}{103}>1.$$

We have that

$$A=\frac{2}{5}+\frac{2}{5}\cdot\frac{4}{7}+\cdots+\frac{2}{5}\cdot\frac{4}{7}\cdot\cdots\cdot\frac{100}{103}>\frac{2}{5}+\frac{2}{5}\cdot\frac{4}{7}\left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\cdots+\left(\frac{2}{3}\right)^{48}\right)=$$

$$=\frac{2}{5}+\frac{8}{35}\cdot\frac{1-\left(\frac{2}{3}\right)^{49}}{\frac{1}{3}}=\frac{2}{5}+\frac{24}{35}\cdot\left(1-\left(\frac{2}{3}\right)^{49}\right)>\frac{2}{5}+\frac{24}{35}\cdot\left(1-\left(\frac{2}{3}\right)^{48}\right)>$$

$$>\frac{2}{5}+\frac{24}{35}\cdot\left(1-\frac{1}{8}\right)=1.$$

Therefore $A>1$.

Let us show that $A<2$*. We need to prove that*

$$\frac{4}{7}+\frac{4}{7}\cdot\frac{6}{9}+\cdots+\frac{4}{7}\cdot\frac{6}{9}\cdot\cdots\cdot\frac{100}{103}<4$$

or

$$\frac{6}{9}+\frac{6}{9}\cdot\frac{8}{11}+\cdots+\frac{6}{9}\cdot\frac{8}{11}\cdot\cdots\cdot\frac{100}{103}<6$$

and so on, hence one needs to notice only that the following inequality holds true

$$\frac{100}{103} < 100.$$

Therefore, we deduce that $1 < A < 2$, *thus* $[A] = 1$.

Problem 8. Let a,b,c,d be real numbers. Given that $a+b+c+d=0$ and $(ab+ac+ad+bc+bd+cd)^2+12 = 6(abc+abd+acd+bcd)$. Find the value of the expression $abc+abd+acd+bcd$.

Solution. *Let* $a = x-1$, $b = y-1$, $c = z-1$, $d = t-1$, *then* $x+y+z+t = 4$ *and* $(xy+yz+zt+xt+xz+yt)^2 = 6(xyz+xyt+yzt+ztx)$. *Therefore* $xyz+xyt+yzt+ztx \geq 0$.

If $xyz+xyt+yzt+ztx > 0$, *we have that*

$$(xy+yz+zt+xt+xz+yt)^2 = 2(x+y+z+t)(xyz+xyt+yzt+ztx)+(xy-zt)^2+(xz)^2+$$

$$+(xt)^2+(yz)^2+(yt)^2 \geq 8(xyz+xyt+yzt+ztx) > 6(xyz+xyt+yzt+ztx).$$

Hence

$$(xy+yz+zt+xt+xz+yt)^2 > 6(xyz+xyt+yzt+ztx),$$

which leads to a contradiction.

We obtain that $xyz+xyt+yzt+ztx = 0$ *and* $xy+yz+zt+xt+xz+yt = 0$, *thus*

$$abc+abd+bcd+acd = xyz+xyt+yzt+ztx-2xy-2xz-2xt-2yz-2yt-$$

$$-2zt+3x+3y+3z+3t-4 = 8.$$

Problem 9. Let M be the greatest number, such that the inequality $a^3+b^3+c^3 \geq 3abc+M|(a-b)(b-c)(c-a)|$ holds true for all non-negative a,b,c. Find the integer part of M.

Solution. *Let us prove that if* a,b,c *are non-negative numbers, then*

$$a^3+b^3+c^3 \geq 3abc+4.25|(a-b)(b-c)(c-a)|.$$

We have that $a^3+b^3+c^3-3abc = 0.5(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2)$. *Therefore, we need to prove that* $(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2) \geq 8.5|a-b||b-c||c-a|$.

Let $\min(a,b,c) = c$. *Without loss of generality, one can assume that* $c = 0$. *It is sufficient to prove the inequality for the numbers* $a-c$, $b-c$, 0. *Hence, we need to prove that* $a^3+b^3 \geq 4.25ab|a-b|$.

Let $\min(a,b) = b$.

If $b \leq \frac{a}{3}$, *then*

$$a^3 + b^3 \geq a^3 \geq \frac{8.5}{9}a^3 \geq 4.25ab|a-b|.$$

If $\frac{a}{3} \leq b \leq \frac{2a}{5}$, *then*

$$a^3 + b^3 \geq \frac{28}{27}a^3 \geq 4.25 \cdot \frac{6}{25}a^3 \geq 4.25ab|a-b|.$$

If $\frac{2a}{5} \leq b$, *then*

$$a^3 + b^3 \geq \frac{133}{125}a^3 \geq 4.25 \cdot \frac{1}{4}a^3 \geq 4.25ab|a-b|.$$

Therefore $M \geq 4.25$. *Note that if we take* $c = 0$, $a = 3$, $b = 1$, *then we have* $28 \geq 6M$. *We deduce that* $M \leq \frac{14}{3}$, *thus* $[M] = 4$.

7.3.5 Problem Set 5

Problem 1. Find the greatest root of the equation

$$x^3 + x = 30(x^2 - 3x + 1)^2.$$

Solution. *The given inequality is equivalent to the following inequality*

$$(2^{x+6} + x)(2^{x-6} - 2^5) \leq 0. \tag{7.54}$$

Note that $f(x) = 2^{x+6} + x$ *and* $g(x) = 2^{x-6} - 2^5$ *are increasing functions. On the other hand, we have that* $f(-4) = 0$ *and* $g(11) = 0$. *Thus, the set of solutions of* (7.90) *is* $[-4, 11]$. *Therefore, the sum of all the integer solutions of the given inequality is equal to 56.*

Problem 2. Find the number of solutions belonging to $[0, 1]$

$$\{x\} + \{2x\} + \{3x\} + \{4x\} = 1,$$

where by $\{x\}$ we denote the fractional part of real number x.

Solution. *Let us consider on a coordinate plane the following points* $A(x,y)$ *and* $B(u,v)$. *We have that* $(x-6)^2 + (y-8)^2 = 1$ *and* $(u-10)^2 + (v-5)^2 = 81$. *On the other hand, we have that* $AB = 15$. *Let us also consider points* $C(6,8)$ *and* $D(10,5)$, *then* $AC = 1$, $BD = 9$, $CD = 5$. *We have obtained that* $AB = AC + CD + BD$, *thus*

points C and D are on the segment AB. Let M be the midpoint of AB, then $MD = 2.5$. *Thus, if point N is the midpoint of BD, then D is the midpoint of MN. Therefore, we obtain that*

$$\frac{\frac{x+u}{2} + \frac{10+u}{2}}{2} = 10,$$

and

$$\frac{\frac{y+v}{2} + \frac{v+5}{2}}{2} = 5.$$

Therefore, $x + 2u = 30$ *and* $y + 2v = 15$. *Hence,* $x + y + 2u + 2v = 45$.

Problem 3. Given that 2015 is written as a sum of some three-digit and one one-digit numbers, such that the digits of each three-digit number are consecutive terms of an arithmetic progression. Find the possible greatest value of one-digit number.

Solution. *Let us write the given equation in the following way*

$$3(3x^3 - 2x)^3 - 2(3x^3 - 2x) \le x. \tag{7.55}$$

Note that $x = 1$ *is a solution of* (7.66).

If $x > 1$, *we have that* $3x^3 - 2x > x$. *Thus,*

$$3(3x^3 - 2x)^3 - 2(3x^3 - 2x) > 3x^3 - 2x > x.$$

Hence, $3(3x^3 - 2x)^3 - 2(3x^3 - 2x) > x$. *Therefore, the greatest solution of the given inequality is equal to 1.*

Problem 4. Let x_1, x_2, x_3 be the roots of the equation $x^3 - 3x^2 + 4x - 65 = 0$. Find the value of the expression $x_1^3 + 3x_2^2 + 3x_3^2 - 4x_2 - 4x_3$.

Solution. *We have that*

$$(x^3 + y^3)(x^2 + xy + y^2) = 100.$$

Hence, $x^6 - y^6 = 100(x - y)$. *In a similar way, we obtain that* $y^6 - z^6 = 50(y - z)$ *and* $z^6 - x^6 = 25(z - x)$. *Summing up the obtained equalities, we deduce that* $4(x - y) + 2(y - z) + z - x = 0$. *Hence,* $3x - 2y = z$. *Therefore,*

$$\frac{30z}{3x - 2y + z} = 15.$$

Problem 5. Let a, b be real numbers. Given that the equation $\cos x + \cos(ax) = b$ has a unique solution. Find the number of solutions of the equation $\cos x + \cos(ax) = -b$.

Solution. *Let us prove that if the following equation*

$$ax^2+bxy+cy^2+dx+ey+f=0, \tag{7.56}$$

has two distinct solutions (x_1,y_1) *and* (x_2,y_2)*, then it has infinitely many solutions.*

Let (x_1,y_1) *and* (x_2,y_2) *be distinct solutions. Without loss of generality, one can assume that* $y_1 \neq y_2$.

Let $a>0$. *As* $ax^2+(by_1+d)x+cy_1^2+ey_1+f=0$ *has a solution, then* $F(y_1)=(by_1+d)^2-4a(cy_1^2+ey_1+f)\geq 0$. *Let us consider*

$$F(y)=(b^2-4ac)y^2+(2bd-4ae)y+d^2-4af.$$

As $F(y_1),F(y_2)\geq 0$ *and* $y_1\neq y_2$*, then for infinitely many* y *we have that* $F(y)\geq 0$. *On the other hand, for all the* y (7.67) *has solutions.*

If $a=0$*, then we have that*

$$(by+d)x+cy^2+ey+f=0. \tag{7.57}$$

If $b=d=0$*, then* (7.62) *has infinitely many solutions.*

If $b^2+d^2\neq 0$*, then*

$$x=-\frac{cy^2+ey+f}{by+d}.$$

Thus, equation (7.62) *has infinitely many solutions. Therefore, equation* (7.67) *has infinitely many solutions. We obtain that* $n=0$ *or* $n=1$*, hence* $n^2-n+50=50$.

Problem 6. Let x,y,z be real numbers. Given that

$$x+y^2+z^3=y+z^2+x^3=z+x^2+y^3=0.$$

Let n be the number of possible values of the expression $x^2y+y^2z+z^2x-xyz-x^2y^2z^2$. Find $n^2-3n+100$.

Solution. *Note that the function*

$$f(x)=x^{215}\sqrt{|x|+1}$$

is an odd and increasing function. Thus, the function $g(x)=f(x-1)+f(x-2)+\cdots+f(x-215)$ *is an increasing function and* $g(108)=f(107)+f(106)+\cdots+f(0)+\cdots+f(-107)=f(107)+f(106)+\cdots+f(-107)=f(107)+f(106)+\cdots+f(0)-f(1)-\cdots-f(107)=0$. *As* $g(x)$ *is an increasing function, then it cannot accept a value equal to 0 at two different points. Therefore, the single root of the given equation is 108.*

Problem 7. Find the sum of all real-valued solutions of the equation

$$x^5 - 12x^4 + 54x^3 - 108x^2 + 81x + 1 = \sqrt{\cos\frac{2\pi x}{3}}.$$

Solution. *We have that $p(x) = (x-1)(x-2)q(x)$, where $q(x)$ is a quadratic polynomial. As $p(x)$ has only non-negative value, then the roots of $q(x)$ are 1 and 2. Thus $p(x) = a(x-1)^2(x-2)^2$. We have that $p(3) = 12$, then $a = 3$. Therefore, $p(4) + p(5) = 540$.*

Problem 8. Let x, y, z be positive numbers. Given that

$$\sqrt{x^2+y^2} + \sqrt{x^2+z^2} + y + z = \sqrt[4]{108} \cdot \sqrt{x(y+z)}.$$

Find the value of the expression $\dfrac{x^2+y^2}{z^2}$.

Solution. *We have that*

$$(x-4)^6 + 3(x-4)^2 \le (a-9+6y-y^2)^3 + 3(a-9+6y-y^2).$$

Let us consider the following function

$$f(x) = x^3 + 3x.$$

Note that if $x_2 > x_1$, then

$$f(x_2) - f(x_1) = (x_2-x_1)(x_2^2 + x_1x_2 + x_1^2 + 3) = (x_2-x_1)\left(\left(x_2 + \frac{x_1}{2}\right)^2 + \frac{3}{4}x_1^2 + 3\right) > 0.$$

Therefore, f is an increasing function.

The given inequality is equivalent to the following inequality

$$f((x-4)^2) \le f(a-9+6y+y^2),$$

or to the following inequality

$$(x-4)^2 \le a-9+6y+y^2.$$

We have that the following inequality

$$(x-4)^2 + (y-3)^2 \le a-18$$

has a unique solution, thus $a = 18$.

Problem 9. Find the greatest possible value of M, such that the inequality

$$M\,|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \le (a^2+b^2+c^2)^2$$

holds true for any non-negative numbers a,b,c.

Solution. *We have that*

$$\sqrt{(1+a)(1+b)}+\sqrt{(1+b)(1+c)}+\sqrt{(1+c)(1+a)} \le \frac{1+a+1+b}{2}+\frac{1+b+1+c}{2}+$$

$$+\frac{1+c+1+a}{2}=3+a+b+c=4.$$

Therefore, $A \le 40$. *Note that, if* $x \ge y \ge z \ge 0$, *then*

$$\sqrt{1+x}+\sqrt{1+y} \ge \sqrt{1+x+z}+\sqrt{1+y-z}. \tag{7.58}$$

Indeed, we need to show that $(1+x)(1+y) \ge (1+x+z)(1+y-z)$ *or* $0 \ge (y-x)z-z^2$. *Obviously, the last inequality holds true.*

Let $a \ge b \ge c$, *according to* (7.63), *we have that*

$$\sqrt{1+a+b}+\sqrt{1+b+c}+\sqrt{1+c+a} \ge \sqrt{1+a+b+c}+\sqrt{1+b}+\sqrt{1+c+a} \ge$$

$$\ge \sqrt{2}+1+\sqrt{1+c+a+b}=1+2\sqrt{2}.$$

Therefore, $A \ge 10+20\sqrt{2} > 38$.

If $a=b=c=\frac{1}{3}$, *then we have that* $[A]=40$.

If $a=b=\frac{1}{2}$, $c=0$, *then we have that* $A=15+10\sqrt{6}$, $[A]=39$.

If $a=1$, $b=c=0$, *then we have that* $A=10+20\sqrt{2}$, $[A]=38$.

Hence, the required sum is equal to 117.

7.3.6 Problem Set 6

Problem 1. Find the sum of all integer solutions of the inequality

$$4^x+x\cdot 2^{x-6}-2^{x+11}-32x \le 0.$$

Solution. *The given inequality is equivalent to the following inequality*

$$(2^{x+6}+x)(2^{x-6}-2^5) \le 0. \tag{7.59}$$

Note that $f(x)=2^{x+6}+x$ *and* $g(x)=2^{x-6}-2^5$ *are increasing functions. On the other hand, we have that* $f(-4)=0$ *and* $g(11)=0$. *Thus, the set of solutions of*

(7.70) *is* $[-4, 11]$. *Therefore, the sum of all the integer solutions of the given inequality is equal to 56.*

Problem 2. Given that

$$x^2 + y^2 - 12x - 16y + 99 = 0,\ u^2 + v^2 - 20u - 10v + 44 = 0,$$

and

$$(x-u)^2 + (y-v)^2 = 225.$$

Find the value of $x + y + 2u + 2v$.

Solution. *Let us consider on a coordinate plane the following points* $A(x, y)$ *and* $B(u, v)$. *We have that* $(x-6)^2 + (y-8)^2 = 1$ *and* $(u-10)^2 + (v-5)^2 = 81$. *On the other hand, we have that* $AB = 15$. *Let us also consider points* $C(6, 8)$ *and* $D(10, 5)$, *then* $AC = 1$, $BD = 9$, $CD = 5$. *We have obtained that* $AB = AC + CD + BD$, *thus points* C *and* D *are on the segment* AB. *Let* M *be the midpoint of* AB, *then* $MD = 2.5$. *Thus, if point* N *is the midpoint of* BD, *then* D *is the midpoint of* MN. *Therefore, we obtain that*

$$\frac{\frac{x+u}{2} + \frac{10+u}{2}}{2} = 10,$$

and

$$\frac{\frac{y+v}{2} + \frac{v+5}{2}}{2} = 5.$$

Therefore, $x + 2u = 30$ *and* $y + 2v = 15$. *Hence,* $x + y + 2u + 2v = 45$.

Problem 3. Find the greatest solution of the inequality

$$27x^9 - 54x^7 + 36x^5 - 10x^3 + x \leq 0.$$

Solution. *Let us write the given equation in the following way*

$$3(3x^3 - 2x)^3 - 2(3x^3 - 2x) \leq x. \tag{7.60}$$

Note that $x = 1$ *is a solution of* (7.66).

If $x > 1$, *we have that* $3x^3 - 2x > x$. *Thus,*

$$3(3x^3 - 2x)^3 - 2(3x^3 - 2x) > 3x^3 - 2x > x.$$

Hence, $3(3x^3 - 2x)^3 - 2(3x^3 - 2x) > x$. *Therefore, the greatest solution of the given inequality is equal to 1.*

Problem 4. Let x, y, z be real numbers, such that

$$x^3 + y^3 = \frac{100}{x^2 + xy + y^2},$$

$$y^3 + z^3 = \frac{50}{y^2 + yz + z^2},$$

$$z^3 + x^3 = \frac{25}{z^2 + zx + x^2}.$$

Find the value of the following expression

$$\frac{30z}{3x - 2y + z}.$$

Solution. *We have that*

$$(x^3 + y^3)(x^2 + xy + y^2) = 100.$$

Hence, $x^6 - y^6 = 100(x - y)$. *In a similar way, we obtain that* $y^6 - z^6 = 50(y - z)$ *and* $z^6 - x^6 = 25(z - x)$. *Summing up the obtained equalities, we deduce that* $4(x - y) + 2(y - z) + z - x = 0$. *Hence,* $3x - 2y = z$. *Therefore,*

$$\frac{30z}{3x - 2y + z} = 15.$$

Problem 5. Let a, b, c, d, e, f be real numbers, such that the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

has exactly n real-valued solutions. Find $n^2 - n + 50$.

Solution. *Let us prove that if the following equation*

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{7.61}$$

has two distinct solutions (x_1, y_1) *and* (x_2, y_2), *then it has infinitely many solutions.*

Let (x_1, y_1) *and* (x_2, y_2) *be distinct solutions. Without loss of generality, one can assume that* $y_1 \neq y_2$.

Let $a > 0$. *As* $ax^2 + (by_1 + d)x + cy_1^2 + ey_1 + f = 0$ *has a solution, then* $F(y_1) = (by_1 + d)^2 - 4a(cy_1^2 + ey_1 + f) \geq 0$. *Let us consider*

$$F(y) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + d^2 - 4af.$$

As $F(y_1), F(y_2) \geq 0$ and $y_1 \neq y_2$, then for infinitely many y we have that $F(y) \geq 0$. On the other hand, for all the y (7.67) has solutions.

If $a = 0$, then we have that

$$(by+d)x + cy^2 + ey + f = 0. \tag{7.62}$$

If $b = d = 0$, then (7.62) has infinitely many solutions.

If $b^2 + d^2 \neq 0$, then

$$x = -\frac{cy^2 + ey + f}{by+d}.$$

Thus, equation (7.62) has infinitely many solutions. Therefore, equation (7.67) has infinitely many solutions. We obtain that $n = 0$ or $n = 1$, hence $n^2 - n + 50 = 50$.

Problem 6. Solve the following equation

$$(x-1)^{215}\sqrt{|x-1|+1} + (x-2)^{215}\sqrt{|x-2|+1} + \cdots + (x-215)^{215}\sqrt{|x-215|+1} = 0.$$

Solution. *Note that the function*

$$f(x) = x^{215}\sqrt{|x|+1}$$

is an odd and increasing function. Thus, the function $g(x) = f(x-1) + f(x-2) + \cdots + f(x-215)$ is an increasing function and $g(108) = f(107) + f(106) + \cdots + f(0) + \cdots + f(-107) = f(107) + f(106) + \cdots + f(-107) = f(107) + f(106) + \cdots + f(0) - f(1) - \cdots - f(107) = 0$. As $g(x)$ is an increasing function, then it cannot accept a value equal to 0 at two different points. Therefore, the single root of the given equation is 108.

Problem 7. Let $p(x)$ be a polynomial of the fourth degree with real coefficients, such that for any real x its value is non-negative. Given that $p(1) = p(2) = 0$, $p(3) = 12$. Find $p(4) + p(5)$.

Solution. *We have that $p(x) = (x-1)(x-2)q(x)$, where $q(x)$ is a quadratic polynomial. As $p(x)$ has only non-negative value, then the roots of $q(x)$ are 1 and 2. Thus $p(x) = a(x-1)^2(x-2)^2$. We have that $p(3) = 12$, then $a = 3$. Therefore, $p(4) + p(5) = 540$.*

Problem 8. Let a be a real number, such that the following inequality

$$(x-4)^6 + 3x^2 + 3y^2 - 24x - 18y + 75 - 3a \leq (a - 9 + 6y - y^2)^3$$

has a unique solution in the set of real numbers. Find a.

Solution. *We have that*

$$(x-4)^6 + 3(x-4)^2 \leq (a-9+6y-y^2)^3 + 3(a-9+6y-y^2).$$

Let us consider the following function

$$f(x) = x^3 + 3x.$$

Note that if $x_2 > x_1$, then

$$f(x_2) - f(x_1) = (x_2 - x_1)(x_2^2 + x_1x_2 + x_1^2 + 3) = (x_2 - x_1)\left(\left(x_2 + \frac{x_1}{2}\right)^2 + \frac{3}{4}x_1^2 + 3\right) > 0.$$

Therefore, f is an increasing function.

The given inequality is equivalent to the following inequality

$$f((x-4)^2) \leq f(a - 9 + 6y + y^2),$$

or to the following inequality

$$(x-4)^2 \leq a - 9 + 6y + y^2.$$

We have that the following inequality

$$(x-4)^2 + (y-3)^2 \leq a - 18$$

has a unique solution, thus $a = 18$.

Problem 9. Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $a + b + c = 1$ and

$$A = 10(\sqrt{(1+a)(1+b)} + \sqrt{(1+b)(1+c)} + \sqrt{(1+c)(1+a)}.$$

Find the sum of all possible values of the integer part of A.

Solution. *We have that*

$$\sqrt{(1+a)(1+b)} + \sqrt{(1+b)(1+c)} + \sqrt{(1+c)(1+a)} \leq \frac{1+a+1+b}{2} + \frac{1+b+1+c}{2} +$$

$$+ \frac{1+c+1+a}{2} = 3 + a + b + c = 4.$$

Therefore, $A \leq 40$. Note that if $x \geq y \geq z \geq 0$, then

$$\sqrt{1+x} + \sqrt{1+y} \geq \sqrt{1+x+z} + \sqrt{1+y-z}. \tag{7.63}$$

Indeed, we need to show that $(1+x)(1+y) \geq (1+x+z)(1+y-z)$ or $0 \geq (y-x)z - z^2$. Obviously, the last inequality holds true.

Let $a \geq b \geq c$, according to (7.63), we have that

$$\sqrt{1+a+b} + \sqrt{1+b+c} + \sqrt{1+c+a} \geq \sqrt{1+a+b+c} + \sqrt{1+b} + \sqrt{1+c+a} \geq$$

$$\geq \sqrt{2}+1+\sqrt{1+c+a+b}=1+2\sqrt{2}.$$

Therefore, $A \geq 10+20\sqrt{2} > 38$.

If $a=b=c=\frac{1}{3}$, *then we have that* $[A]=40$.

If $a=b=\frac{1}{2}$, $c=0$, *then we have that* $A=15+10\sqrt{6}$, $[A]=39$.

If $a=1$, $b=c=0$, *then we have that* $A=10+20\sqrt{2}$, $[A]=38$.

Hence, the required sum is equal to 117.

7.3.7 Problem Set 7

Problem 1. Given that $x-\dfrac{1}{x}=1$. Find the value of the expression $\sqrt{5}|x^8-\dfrac{1}{x^8}|$.

Solution. *We have the* $x^2-2+\dfrac{1}{x^2}=1$, *thus* $|x+\dfrac{1}{x}|=\sqrt{5}$ *and* $x^2+\dfrac{1}{x^2}=3$.

Hence $x^4+\dfrac{1}{x^4}=7$. *Therefore,*

$$\sqrt{5}|x^8-\frac{1}{x^8}|=\sqrt{5}|x^4+\frac{1}{x^4}||x^4-\frac{1}{x^4}|=$$

$$\sqrt{5}|x^4+\frac{1}{x^4}||x^2+\frac{1}{x^2}||x-\frac{1}{x}||x+\frac{1}{x}|=\sqrt{5}\cdot 7\cdot 3\sqrt{5}=105.$$

Problem 2. Solve the equation

$$\cos 2\pi x-4\cos \pi x+2x^2-24x+75=0.$$

Solution. *We have that* $2(\cos \pi x-1)^2+2(x-6)^2=0$. *Thus*

$$\begin{cases}\cos \pi x-1=0\\ x=6.\end{cases}$$

Therefore, $x=6$.

Problem 3. Consider the numbers 200, 201, ..., 400. At most, how many numbers one can choose among those numbers, such that they make a geometric progression?

Solution. *The following geometric progression 216, 252, 294, 343 has four terms. Let us show that from the given numbers (in the initial problem) one cannot choose five numbers, such that they make a geometric progression. We proceed by contradiction argument. Assume that* $b_1, b_2, \ldots, b_5$ *is a geometric progression and* $q>1$, $b_1>200$, $b_5 \leq 400$. *Thus,* $q^4=\dfrac{b_5}{b_1}\leq 2$. *Hence,* $q=1+\dfrac{m}{n}$, *where*

$m,n \in \mathbb{N}$, $(m,n)=1$. According to Bernoulli's inequality, we deduce that $1+\frac{4m}{n} < \left(1+\frac{m}{n}\right)^4 \le 2$. *Therefore,* $\frac{m}{n} < \frac{1}{4}$ *and* $n \ge 5$. *We have that* $b_5 = b_1 q^4 \in \mathbb{N}$. *Thus,* $n^4 \mid b_1$. *This leads to a contradiction.*

Problem 4. Let A, B, C be pairwise disjoint sets. Let the number of elements of the union of the sets A,B,C is divisible by 80. Given that the pairwise arithmetic means of the sets $A\cup B\cup C, A\cup B, B\cup C, C\cup A$ are equal to 3, -1, 13, 6, respectively. Find the number of elements of the set $A\cup B\cup C$.

Solution. *Let us denote by $S(M)$ the sum of all the elements of the finite set M and by $n(M)$ the number of elements. According to the assumptions of the problem, we have that*

$$S(A)+S(B)+S(C)=3(n(A)+n(B)+n(C)), \tag{7.64}$$

$$S(A)+S(B)=-(n(A)+n(B)), \tag{7.65}$$

$$S(B)+S(C)=13(n(B)+n(C)), \tag{7.66}$$

$$S(A)+S(C)=6(n(A)+n(C)). \tag{7.67}$$

Summing up the equations (7.90), (7.66), (7.67), we obtain that

$$2(S(A)+S(B)+S(C))=5n(A)+12n(B)+19n(C).$$

From (7.89), we deduce that

$$6n(A)+6n(B)+6n(C)=5n(A)+12n(B)+19n(C).$$

Hence, we have that $n(A)=6n(B)+13n(C)$. *Therefore,* $7 \mid 7(n(B)+2n(C)) = n(A)+n(B)+n(C)$. *We obtain that* $560 \mid n(A)+n(B)+n(C)$. *Thus,* $n(A)+n(B)+n(C)=560$.

Problem 5. Solve the inequality

$$\log_{0.3}\frac{4^{x+10}-2^{x+12}+5}{4^{2\sqrt{x}+9}-2^{2\sqrt{x}+11}+5} \ge \frac{1}{2^{2\sqrt{x}+9}-1}-\frac{1}{2^{x+10}-1}.$$

Solution. *We have that* $x \ge 0$, *thus* $2^{x+10}-1 \ge 2^{2\sqrt{x}+9}-1 > 0$. *Note that the given inequality is equivalent to the following inequality*

$$\log_{0.3}\frac{(2^{x+10}-2)^2+1}{(2^{2\sqrt{x}+9}-2)^2+1} \ge \frac{1}{2^{2\sqrt{x}+9}-1}-\frac{1}{2^{x+10}-1}. \tag{7.68}$$

On the other hand, we have that

$$0 \geq \log_{0.3} \frac{(2^{x+10}-2)^2+1}{(2^{2\sqrt{x}+9}-2)^2+1},$$

and

$$\frac{1}{2^{2\sqrt{x}+9}-1} - \frac{1}{2^{x+10}-1} \geq 0.$$

Therefore, the inequality (7.68) is equivalent to the equation $2^{2\sqrt{x}+9}-1 = 2^{x+10}-1$. *Hence, we deduce that* $2\sqrt{x}+9 = x+10$. *Thus,* $(\sqrt{x}-1)^2 = 0$. *We obtain that* $x = 1$.

Problem 6. Let $x > 0, y > 0, z > 0, t > 0$ be real numbers. Find the greatest possible value of the following expression

$$\frac{xyzt(x^3+yzt)(y^3+ztx)(z^3+txy)(t^3+xyz)}{\sqrt{(x^8+y^4z^4)(y^8+z^4t^4)(z^8+t^4x^4)(t^8+x^4y^4)}}.$$

Solution. *We have that*

$$x^8+y^4z^4 = \frac{x^9}{x\cdot 1} + \frac{y^3z^3t^3}{\dfrac{t^3}{yz}\cdot 1} \geq \frac{(x^3+yzt)^3}{\left(x+\dfrac{t^3}{yz}\right)\cdot 2}.$$

In a similar way, one can deduce that

$$y^8+z^4t^4 \geq \frac{(y^3+ztx)^3}{\left(y+\dfrac{x^3}{zt}\right)\cdot 2},$$

$$z^8+t^4x^4 \geq \frac{(z^3+txy)^3}{\left(z+\dfrac{y^3}{tx}\right)\cdot 2},$$

and

$$t^8+x^4y^4 \geq \frac{(t^3+xyz)^3}{\left(t+\dfrac{z^3}{xy}\right)\cdot 2}.$$

Multiplying all these inequalities, we obtain that

$$4 \geq \frac{xyzt(x^3+yzt)(y^3+ztx)(z^3+xyt)(t^3+xyz)}{\sqrt{(x^8+y^4z^4)(y^8+z^4t^4)(z^8+t^4x^4)(t^8+x^4y^4)}} = A.$$

Note that if $x = y = z = t = 1$, then $A = 4$. Therefore, the greatest value of the given expression is equal to 4.

Problem 7. Let x, y, z be real numbers, such that $67[y] + 39\{z\} - 5x^2 = 408.3$, $39[z] + 24\{x\} - 5y^2 = -97.85$ and $24[x] + 67\{y\} - 5y^2 = 18.45$. Find the value of the sum $x + y + z$. Here, we denote by $\{x\}$ the fractional part and $[x]$ the integer part of a real number x.

Solution. *Summing up the given equations, we deduce that*

$$(x - 2.4)^2 + (y - 6.7)^2 + (z - 3.9)^2 = 0.$$

Thus, $x = 2.4$, $y = 6.7$, $z = 3.9$. One can easily verify that the obtained triple is a solution of the given system. Hence, $x + y + z = 13$.

Problem 8. Find the number of the real solutions of the following equation

$$(\sqrt{3} + x)^2 + (1 - \sqrt{3}x)^2 = (\sqrt{3} + x)^2 \cdot (1 - \sqrt{3}x)^2.$$

Solution. *We have that*

$$\frac{1}{(\sqrt{3} + x)^2} + \frac{1}{(1 - \sqrt{3}x)^2} = 1.$$

Hence, if $\dfrac{1}{\sqrt{3} + x} = \cos t$, $\dfrac{1}{1 - \sqrt{3}x} = \sin t$. *We deduce that*

$$\frac{\sqrt{3}}{\cos t} + \frac{1}{\sin t} = 4$$

and

$$\sin\left(t + \frac{\pi}{6}\right) = \sin 2t.$$

Therefore, the following numbers

$$\frac{1}{\cos\frac{3\pi}{18}} - \sqrt{3}, \frac{1}{\cos\frac{5\pi}{18}} - \sqrt{3}, \frac{1}{\cos\frac{17\pi}{18}} - \sqrt{3}, \frac{1}{\cos\frac{29\pi}{18}} - \sqrt{3},$$

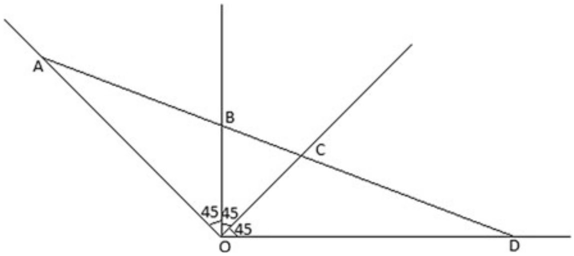
are the solutions of the given equation. Hence, the given equation of the fourth degree has four real solutions.

Problem 9. Let $a > 0$, $b > 0$, $c > 0$, $d > 0$ be real numbers, such that

$$\sqrt{a^2 + b^2 - \sqrt{2}ab} + \sqrt{b^2 + c^2 - \sqrt{2}bc} + \sqrt{c^2 + d^2 - \sqrt{2}cd} = \sqrt{a^2 + d^2 + \sqrt{2}ad}.$$

Find the value of the following expression $\dfrac{(a+c)(b+d)}{ad}$.

Solution. *Let us consider the rays OA, OB, OC, OD, such that* $\angle AOB = \angle BOC = \angle COD = 45^\circ$, $OA = a$, $OB = b$, $OC = c$, $OD = d$.



According to the law of cosines, the given equation can rewritten in the following way $AB + BC + CD = AD$. *Hence, points* A, B, C, D *are on the same line. On the other hand, from the following equations* $S_{AOB} + S_{BOC} = S_{AOC}$, $S_{BOC} + S_{COD} = S_{BOD}$, *we obtain that* $\dfrac{\sqrt{2}}{b} = \dfrac{1}{a} + \dfrac{1}{c}$ *and* $\dfrac{\sqrt{2}}{c} = \dfrac{1}{b} + \dfrac{1}{d}$. *Therefore,*

$$\frac{(a+c)(b+d)}{ad} = \frac{\sqrt{2}ac}{b} \cdot \frac{\sqrt{2}bd}{c} \cdot \frac{1}{ad} = 2.$$

7.3.8 Problem Set 8

Problem 1. Let a, b, c be the terms of an arithmetic progression. Given that the numbers $\dfrac{1}{c}, \dfrac{1}{b+1}, \dfrac{1}{a+2}$ are the terms of an arithmetic progression. Find the value of the difference $b-a$.

Solution. *We have that*

$$\begin{cases} 2(b+1) = c + (a+2), \\ \dfrac{2}{b+1} = \dfrac{1}{c} + \dfrac{1}{a+2}. \end{cases}$$

Thus, it follows that $c = b+1 = a+2$.

Therefore, $b - a = 1$.

Problem 2. Let $x^2 + y^2 = \sqrt{3} + xy$. Find the value of the following expression

$$x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4.$$

Solution. *We have that*

$$x^2 - xy + y^2 = \sqrt{3}.$$

Thus, it follows that

$$(x^2 - xy + y^2)^2 = 3.$$

Therefore,

$$x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4 = 3.$$

Problem 3. Consider an increasing sequence of positive integers, such that the first term is equal to 2 and the last term is equal to 9. Given that any three consecutive terms of this sequence make either an arithmetic progression or a geometric progression. At least, how many terms can have this sequence?

Solution. *Note that the sequence 2, 4, 6, 9 satisfies the assumptions of the problem.*

On the other side, a sequence satisfying the assumptions of the problem cannot have only three terms.

Thus, the answer is 4.

Problem 4. Solve the equation

$$x + 3^5 = 108\sqrt[4]{x}.$$

Solution. *We have that*

$$x + 3^5 = x + 3^4 + 3^4 + 3^4 \geq 4 \cdot \sqrt[4]{3^4 \cdot 3^4 \cdot 3^4 \cdot x} = 108\sqrt[4]{x}.$$

Thus, it follows that the equality holds true when $x = 3^4$.

Problem 5. Given that $x^2 + \frac{1}{x} = 3$. Find the value of the following expression

$$9x^2 + 3x + \frac{1}{x^3}.$$

Solution. *We have that*

$$x^2 + \frac{1}{x} = 3.$$

Thus, it follows that

$$x^3 = 3x - 1,$$

and

$$\left(x^2 + \frac{1}{x}\right)^3 = 27.$$

Hence, we deduce that

$$\frac{1}{x^3} = 24 - (x^3)^2 - 3x^3 = 24 - (3x-1)^2 - 3(3x-1).$$

We obtain that

$$\frac{1}{x^3} = 26 - 9x^2 - 3x.$$

Therefore,

$$9x^2 + 3x + \frac{1}{x^3} = 26.$$

Problem 6. Consider numbers a, b, c, such that

$$\frac{a^3}{4a^2+2ab+b^2} + \frac{b^3}{4b^2+2bc+c^2} + \frac{c^3}{4c^2+2ca+a^2} = 10$$

and

$$\frac{b^3}{4a^2+2ab+b^2} + \frac{c^3}{4b^2+2bc+c^2} + \frac{a^3}{4c^2+2ca+a^2} = 12.$$

Find the value of the sum $a+b+c$.

Solution. *Note that*

$$\frac{8a^3}{4a^2+2ab+b^2} - \frac{b^3}{4a^2+2ab+b^2} = \frac{(2a)^3-b^3}{4a^2+2ab+b^2} = 2a-b.$$

If we multiply both sides of the first equation by 8 and subtract the second equation, then we deduce that

$$(2a-b)+(2b-c)+(2c-a) = 80-12.$$

Therefore, we obtain that $a+b+c = 68$.

Problem 7. Find the number of solutions of the following equation

$$[x] + 2\{x^2\} = [x^3],$$

where by $[a]$ we denote the integer part of real number a and by $\{a\}$ we denote the fractional part.

Solution. *Note that*

$$2\{x^2\} = [x^3] - [x]$$

is an integer number and $0 \le 2\{x^2\} < 2$. *Thus, either* $\{x^2\} = 0$ *or* $\{x^2\} = 0.5$.

If $\{x^2\} = 0$, *then* $[x^3] - [x] = 0$.

We have that $x^2 \in \mathbb{Z}$. *Hence, if* $x^2 \geq 2$, $|x^3 - x| = |x| \cdot |x^2 - 1| \geq |x| > 1$. *Then,* $|[x^3] - [x]| \geq 1$.

Note that $x = 0$, $x = 1$ *and* $x = -1$ *are the solutions of the given equation.*

If $\{x^2\} = 0.5$, *then* $[x^3] - [x] = 1$.

If $x^2 \geq 2.5$, *then*

$$|x^3 - x| = |x| \cdot |x^2 - 1| \geq 1.5\sqrt{2.5} > 2.$$

Thus, it follows that $|[x^3] - [x]| \geq 2$.

Therefore, $x^2 = 0.5$ *or* $x^2 = 1.5$. *One can easily verify that none of the solutions of those equation is a solution of the given equation.*

Hence, the given equation has three solutions.

Problem 8. Consider numbers x, y, z, such that

$$x^3 + x(y^2 + yz + z^2) = 76,$$

$$y^3 + y(x^2 + xz + z^2) = 34,$$

and

$$z^3 + z(x^2 + xy + y^2) = -29.$$

Find the value of the expression $\dfrac{30y}{3x + y + 2z}$.

Solution. *From the given equations, we deduce that*

$$x^3(y - z) + x(y^3 - z^3) = 76(y - z),$$

$$y^3(z - x) + y(z^3 - x^3) = 34(z - x),$$

$$z^3(x - y) + z(x^3 - y^3) = -29(x - y).$$

Summing up these three equations, we deduce that

$$105y - 63x - 42z = 0.$$

Thus, it follows that

$$5y = 3x + 2z,$$

and

$$\frac{30y}{3x + y + 2z} = \frac{30y}{6y} = 5.$$

Problem 9. Let $p(x)$ be n-th degree polynomial with integer coefficients. Given that $p\left(2^{\frac{1}{5}}+2^{-\frac{1}{5}}\right)=2015$. Find the smallest possible value of n.

Solution. *At first, let us construct a polynomial $p(x)$ with integer coefficients, such that $p(a)=2015$, $a=b+\frac{1}{b}$, where $b=2^{\frac{1}{5}}$. For any positive integer n greater than 1, we have that*

$$b^n+\frac{1}{b^n}=\left(b^{n-1}+\frac{1}{b^{n-1}}\right)\left(b+\frac{1}{b}\right)-\left(b^{n-2}+\frac{1}{b^{n-2}}\right). \tag{7.69}$$

Hence, from (7.83), it follows that

$$b^5+\frac{1}{b^5}=\left(b^4+\frac{1}{b^4}\right)\left(b+\frac{1}{b}\right)-\left(b^3+\frac{1}{b^3}\right)=\left(\left(b^3+\frac{1}{b^3}\right)\left(b+\frac{1}{b}\right)-\right.$$

$$\left.\left(b^2+\frac{1}{b^2}\right)\right)\left(b+\frac{1}{b}\right)-\left(\left(b^2+\frac{1}{b^2}\right)\left(b+\frac{1}{b}\right)-\left(b+\frac{1}{b}\right)\right)=$$

$$=\left(b^3+\frac{1}{b^3}\right)\left(b+\frac{1}{b}\right)^2-2\left(b^2+\frac{1}{b^2}\right)\left(b+\frac{1}{b}\right)+\left(b+\frac{1}{b}\right)=$$

$$=\left(\left(b+\frac{1}{b}\right)\left(b^2+\frac{1}{b^2}\right)-\left(b+\frac{1}{b}\right)\right)\left(b+\frac{1}{b}\right)^2-2\left(\left(b+\frac{1}{b}\right)^2-2\right)\left(b+\frac{1}{b}\right)+$$

$$+\left(b+\frac{1}{b}\right)=\left(\left(b+\frac{1}{b}\right)^3-3\left(b+\frac{1}{b}\right)\right)\left(b+\frac{1}{b}\right)^2-2\left(b+\frac{1}{b}\right)^3+5\left(b+\frac{1}{b}\right)=$$

$$=\left(b+\frac{1}{b}\right)^5-5\left(b+\frac{1}{b}\right)^3+5\left(b+\frac{1}{b}\right).$$

We have obtained that

$$b^5+\frac{1}{b^5}=a^5-5a^3+5a.$$

On the other hand,

$$b^5+\frac{1}{b^5}=\frac{5}{2}.$$

Therefore,

$$2a^5-10a^3+10a-5=0.$$

Thus, $p(a) = 2015$*, where* $p(x) = 2x^5 - 10x^3 + 10x + 2010$.

Now, let us prove that there does not exists a polynomial $s(x)$ *with smaller degree with integer coefficients, such that* $s(a) = 2015$.

Let k *be the smallest degree of polynomial* $s(x)$ *with integer coefficients, such that* $s(a) = 2015$.

If $k < 5$*, then*

$$2x^5 - 10x^3 + 10x - 5 = t(x)q(x) + r(x),$$

where $t(x) = s(x) - 2015$.

Obviously, polynomials $q(x)$ *and* $r(x)$ *have rational coefficients. Moreover,* $r(a) = -t(a)q(a) = 0$. *This contradicts to the definition of* k*, if* $r(x)$ *is not zero polynomial.*

Therefore, it follows that

$$2x^5 - 10x^3 + 10x - 5 = \left(x + \frac{m}{n}\right)(2x^4 + r_1x^3 + r_2x^2 + r_3x + r_4),$$

or

$$2x^5 - 10x^3 + 10x - 5 = (x^2 + r_1x + r_2)(2x^3 + r_3x^2 + r_4x + r_5),$$

where $m \in \mathbb{Z}, n \in \mathbb{N}, (m,n) = 1, r_1, r_2, r_3, r_4, r_5 \in \mathbb{Q}$.

If

$$2x^5 - 10x^3 + 10x - 5 = \left(x + \frac{m}{n}\right)(2x^4 + r_1x^3 + r_2x^2 + r_3x + r_4), \tag{7.70}$$

then

$$2\left(-\frac{m}{n}\right)^5 - 10\left(-\frac{m}{n}\right)^3 + 10\left(-\frac{m}{n}\right) - 5 = 0.$$

Thus,

$$2m^5 - 10m^3n^2 + 10mn^4 = -5n^5,$$

we obtain that $m \mid 5$ *and* $n \mid 2$.

Hence,

$$\frac{m}{n} \in \left\{ \pm 1; \pm 5; \pm\frac{1}{2}; \pm\frac{5}{2} \right\}.$$

One can easily verify that none of the numbers

$$\pm 1; \pm 5; \pm\frac{1}{2}; \pm\frac{5}{2}$$

is a root of polynomial $2x^5 - 10x^3 + 10x - 5$.

Hence, (7.84) *cannot hold true.*

If

$$2x^5 - 10x^3 + 10x - 5 = (2x^2 + r_1 x + r_2)(x^3 + r_3 x^2 + r_4 x + r_5), \tag{7.71}$$

then by method of equalization of the coefficients corresponding to the same degrees of x, we obtain that

$$r_1 + 2r_3 = 0, 2r_4 + r_1 r_3 + r_2 = -10, 2r_5 + r_1 r_4 + r_2 r_3 = 0, r_1 r_5 + r_2 r_4 = 10, r_2 r_5 = -5.$$

From the first three equations, we deduce that

$$r_1 = -2r_3, r_2 = 2r_3^2 - 2r_4 - 10, r_5 = 2r_3 r_4 - r_3^3 + 5r_3,$$

and from the fourth equation, it follows that

$$r_4^2 + (r_3^2 + 5)r_4 + 5 - r_3^4 + 5r_3^2 = 0, D = 5(r_3^2 - 1)^2.$$

Therefore,

$$r_4 = \frac{-r_3^2 - 5 \pm \sqrt{5}(r_3^2 - 1)}{2}, r_3, r_4 \in \mathbb{Q}.$$

Thus,

$$r_3^2 = 1, r_4 = -\frac{r_3^2 + 5}{2} = 3, r_2 = 2 - 6 - 10 = -14, r_5 = 6r_3 - r_3 + 5r_3 = 10r_3.$$

On the other hand, $r_2 r_5 = -140 r_3 \neq -5$.

Hence, (7.85) *does not hold true.*

We have obtained that the smallest possible degree is equal to 5.

7.3.9 Problem Set 9

Problem 1. Evaluate the expression

$$\frac{(1^2 - 1 \cdot 100 + 100^2) + (2^2 - 2 \cdot 99 + 99^2) + \cdots + (50^2 - 50 \cdot 51 + 51^2)}{50^2}.$$

Solution. *We have that*

$$101 \cdot (1^2 - 1 \cdot 100 + 100^2) + 101 \cdot (2^2 - 2 \cdot 99 + 99^2) + \cdots + 101 \cdot (50^2 - 50 \cdot 51 + 51^2) =$$

$$= 1^3 + 100^3 + 2^3 + 99^3 + \cdots + 50^3 + 51^3 = (1 + 2 + \cdots + 100)^2 = 50^2 \cdot 101^2.$$

Therefore, we obtain that

$$\frac{(1^2-1\cdot 100+100^2)+(2^2-2\cdot 99+99^2)+\cdots+(50^2-50\cdot 51+51^2)}{50^2}=101.$$

Problem 2. Given that

$$\frac{x^2}{x+1}+\frac{y^2}{y+2}+\frac{z^2}{z+3}=x+y+z+106.$$

Find the value of the following expression

$$\frac{1}{4x+4}+\frac{1}{y+2}+\frac{9}{4z+12}.$$

Solution. *We have that*

$$\frac{x^2-1}{x+1}+\frac{1}{x+1}+\frac{y^2-4}{y+2}+\frac{4}{y+2}+\frac{z^2-9}{z+3}+\frac{9}{z+3}=x+y+z+106.$$

Therefore,

$$\frac{1}{x+1}+\frac{4}{y+2}+\frac{9}{z+3}=112.$$

Hence, we obtain that

$$\frac{1}{4(x+1)}+\frac{4}{y+2}+\frac{9}{4z+12}=28.$$

Problem 3. Solve the equation

$$\sqrt{x^2-4x+3}+7\sqrt{-x^2+6x-8}=7+\sqrt{x^3-10x^2+31x-30}.$$

Solution. *We have that*

$$\begin{cases} x^2-4x+3\geq 0,\\ -x^2+6x-8\geq 0,\\ x^3-10x^2+31x-30\geq 0.\end{cases}$$

The solution of this system is $x=3$, as

$$x^3-10x^2+31x-30=(x-3)(x^2-7x+10)=(x-2)(x-3)(x-5).$$

One can easily verify that $x=3$ is a solution.

Problem 4. Given that

$$a(a^2-9b+9)=(b+2)(b^2-5b+13).$$

Find the total number of the possible values of $a-b$.

Solution. *We have that*

$$a^3+(1-b)^3+(-3)^3-3a(1-b)(-3)=0.$$

Therefore,

$$(a+(1-b)+(-3))((a-(1-b))^2+(a+3)^2+(1-b+3)^2)=0.$$

Hence, $a-b=2$ or

$$\begin{cases} a=1-b, \\ a=-3, \\ b=4. \end{cases}$$

Thus, we obtain that $a-b=2$ or $a-b=-7$.

Problem 5. Given that $a+b+c=0$, $a^2+b^2+c^2=14$, $a^3+b^3+c^3=21$. Find the value of $a^6+b^6+c^6$.

Solution. *Consider the following polynomial $p(x)=(x-a)(x-b)(x-c)$. Note that*

$$p(x)=x^3-(a+b+c)x^2+(ab+bc+ac)x-abc.$$

On the other hand,

$$a^2+b^2+c^2+2(ab+bc+ac)=(a+b+c)^2.$$

Hence, $ab+bc+ac=-7$. Thus, we deduce that

$$p(x)=x^3-7x-abc.$$

Using that $p(a)+p(b)+p(c)=0$, it follows that $abc=7$. We obtain that

$$p(x)=x^3-7x-7.$$

We have that

$$ap(a)+bp(b)+cp(c)=0.$$

It follows that

$$a^4+b^4+c^4=7(a^2+b^2+c^2)=98.$$

On the other hand, using the condition

$$a^2p(a)+b^2p(b)+c^2p(c)=0,$$

we obtain that

$$a^5+b^5+c^5=7(a^3+b^3+c^3)+7(a^2+b^2+c^2)=147+98=245.$$

Now, using the condition

$$a^3p(a)+b^3p(b)+c^3p(c)=0,$$

we obtain that

$$a^6+b^6+c^6=7(a^4+b^4+c^4)+7(a^3+b^3+c^3)=7\cdot 98+7\cdot 21=7\cdot 119=833.$$

Problem 6. Evaluate the expression

$$\left[\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{\ldots\sqrt{2014+\sqrt{2015}}}}}}\right].$$

Solution. *Consider* $x_1=\sqrt{2015}$, $x_2=\sqrt{2014+\sqrt{2015}}$,

$$x_3=\sqrt{2013+\sqrt{2014+\sqrt{2015}}},$$

$$\ldots,$$

$$x_{2015}=\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{\ldots\sqrt{2014+\sqrt{2015}}}}}}.$$

Note that $x_i>1$, $i=1,2,\ldots,2015$.

Let us prove that $x_i<2017-i$, $i=1,\ldots,2015$. *Proof by mathematical induction.*

Basis. *If* $i=1$, *then obviously it holds true* $\sqrt{2015}<2016$.

Inductive step. *If the statement holds true for* $1\le k<2015$, *that is* $x_k<2017-k$, *then* $x_{k+1}<2017-(k+1)$.

We have that

$$x_{k+1}=\sqrt{2015-k+x_k}<\sqrt{2015-k+2017-k}<2016-k,$$

as

$$2 \leq 2016 - k.$$

Since both the basis and the inductive step have been proven, by mathematical induction, the statement holds true for all positive integers k.

Hence, we deduce that $x_{2015} < 2$. Therefore, $[x_{2015}] = 1$.

Problem 7. Let a, b, c be pairwise distinct real numbers. Given that

$$(b-c)\sqrt[3]{a^3+1} + (c-a)\sqrt[3]{b^3+1} + (a-b)\sqrt[3]{c^3+1} = 0.$$

Find the total number of the possible values of the following expression

$$(a^3+1)(b^3+1)(c^3+1) - (abc+1)^3.$$

Solution. *We have that*

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx).$$

Therefore, it follows that

$$(b-c)^3(a^3+1) + (c-a)^3(b^3+1) + (a-b)^3(c^3+1) =$$

$$= 3(b-c)(c-a)(a-b)\sqrt[3]{(a^3+1)(b^3+1)(c^3+1)}.$$

On the other hand, we have that

$$(b-c)^3 + (c-a)^3 + (a-b)^3 = 3(a-b)(b-c)(c-a).$$

Thus, it follows that

$$a^3(b-c)^3 + b^3(c-a)^3 + c^3(a-b)^3 = 3abc(b-c)(c-a)(a-c).$$

Hence, we obtain that

$$abc + 1 = \sqrt[3]{(a^3+1)(b^3+1)(c^3+1)}.$$

Therefore,

$$(a^3+1)(b^3+1)(c^3+1) - (abc+1)^3 = 0.$$

Problem 8. Let sequence $x_1, x_2, \ldots, x_{100}$ be such that $x_1 = x_{100} = 2015$ and

$$x_{i+1} = x_i^3 - x_{i-1}^3 + 2x_i - x_{i-1}, i = 2, 3, \ldots, 99.$$

Find the number of all positive integers n, $(1 \leq n \leq 100)$, such that $x_n = 2015$.

Solution. *Note that*

$$x_{i+1}-x_i=x_i^3-x_{i-1}^3+x_i-x_{i-1}=(x_i-x_{i-1})(x_i^2+x_ix_{i-1}+x_{i-1}^2+1).$$

On the other hand,

$$x_i^2+x_ix_{i-1}+x_{i-1}^2+1=\left(x_i+\frac{x_{i-1}}{2}\right)^2+\frac{3}{4}x_{i-1}^2+1>0.$$

Thus, if $x_1\leq x_2$*, then for* $i=2$ *we obtain that* $x_2\leq x_3$*, for* $i=3$ *it follows that* $x_3\leq x_4$ *and so on. Hence, if* $x_1\leq x_2$*, then we deduce that* $x_1\leq x_2\leq\cdots\leq x_{100}$*. In a similar way, if* $x_1\geq x_2$*, then* $x_1\geq x_2\geq\cdots\geq x_{100}$*. Using the condition* $x_1=x_{100}=2015$*, we obtain that* $x_1=x_2=\cdots=x_{100}=2015$.

Problem 9. Let a,b,c be non-negative numbers, such that $a^2+b^2+c^2\leq 6$. Find the possible greatest value of $\frac{c(a+b)}{ab+3}$.

Solution. *Let us prove that* $\frac{c(a+b)}{ab+3}\leq 1$*. Proof by contradiction argument. Assume that* $\frac{c(a+b)}{ab+3}>1$*. Denote* $a=R\cos\phi$ *and* $b=R\sin\phi$*, where* $R\geq 0$ *and* $0\leq\phi\leq\frac{\pi}{2}$*. Using that* $a^2+b^2+c^2\leq 6$*, we obtain that* $R\leq\sqrt{6}$ *and* $c\leq\sqrt{6-R^2}$*.*

Therefore, $c(a+b)>ab+3$*.*

Thus, it follows that

$$R\sqrt{6-R^2}(\cos\phi+\sin\phi)>R^2\sin\phi\cos\phi+3.$$

We deduce that

$$R\sqrt{6-R^2}(\sin\phi+\cos\phi)>R^2\frac{(\sin\phi+\cos\phi)^2-1}{2}+3.$$

Hence,

$$(R(\sin\phi+\cos\phi)-\sqrt{6-R^2})^2<0.$$

This leads to a contradiction.

We have obtained that $\frac{c(a+b)}{ab+3}\leq 1$*. On the other hand, if* $a=0$*,* $b=\sqrt{3}$ *and* $c=\sqrt{3}$*, then* $\frac{c(a+b)}{ab+3}=1$*. Hence, the greatest possible value of* $\frac{c(a+b)}{ab+3}$ *is equal to 1.*

7.3.10 Problem Set 10

Problem 1. Solve the equation

$$\frac{1}{x(x-1)}+\frac{2}{(x-1)(x-3)}+\frac{3}{(x-3)(x-6)}+\frac{4}{(x-6)(x-10)}=-0.4.$$

Solution. *We have that*

$$\frac{1}{x(x-1)}+\frac{2}{(x-1)(x-3)}+\frac{3}{(x-3)(x-6)}+\frac{4}{(x-6)(x-10)}=\frac{1}{x-1}-\frac{1}{x}+\frac{1}{x-3}-$$

$$-\frac{1}{x-1}+\frac{1}{x-6}-\frac{1}{x-3}+\frac{1}{x-10}-\frac{1}{x-6}=\frac{1}{x-10}-\frac{1}{x}=\frac{10}{(x-10)x}.$$

Hence, we obtain that

$$\frac{10}{(x-10)x}=-0.4.$$

Therefore, it follows that $x=5$.

Problem 2. Given that $x^3-\frac{1}{x}=2$. Find the value of the following expression

$$-x^4+3x^3+2x-\frac{3}{x}.$$

Solution. *We have that* $x^4=2x+1$ *and* $x^3=2+\frac{1}{x}$. *Thus, it follows that*

$$-x^4+3x^3+2x-\frac{3}{x}=-(2x+1)+3\left(2+\frac{1}{x}\right)+2x-\frac{3}{x}=5.$$

Problem 3. Find the number of solutions of the equation

$$\sqrt{[x]}+\sqrt{\{x\}}=x,$$

where $[x]$ is the integer part and $\{x\}$ is the fractional part of a real number x.

Solution. *Note that* $x_0\geq 0$, *where* x_0 *is a solution of the given equation. We have that*

$$[x_0]+2\sqrt{[x_0]\{x_0\}}+\{x_0\}=x_0^2.$$

Thus, it follows that

$$x_0^2-x_0=2\sqrt{[x_0]\{x_0\}}\leq[x_0]+\{x_0\}=x_0.$$

Therefore, $0\leq x_0\leq 2$.

Note that $x_0 = 0$ and $x_0 = 1$ satisfy the assumptions of the problem.

If $0 < x_0 < 1$, then

$$\sqrt{[x_0]} + \sqrt{\{x_0\}} = \sqrt{x_0} > x_0.$$

This leads to a contradiction.

If $1 < x_0 < 2$, then

$$\sqrt{[x_0]} + \sqrt{\{x_0\}} = 1 + \sqrt{\{x_0\}} > 1 + \{x_0\} = x_0.$$

This leads to a contradiction.

If $x_0 = 2$, then this case leads to a contradiction.

Hence, the given equation has two solutions.

Problem 4. Find the value of the expression

$$\left(\frac{1}{1-a+a^2} - \frac{1}{1+a+a^2} - \frac{2a}{1-a^2+a^4} + \frac{4a^3}{1-a^4+a^8}\right) : \frac{a^7}{1+a^8+a^{16}}.$$

Solution. *We have that*

$$\frac{1}{1-a+a^2} - \frac{1}{1+a+a^2} - \frac{2a}{1-a^2+a^4} + \frac{4a^3}{1-a^4+a^8} = \frac{2a}{(1+a^2)^2-a^2} - \frac{2a}{1-a^2+a^4} +$$

$$+\frac{4a^3}{1-a^4+a^8} = \frac{-4a^3}{(1+a^4)^2-(a^2)^2} + \frac{4a^3}{1-a^4+a^8} = \frac{8a^7}{(1+a^8)^2-(a^4)^2} = \frac{8a^7}{1+a^8+a^{16}}.$$

Therefore, we obtain that

$$\left(\frac{1}{1-a+a^2} - \frac{1}{1+a+a^2} - \frac{2a}{1-a^2+a^4} + \frac{4a^3}{1-a^4+a^8}\right) : \frac{a^7}{1+a^8+a^{16}} = 8.$$

Problem 5. Consider a sequence (x_n), such that $x_1 = 2$ and

$$x_{n+1} = x_n + \sqrt{2x_{n+1} + 2x_n}, n = 1, 2, \dots$$

Find x_{31}.

Solution. *Note that $x_{n+1} \geq x_n$, $n = 1, 2, \dots$. We have that $x_1 = 2$. Thus, it follows that $x_n \geq 2$, $n = 1, 2, \dots$*

On the other hand,

$$x_{n+1}^2 - 2(x_n + 1)x_{n+1} + x_n^2 - 2x_n = 0.$$

We deduce that

$$x_{n+1} = x_n + 1 + \sqrt{4x_n + 1},\ n = 1, 2, \dots$$

Therefore, sequence (x_n) is unique.

Note that sequence $x_n = n(n+1)$ *satisfies assumptions of the problem. In this case, we have that* $x_1 = 2$. *We need to prove that*

$$(n+1)(n+2) = n(n+1) + \sqrt{2(n+1)(n+2)+2n(n+1)},\ n = 1,2,\dots$$

This holds true, as it is equivalent to

$$2n+2 = \sqrt{2(n+1)(n+2)+2n(n+1)}, n = 1,2,\dots$$

We obtain that

$$x_{31} = 31 \cdot 32 = 992.$$

Problem 6. Given that the circle ω and the graph of function $y = x^3 - 9x^2 + 48x - 100$ intersect at points $A_i(x_i, y_i)$, $i = 1,2,\dots,6$. Find the value of $x_1 + x_2 + \dots + x_6$.

Solution. *Let the equation of circle* ω *be*

$$(x-x_0)^2 + (y-y_0)^2 = R^2.$$

Therefore, x_1, x_2, $\dots$, x_6 *are the roots of the equation*

$$(x-x_0)^2 + (x^3 - 9x^2 + 48x - 100 - y_0)^2 - R^2 = 0.$$

Let us rewrite this equation in the following way

$$x^6 - 18x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0.$$

According to the Vieta's theorem

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18.$$

Problem 7. Given that

$$\begin{cases} x^2 + xy + y^2 = 169, \\ y^2 + yz + z^2 = 400, \\ x^2 + xz + z^2 = 441. \end{cases}$$

Find the value of $\dfrac{231y + 41z}{x}$.

Solution. *We have that*

$$\begin{cases} 169(x-y) = x^3 - y^3, \\ 400(y-z) = y^3 - z^3, \\ 441(z-x) = z^3 - x^3. \end{cases}$$

Summing up these equations, we obtain that

$$169(x-y)+400(y-z)+441(z-x)=0.$$

Therefore, it follows that

$$\frac{231y+41z}{x}=272.$$

Problem 8. Let x_1, x_2, x_3 be pairwise distinct real numbers and roots of the equation

$$x^4-6x^3+11x^2+bx+c=0.$$

Given that

$$x_1+2x_2+3x_3=14.$$

Find $|b|+|c|$.

Solution. *Let x_1, x_2, x_3, x_4 be the roots of the equation*

$$x^4-6x^3+11x^2+bx+c=0.$$

According to the Vieta's theorem

$$x_1+x_2+x_3+x_4=6,$$

and

$$x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4=11.$$

Note that

$$(x_1-1)^2+(x_2-2)^2+(x_3-3)^2+x_4^2=x_1^2+x_2^2+x_3^2+x_4^2-2(x_1+2x_2+3x_3)+14=$$

$$=(x_1+x_2+x_3+x_4)^2-2(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)-2\cdot 14+14=$$

$$=36-22-14=0.$$

Thus, it follows that $x_1=1$, $x_2=2$, $x_3=3$, $x_4=0$.

According to the Vieta's theorem

$$b=-(x_1x_2x_3+x_1x_2x_4+x_2x_3x_4+x_1x_3x_4)=-6,$$

and

$$c=x_1x_2x_3x_4=0.$$

Therefore, we obtain that

$$|b|+|c|=6.$$

Problem 9. Let x, y, z be real numbers. Given that the smallest value of

$$\frac{(x^2-x+1)(y^2-y+1)(z^2-z+1)}{(xyz)^2-xyz+1}.$$

is equal to m. Find the value of $(m+3)^2$.

Solution. *Let $x_0+\frac{1}{x_0}=1+\sqrt{3}$ and $x=y=z=x_0$, then*

$$\frac{(x^2-x+1)(y^2-y+1)(z^2-z+1)}{(xyz)^2-xyz+1}=\frac{\left(x_0+\frac{1}{x_0}-1\right)\left(x_0+\frac{1}{x_0}-1\right)\left(x_0+\frac{1}{x_0}-1\right)}{x_0^3+\frac{1}{x_0^3}-1}=$$

$$=\frac{3\sqrt{3}}{(1+\sqrt{3})^3-3(1+\sqrt{3})-1}=2\sqrt{3}-3.$$

Let us now prove that the smallest value of

$$\left(1+\frac{2}{\sqrt{3}}\right)(x^2-x+1)(y^2-y+1)(z^2-z+1)\geq(xyz)^2-xyz+1,$$

is equal to $2\sqrt{3}-3$. Therefore, $(m+3)^2=12$.

If one of the numbers x, y, z is equal to 0, for example $z=0$, then

$$\frac{(x^2-x+1)(y^2-y+1)(z^2-z+1)}{(xyz)^2-xyz+1}=(x^2-x+1)(y^2-y+1)\geq\frac{3}{4}\cdot\frac{3}{4}>2\sqrt{3}-3.$$

Without loss of generality, one can assume that $yz>0$.

Denote by $k=1+\frac{2}{\sqrt{3}}$, $A=(y^2-y+1)(z^2-z+1)$, $B=yz$, then one needs to prove that

$$(kA-B^2)x^2-(kA-B)x+kA-1\geq 0. \tag{7.72}$$

Note that

$$kA\geq k\cdot\frac{3}{4}y^2\cdot\frac{3}{4}z^2>B^2.$$

Hence, if we prove that

$$D=(kA-B)^2-4(kA-B^2)(kA-1)\leq 0,$$

then (7.98) holds true.

We need to prove that

$$3k^2A^2 - 2k(2B^2 + 2 - B)A + 3B^2 \geq 0.$$

It is sufficient to prove that

$$A \geq \frac{2B^2 + 2 - B + 2\sqrt{(B-1)^2(B^2+B+1)}}{3k}. \tag{7.73}$$

Note that

$$A = (y+z)^2 - (B+1)(y+z) + B^2 - B + 1. \tag{7.74}$$

Let $t = \sqrt{B} + \frac{1}{\sqrt{B}}$. *If* $t > 4$, *then from* (7.74), *we obtain that*

$$A \geq \left(\frac{B+1}{2}\right)^2 - (B+1)\frac{B+1}{2} + B^2 - B + 1 = \frac{3}{4}(B^2 - 2B + 1).$$

Now, let us prove that

$$\frac{9k}{4}(B^2 - 2B + 1) \geq 2B^2 + 2 - B + 2\sqrt{(B-1)^2(B^2+B+1)},$$

$$\frac{9k}{4}\left(B + \frac{1}{B} - 2\right) \geq 2\left(B + \frac{1}{B}\right) - 1 + 2\sqrt{\left(B + \frac{1}{B} - 2\right)\left(B + \frac{1}{B} + 1\right)},$$

$$\frac{9k}{4}(t^2 - 4) \geq 2t^2 - 5 + 2\sqrt{(t^2-4)(t^2-1)},$$

this holds true, as

$$\sqrt{(t^2-4)(t^2-1)} < \frac{t^2 - 4 + t^2 - 1}{2},$$

and

$$\frac{9k}{4}(t^2 - 1) > 4t^2 - 10.$$

Indeed, we have that

$$\frac{9k}{4}(t^2-1) - (4t^2 - 10) = \left(\frac{9k}{4} - 4\right)t^2 + 10 - \frac{9k}{4} > \left(\frac{9k}{4} - 4\right) \cdot 16 + 10 - \frac{9k}{4} > 0.$$

Therefore, (7.73) *holds true.*

If $t \leq 4$, *then* $y + z \geq 2\sqrt{yz} = 2\sqrt{B} \geq \frac{B+1}{2}$, *then from* (7.74) *we obtain that*

$$A \geq (2\sqrt{B})^2 - (B+1) \cdot 2\sqrt{B} + B^2 - B + 1 = B^2 + 3B + 1 - 2(B+1)\sqrt{B}.$$

It is sufficient to prove that

$$3k(B^2+3B+1-2(B+1)\sqrt{B}) \geq 2B^2+2-B+2\sqrt{(B^2-2B+1)(B^2+B+1)},$$

$$3k(t-1)^2 \geq 2t^2-5+2\sqrt{(t^2-4)(t^2-1)},$$

or

$$(9+4\sqrt{3})t^4-4\sqrt{3}(8+5\sqrt{3})t^3+(144+84\sqrt{3})t^2-8(11\sqrt{3}+18)t+60+32\sqrt{3} \geq 0,$$

$$(t-1-\sqrt{3})^2((9+4\sqrt{3})t^2-(18+6\sqrt{3})t+12+2\sqrt{3}) \geq 0.$$

This inequality holds true, as

$$D_1 = (9+3\sqrt{3})^2-(9+4\sqrt{3})(12+2\sqrt{3}) < 0.$$

This ends the proof. Hence, $(m+3)^2 = 12$.

7.3.11 Problem Set 11

Problem 1. Let (b_n) be a geometric progression. Given that

$$b_3+\sqrt[3]{2}b_7+\sqrt[3]{4}b_{20} = 0.$$

Find the value of the expression

$$\frac{b_3^3+2b_7^3+4b_{20}^3}{b_{10}^3}.$$

Solution. *Let us use the following formula*

$$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-xz-yz).$$

Thus, it follows that

$$b_3^3+2b_7^3+4b_{20}^3 = 3\cdot b_3\cdot\sqrt[3]{2}b_7\cdot\sqrt[3]{4}b_{20} = 6b_1q^2b_1q^6b_1q^{19} = 6(b_1q^9)^3 = 6b_{10}^3.$$

Therefore,

$$\frac{b_3^3+2b_7^3+4b_{20}^3}{b_{10}^3} = 6.$$

Problem 2. Solve the equation

$$\sqrt{-2x^2+12x-15-2\sqrt{2}}+\sqrt{-3x^2+18x-21-4\sqrt{2}}=1.$$

Solution. *Note that*

$$\sqrt{-2x^2+12x-15-2\sqrt{2}}=\sqrt{3-2\sqrt{2}-2(x-3)^2}\leq\sqrt{2}-1,$$

and

$$\sqrt{-3x^2+18x-21-4\sqrt{2}}=\sqrt{6-4\sqrt{2}-3(x-3)^2}\leq 2-\sqrt{2}.$$

Therefore,

$$\sqrt{-2x^2+12x-15-2\sqrt{2}}+\sqrt{-3x^2+18x-21-4\sqrt{2}}\leq 1.$$

The equality sign holds true only, if $x=3$.

Problem 3. Find the number of solutions of the equation

$$20\sqrt{[x]}-\sqrt{\{x\}}=x,$$

where we denote by $[x]$ the integer part and by $\{x\}$ the fractional part of a real number x.

Solution. *We have that $[x]\geq 0$, thus $x\geq 0$.*

If $x>400$, then $x=\sqrt{x}\cdot\sqrt{x}>20\sqrt{x}\geq 20\sqrt{[x]}\geq 20\sqrt{[x]}-\sqrt{\{x\}}$. Therefore, the roots of the given equation belong to $[0,400]$.

Let $[x]=k$, where $k=0,1,\ldots,400$. We have that $20\sqrt{k}-x=\sqrt{x-k}$. Hence,

$$x=\frac{40\sqrt{k}+1-\sqrt{80\sqrt{k}-4k+1}}{2}.$$

On the other hand, from the condition $[x]=k$, we obtain that

$$\frac{40\sqrt{k}+1-\sqrt{80\sqrt{k}-4k+1}}{2}<k+1.$$

Thus, it follows that $40\sqrt{k}-2k\leq 4$. We obtain that either $\sqrt{k}\leq 10-\sqrt{98}$ or $\sqrt{k}\geq 10+\sqrt{98}$. Therefore,

$$x=\frac{40\sqrt{k}+1-\sqrt{80\sqrt{k}-4k+1}}{2},$$

where $k=0,396,397,398,399,400$.

Hence, the given equation has six solutions.

Problem 4. Let $p(x)$ be n-th degree polynomial with integer coefficients, such that

$$p(1+\sqrt{2}) = 54+29\sqrt{2},$$

$$p(1+\sqrt{3}) = 89+44\sqrt{3},$$

and

$$p(1+\sqrt{5}) = 189+80\sqrt{5}.$$

Find the smallest possible value of n.

Solution. *Consider the following polynomial*

$$q(x) = p(x) - x^5 - 13.$$

Note that it has integer coefficients and $q(1+\sqrt{2}) = 0$, $q(1+\sqrt{3}) = 0$, $q(1+\sqrt{5}) = 0$. *Therefore,* $q(1-\sqrt{2}) = 0$, $q(1-\sqrt{3}) = 0$, $q(1-\sqrt{5}) = 0$.

Thus, it follows that either $q(x) \equiv 0$ *or the degree of* $q(x)$ *is not less, than 6. Hence, either* $p(x) = x^5 + 13$ *or the degree of* $p(x)$ *is not less than 6. We obtain that the smallest possible value of* n *is equal to 5.*

Problem 5. Find the value of the expression

$$\left(\frac{y(x-z)}{xy+yz-2xz} + \frac{z(y-x)}{xz+yz-2xy} + \frac{x(z-y)}{xy+zx-2yz}\right)\left(\frac{xy+yz-2xz}{y(x-z)} + \right.$$

$$\left. + \frac{xz+yz-2xy}{z(y-x)} + \frac{xy+zx-2yz}{x(z-y)}\right).$$

Solution. ***Lemma 7.13.*** *If* $u+v+w=0$ *and* $(u-v)(v-w)(w-u) \neq 0$, *then*

$$\frac{u}{v-w} + \frac{v}{w-u} + \frac{w}{u-v} = -9 \cdot \frac{uvw}{(v-w)(w-u)(u-v)}.$$

Proof. We have that

$$\frac{u}{v-w} + \frac{v}{w-u} + \frac{w}{u-v} = \frac{u(w-u)(u-v) + v(v-w)(u-v) + w(v-w)(w-u)}{(v-w)(w-u)(u-v)} =$$

$$= \frac{-u^3 - v^3 - w^3 - 3uvw + u^2(w+v) + v^2(u+w) + w^2(u+v)}{(v-w)(w-u)(u-v)} =$$

$$= \frac{-2(u^3+v^3+w^3) - 3uvw}{(v-w)(w-u)(u-v)} = -9 \cdot \frac{uvw}{(v-w)(w-u)(u-v)}.$$

This ends the proof of the lemma.

According to the lemma, it follows that

$$\frac{y(x-z)}{xy+yz-2xz}+\frac{z(y-x)}{xz+yz-2xy}+\frac{x(z-y)}{xy+zx-2yz}=$$

$$=-9\cdot\frac{y(x-z)\cdot z(y-x)\cdot x(z-y)}{(xy+yz-2xz)(xz+yz-2xy)(xy+zx-2yz)},$$

and

$$\frac{xy+yz-2xz}{y(x-z)}+\frac{xz+yz-2xy}{z(y-x)}+\frac{xy+zx-2yz}{x(z-y)}=$$

$$=-3\Big(\frac{xy+yz-2xz}{-3y(x-z)}+\frac{xz+yz-2xy}{-3z(y-x)}+\frac{xy+zx-2yz}{-3x(z-y)}\Big)=$$

$$-3\Big(-9\cdot\frac{(xy+yz-2xz)(xz+yz-2xy)(xy+zx-2yz)}{-27y(x-z)z(y-x)x(z-y)}\Big).$$

Thus, we deduce that

$$\Big(\frac{y(x-z)}{xy+yz-2xz}+\frac{z(y-x)}{xz+yz-2xy}+\frac{x(z-y)}{xy+zx-2yz}\Big)\Big(\frac{xy+yz-2xz}{y(x-z)}+$$

$$+\frac{xz+yz-2xy}{z(y-x)}+\frac{xy+zx-2yz}{x(z-y)}\Big)=9.$$

Problem 6. Given that x, y, z are real numbers, such that

$$x^2-xy=y^2-yz=z^2-zx.$$

Find the number of possible values of the expression

$$(x+y-z)(y+z-x)(x+z-y)-xyz.$$

Solution. *Note that any two among the numbers x,y,z are equal, then all of them are equal. Indeed, assume that $x=y$, then either $y=z$ or $y=0$. Hence, $z=0$.*

If $x=y=z$, then

$$(x+y-z)(y+z-x)(x+z-y)-xyz=x^3-x^3=0.$$

If $(x-y)(y-z)(z-x)\neq 0$, then note that

$$(x+y-z)(x-y)=x(y-z),$$

$$(y+z-x)(y-z)=y(z-x),$$

and

$$(x+z-y)(z-x)=z(x-y).$$

Multiplying the last three equations, we obtain that

$$(x+y-z)(y+z-x)(x+z-y)-xyz=0.$$

Problem 7. Let a, b, c be real numbers, such that the following inequality

$$axy+byz+cxz \leq x^2+y^2+z^2,$$

holds true for any real-valued variables x, y, z. Find the greatest possible value of the expression

$$a^2+b^2+c^2+abc.$$

Solution. *We have that the following inequality*

$$z^2-(by+cx)z+x^2+y^2-axy \geq 0,$$

holds true for any z. Therefore,

$$D=(by+cx)^2-4(x^2+y^2-axy) \leq 0,$$

for any x, y.

If $y=0$, $x=1$, we obtain that $c^2 \leq 4$.

We have that

$$(4-c^2)x^2-2(2a+bc)xy+(4-b^2)y^2 \geq 0.$$

If $y=1$, $x=\dfrac{2a+bc}{4-c^2}$, $(c^2 \neq 4)$.

Thus, it follows that

$$(2a+bc)^2-2(2a+bc)^2+(4-b^2)(4-c^2) \geq 0.$$

Hence,

$$-4a^2-4b^2-4c^2-4abc+16 \geq 0.$$

We obtain that

$$a^2+b^2+c^2+abc \leq 4.$$

If $c^2=4$, $b^2=4$ and $bc=-2a$, then

$$a^2+b^2+c^2+abc=4.$$

Thus, the greatest possible value of $a^2+b^2+c^2+abc$ *is equal to 4.*

Problem 8. Given that $a_1, a_2, \ldots, a_{101}$ are nonzero numbers, such that any of the polynomials

$$a_{i_1}x^{100}+a_{i_2}x^{99}+\cdots+a_{i_{101}},$$

has an integer root, where $i_1, i_2, \ldots, i_{101}$ is a random permutation of numbers $1, 2, \ldots, 101$. Find the number of possible values of the sum $a_1+a_2+\cdots+a_{101}$.

Solution. *If 1 is a root for any of those polynomials, then*

$$a_1+a_2+\cdots+a_{101}=0.$$

If 1 is not a root for any of those polynomials, then without loss of generality one can assume that

$$|a_{101}|=\max(|a_1|,|a_2|,\ldots,|a_{101}|).$$

Consider the polynomial

$$a_{101}x^{100}+a_{i_1}x^{99}+\cdots+a_{i_{100}},$$

where the numbers $i_1, i_2, \ldots, i_{100}$ *are some permutation of numbers* $1,2,\ldots,100$. *Let integer k be a root of this polynomial.*

If $|k|\geq 2$*, then*

$$|a_{i_{100}}|\geq |a_{101}||k|^{100}-|a_{i_1}||k|^{99}-\cdots-|a_{i_{99}}||k|\geq 2(|a_{101}||k|^{99}-|a_{i_1}||k|^{98}-\cdots-|a_{i_{99}}|)\geq$$

$$\geq 2((2\cdot|a_{101}|-|a_{i_1}|)|k|^{98}-|a_{i_2}||k|^{97}-\cdots-|a_{i_{99}}|)\geq$$

$$\geq 2(|a_{101}||k|^{98}-|a_{i_1}||k|^{97}-\cdots-|a_{i_{99}}|)\geq\cdots\geq 2|a_{101}|.$$

Thus, it follows that

$$|a_{i_{100}}|\geq 2|a_{101}|.$$

This leads to a contradiction.

Hence, $k=-1$*, then*

$$a_{101}-a_{i_1}+a_{i_2}-\cdots+a_{i_{100}}=0.$$

Summing up all obtained equations, we deduce that $a_{101}=0$*. This leads to a contradiction.*

Problem 9. Find the possible smallest value of M, if given that the following inequality

$$12(abc+abd+acd+bcd)-(ab+ac+ad+bc+bd+cd)^2\leq M,$$

holds true for any real numbers a, b, c, d, where $a+b+c+d=0$.

Solution. *If $a=b=c=-2$ and $d=6$, then*

$$M \geq 12\cdot 64-24^2=192.$$

Let us prove that if $a+b+c+d=0$, then

$$192 \geq 12(abc+abd+acd+bcd)-(ab+ac+ad+bc+bd+cd)^2.$$

Let $a+b=x$, $a+c=y$, $b+c=z$, then

$$ab+ac+ad+bc+bd+cd=ab+bc+ac-(a+b+c)^2=-\frac{x^2+y^2+z^2}{2},$$

and

$$abc+abd+acd+bcd=-xyz.$$

Let us prove that

$$A=192+\frac{x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2}{4} \geq -12xyz.$$

We have that

$$A=\frac{x^4}{4}+\frac{y^4}{4}+\frac{z^4}{4}+\frac{x^2y^2}{4}+\frac{x^2y^2}{4}+\frac{x^2z^2}{4}+\frac{x^2z^2}{4}+\frac{y^2z^2}{4}+\frac{y^2z^2}{4}+64+64+64 \geq$$

$$\geq 12\sqrt[12]{x^{12}y^{12}z^{12}}=12|xyz| \geq -12xyz.$$

Thus, it follows that $A \geq -12xyz$.

Therefore, we obtain that the smallest value of M is equal to 192.

7.3.12 Problem Set 12

Problem 1. Find the number of positive solutions of the equation

$$x^5-80x+128=0.$$

Solution. *Let x_0 be a positive root of the given equation, then*

$$80x_0=x_0^5+128=x_0^5+2^5+2^5+2^5+2^5 \geq 5\cdot\sqrt[5]{x_0^5\cdot 2^5\cdot 2^5\cdot 2^5\cdot 2^5}=80x_0.$$

Thus, it follows that $x_0=2$.

Therefore, the unique positive solution of the given equation is $x_0 = 2$*.*

Problem 2. Let x, y be real numbers. Find the greatest possible value of the expression

$$3(xy - 2x^2 - 2y^2 - x - y + 2).$$

Solution. *Note that, if* $y = const$*, then the possible greatest value of the expression*

$$-2x^2 + (y-1)x - 2y^2 - y + 2$$

is obtained for $x = \dfrac{y-1}{4}$*. In this case, we have that*

$$-2x^2 + (y-1)x - 2y^2 - y + 2 = \frac{1}{8}(-15y^2 - 10y + 17).$$

On the other hand, the greatest possible value of the expression $-15y^2 - 10y + 17$ *is obtained, if* $y = -\dfrac{1}{3}$*. Therefore, the greatest possible value of the given equation is equal to* $3 \cdot \dfrac{1}{8} \cdot \dfrac{56}{3} = 7$*.*

Problem 3. Given that

$$\frac{x}{x^2 + 3x + 1} = \frac{15 + \sqrt{130}}{19}.$$

Find the value of the following expression

$$\frac{x^2}{x^4 - 3x^2 + 1}.$$

Solution. *Note that*

$$x + \frac{1}{x} = \frac{19}{15 + \sqrt{130}} - 3 = -\frac{\sqrt{130}}{5}.$$

Thus, it follows that

$$x^2 + \frac{1}{x^2} = \frac{16}{5}.$$

We obtain that

$$x^2 - 3 + \frac{1}{x^2} = \frac{1}{5}.$$

Therefore,

$$\frac{x^2}{x^4 - 3x^2 + 1} = 5.$$

Problem 4. Let a,b,c be positive numbers, such that the numbers

$$\frac{1}{a^2-ab+b^2}+\frac{1}{b^2-bc+c^2}+\frac{1}{c^2-ca+a^2},\ a+b+c,\ \frac{c}{a^3+b^3}+\frac{a}{b^3+c^3}+\frac{b}{c^3+a^3}$$

are consequent terms of an arithmetic progression. Find the value of the following expression

$$\frac{1}{a^3+b^3}+\frac{1}{b^3+c^3}+\frac{1}{c^3+a^3}.$$

Solution. *Note that*

$$(a+b+c)\left(\frac{1}{a^3+b^3}+\frac{1}{b^3+c^3}+\frac{1}{c^3+a^3}\right)=\frac{1}{a^2-ab+b^2}+\frac{c}{a^3+b^3}+$$

$$+\frac{1}{b^2-bc+c^2}+\frac{a}{b^3+c^3}+\frac{1}{c^2-ca+a^2}+\frac{b}{c^3+a^3}=2(a+b+c).$$

Thus, it follows that

$$\frac{1}{a^3+b^3}+\frac{1}{b^3+c^3}+\frac{1}{c^3+a^3}=2.$$

Problem 5. Find the greatest even number n smaller than 30, such as one can arrange 30 numbers on a circle in such a way that the product of any consequently written n numbers is negative.

Solution. *If $n=28$, then if the given assumption holds true, then the products of any two consequently written numbers are of the same sign. Therefore, the product of any consequently written 28 numbers is positive. This leads to a contradiction.*

Now, let us provide an example of 30 numbers satisfying the assumptions of the problem, such that $n=26$. For example, the numbers 1 and -1 written consequently.

Problem 6. Find the value of the expression

$$\frac{1}{\sqrt[3]{\sqrt[3]{2}-1}}\left(\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\cdots-\sqrt[3]{\frac{128}{9}}+\sqrt[3]{\frac{256}{9}}\right).$$

Solution. *Using the formula of the sum of geometric progression (taking the first 9 terms), we obtain that*

$$\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\cdots-\sqrt[3]{\frac{128}{9}}+\sqrt[3]{\frac{256}{9}}=\frac{\sqrt[3]{\frac{1}{9}}\left((-\sqrt[3]{2})^9-1\right)}{-\sqrt[3]{2}-1}=\frac{\sqrt[3]{81}}{\sqrt[3]{2}+1}.$$

Thus, it follows that

$$\frac{1}{\sqrt[3]{\sqrt[3]{2}-1}}\left(\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\cdots-\sqrt[3]{\frac{128}{9}}+\sqrt[3]{\frac{256}{9}}\right)=\frac{\sqrt[3]{81}}{\sqrt[3]{(\sqrt[3]{2}-1)(\sqrt[3]{2}+1)^3}}=$$

$$=\frac{\sqrt[3]{81}}{\sqrt[3]{(\sqrt[3]{4}-1)(\sqrt[3]{4}+\sqrt[3]{16}+1)}}=3.$$

Problem 7. Let x, y be rational numbers, such that

$$x^2-16xy+19y^2+16x-8y-16=0.$$

Find the value of $x+y$.

Solution. *Let us rewrite the given equation in the following form*

$$5(2y-x)^2=(2x-y-4)^2.$$

Thus, it follows that

$$\sqrt{5}|2y-x|=|2x-y-4|.$$

We have that x and y are rational numbers, therefore $2y-x=0$ *and* $2x-y-4=0$. *We obtain that*

$$2y-x+2x-y-4=0.$$

Hence, $x+y=4$.

Problem 8. Find the number of solutions of the equation

$$\sqrt[4]{x^4-x^2-2x+18}+\sqrt{x^4-2x^3+2x^2-2x+3}=\frac{\sqrt{2}}{\sqrt{2}-1}.$$

Solution. *Let us rewrite the given equation in the following way*

$$\sqrt[4]{(x^2-1)^2+(x-1)^2+16}+\sqrt{(x^2-x)^2+(x-1)^2+2}=2+\sqrt{2}.$$

Note that

$$\sqrt[4]{(x^2-1)^2+(x-1)^2+16}\geq\sqrt[4]{16}=2,$$

and

$$\sqrt{(x^2-x)^2+(x-1)^2+2}\geq\sqrt{2}.$$

Therefore, the unique solution of the given equation is $x=1$.

Problem 9. Find the value of the expression

$$\frac{\sqrt{\sqrt{2016}+\sqrt{1}}+\sqrt{\sqrt{2016}+\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}+\sqrt{2015}}}{\sqrt{\sqrt{2016}-\sqrt{1}}+\sqrt{\sqrt{2016}-\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}-\sqrt{2015}}}-$$

$$-\frac{\sqrt{\sqrt{2016}-\sqrt{1}}+\sqrt{\sqrt{2016}-\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}-\sqrt{2015}}}{\sqrt{\sqrt{2016}+\sqrt{1}}+\sqrt{\sqrt{2016}+\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}+\sqrt{2015}}}.$$

Solution. *We have that*

$$\sqrt{\sqrt{2016}+\sqrt{n}}+\sqrt{\sqrt{2016}-\sqrt{n}}=\sqrt{\left(\sqrt{\sqrt{2016}+\sqrt{n}}+\sqrt{\sqrt{2016}-\sqrt{n}}\right)^2}=$$

$$=\sqrt{2}\cdot\sqrt{\sqrt{2016}+\sqrt{2016-n}},$$

where $n=1,2,\dots,2015$*. Summing up all these 2015 equalities, we obtain that*

$$\frac{\sqrt{\sqrt{2016}+\sqrt{1}}+\sqrt{\sqrt{2016}+\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}+\sqrt{2015}}}{\sqrt{\sqrt{2016}-\sqrt{1}}+\sqrt{\sqrt{2016}-\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}-\sqrt{2015}}}=\frac{1}{\sqrt{2}-1},$$

and

$$\frac{\sqrt{\sqrt{2016}-\sqrt{1}}+\sqrt{\sqrt{2016}-\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}-\sqrt{2015}}}{\sqrt{\sqrt{2016}+\sqrt{1}}+\sqrt{\sqrt{2016}+\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}+\sqrt{2015}}}=\sqrt{2}-1.$$

Therefore,

$$\frac{\sqrt{\sqrt{2016}+\sqrt{1}}+\sqrt{\sqrt{2016}+\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}+\sqrt{2015}}}{\sqrt{\sqrt{2016}-\sqrt{1}}+\sqrt{\sqrt{2016}-\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}-\sqrt{2015}}}-$$

$$-\frac{\sqrt{\sqrt{2016}-\sqrt{1}}+\sqrt{\sqrt{2016}-\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}-\sqrt{2015}}}{\sqrt{\sqrt{2016}+\sqrt{1}}+\sqrt{\sqrt{2016}+\sqrt{2}}+\cdots+\sqrt{\sqrt{2016}+\sqrt{2015}}}=2.$$

Problem 10. Find the greatest possible value of k, such that the following inequality

$$k\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq\left(\frac{1+abc}{\sqrt{abc}}\right)^3.$$

holds true for any positive numbers a,b,c.

Solution. *If* $a=b=c=1$*, then we obtain that* $k\leq 8$.

Let us prove that if $k = 8$, then the given equation holds true for any positive numbers a, b, c.

We have that $a, b, c > 0$, therefore at least two factors of the product on the left side are positive. In the case, if only one factor is non-positive, then the inequality obviously holds true. If all the factors of the left side are positive, then we have that

$$3 + 3abc = b\Big(a - 1 + \frac{1}{b}\Big) + c\Big(b - 1 + \frac{1}{c}\Big) + a\Big(c - 1 + \frac{1}{a}\Big) +$$

$$+bc\Big(a - 1 + \frac{1}{b}\Big) + ac\Big(b - 1 + \frac{1}{c}\Big) + ab\Big(c - 1 + \frac{1}{a}\Big).$$

Using the AM-GM inequality, we obtain that

$$3 + 3abc \geq 6\sqrt[6]{a^3b^3c^3\left(\Big(a - 1 + \frac{1}{b}\Big)\Big(b - 1 + \frac{1}{c}\Big)\Big(c - 1 + \frac{1}{a}\Big)\right)^2}.$$

Note that this inequality is equivalent to the given inequality.

Therefore, the greatest possible value of k is equal to 8.

Problem 11. Let $x_1, x_2, \dots, x_{100}$ be positive integers, such that $x_1 \leq x_2 \leq \cdots \leq x_{100}$ and $x_1 + x_2 + \cdots + x_{100} = 2016$. Given that the value of the expression $p(x_1) + p(x_2) + \cdots + p(x_{100})$ is the smallest possible, where $p(x) = x^3 - 6x^2 + 11x + 16$. Find the value of

$$x_{61} + x_{62} + \cdots + x_{100}.$$

Solution. *Note that*

$$p(x) = (x-1)(x-2)(x-3) + 22.$$

Let us prove that $x_{100} - x_1 \leq 1$. Indeed, if $x_{100} - x_1 \geq 2$, then consider $x_1 + 1, x_2, \dots, x_{99}, x_{100} - 1$ positive integers. Note that $x_1 + 1 + x_2 + \cdots + x_{99} + x_{100} - 1 = 2016$.

On the other hand,

$$p(x_1) + p(x_{100}) - p(x_1 + 1) - p(x_{100} - 1) = 3(x_{100} - 2)(x_{100} - 3) - 3(x_1 - 1)(x_1 - 2) > 0,$$

as $x_{100} - 2 > x_1 - 1 \geq 0$, $x_{100} - 3 > x_1 - 2$ and $x_{100} - 3 > 0$.

Therefore,

$$p(x_1) + p(x_2) + \cdots + p(x_{100}) > p(x_1 + 1) + p(x_2) + \cdots + p(x_{99}) + p(x_{100} - 1).$$

This leads to a contradiction, with the condition that the value of the expression $p(x_1) + p(x_2) + \cdots + p(x_{99}) + p(x_{100})$ is the smallest possible.

Let $x_1 = \cdots = x_k = a$ and $x_{k+1} = \cdots = x_{100} = a + 1$, we have that

$$ka + (100 - k)(a + 1) = 2016.$$

Thus, it follows that

$$k = 100(a + 1) - 2016.$$

Hence, we obtain that $k = 84$, $a = 20$.

Therefore,

$$x_{61} + x_{62} + \cdots + x_{100} = 40 \cdot 20 + 16 = 816.$$

Problem 12. Given that a circle with the radius 10 intersects hyperbola $y = \frac{3}{x}$ at point $A_i(x_i, y_i)$, where $i = 1, 2, 3, 4$. Find the value of the following expression

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2.$$

Solution. *Let the circle*

$$(x - x_0)^2 + (y - y_0)^2 = 10^2$$

intersects hyperbola $y = \frac{3}{x}$ *at point* $A_i(x_i, y_i)$, $i = 1, 2, 3, 4$. *Hence, the numbers* x_1, x_2, x_3, x_4 *are roots of the equation*

$$(x - x_0)^2 + \left(\frac{3}{x} - y_0\right)^2 = 100.$$

Let us rewrite this equation in the following way

$$x^4 - 2x_0x^3 + (x_0^2 + y_0^2 - 100)x^2 - 6y_0x + 9 = 0.$$

According to the Vieta's theorem, it follows that

$$2x_0 = x_1 + x_2 + x_3 + x_4,$$

$$x_0^2 + y_0^2 - 100 = x_1x_2 + x_1x_3 + \cdots + x_3x_4,$$

$$6y_0 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4,$$

$$9 = x_1x_2x_3x_4.$$

Hence, we obtain that

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{2}\right)^2 + \left(\frac{x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4}{6}\right)^2 - 100 =$$

$$= x_1x_2 + x_1x_3 + \cdots + x_3x_4,$$

$$(x_1+x_2+x_3+x_4)^2+\left(\frac{3}{x_1}+\frac{3}{x_2}+\frac{3}{x_3}+\frac{3}{x_4}\right)^2=$$

$$=400+4x_1x_2+4x_1x_3+4x_1x_4+4x_2x_3+4x_2x_4+4x_3x_4.$$

Therefore,

$$x_1^2+x_2^2+x_3^2+x_4^2+y_1^2+y_2^2+y_3^2+y_4^2=400.$$

7.3.13 Problem Set 13

Problem 1. Find the number of real solutions of the equation

$$\sin(\pi x)=\sin^3(\pi x)+\sqrt{-x^3+3x^2-2x}.$$

Solution. *Note that if x_0 is a solution of the given equation, then*

$$\sin(\pi x_0)(1-\sin(\pi x_0))(1+\sin(\pi x_0))=\sqrt{(x_0-1)(2-x_0)x_0}, \tag{7.75}$$

and either $x_0 \le 0$ or $1 \le x_0 \le 2$.

If $1<x_0<2$, then $\sin(\pi x_0)<0$, $1-\sin(\pi x_0)>0$ and $1+\sin(\pi x_0)\ge 0$. On the other hand, $\sqrt{(x_0-1)(2-x_0)x_0}>0$. Therefore, (7.75) does not hold true.

If either $x_0=0$ or $x_0=1$ or $x_0=2$, then (7.75) holds true.

If $x_0<0$, then let us consider two cases:

a) If $-1\le x_0<0$, then $\sin(\pi x_0)\le 0$, $1-\sin(\pi x_0)\ge 0$ and $1+\sin(\pi x_0)\ge 0$. On the other hand $\sqrt{(x_0-1)(2-x_0)x_0}>0$. Therefore, (7.75) does not hold true.

b) If $x_0<-1$, then $1-x_0>2$, $2-x_0>3$ and $-x_0>1$. Thus, it follows that $\sqrt{(x_0-1)(2-x_0)x_0}>\sqrt{6}>1$ and

$$|\sin(\pi x_0)(1-\sin^2(\pi x_0))|=|\sin(\pi x_0)|\cdot|\cos^2(\pi x_0)|\le 1.$$

Therefore, (7.75) does not hold true. Hence, the given equation has three real solutions.

Problem 2. One writes (in a random way) in the squares of 3×3 grid square the numbers $1,\dots,9$, such that in any square is written only one number. Then, consider (eight) sums of three numbers written in the same row, column or diagonal. At most, how many sums among those eight sums can be a square of a positive integer?

Solution. *Let us provide an example, such that six of those sums are squares of positive integers.*

1	8	4
6	3	7
2	9	5

Now, let us prove that more than six of those sums cannot be squares of positive integers. We proceed by contradiction argument. Therefore, either all three row sums or all three column sums are squares of positive integers. Note that, all three row(column) sums cannot be squares of positive integers, as $a^2+b^2+c^2=45$, if one of the numbers a,b,c is equal to 2. This leads to a contradiction.

Hence, at most six sums among those eight sums can be a square of a positive integer.

Problem 3. Evaluate the expression

$$\left[\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\dots\cdot\frac{100}{99}\right],$$

where by $[x]$ we denote the integer part of a real number x.

Solution. *Note that if $a>1$, then*

$$\frac{a+1}{a}<\frac{a}{a-1}.$$

Thus, it follows that

$$A=\frac{10}{9}\cdot\frac{12}{11}\cdot\dots\cdot\frac{100}{99}<\frac{9}{8}\cdot\frac{11}{10}\cdot\dots\cdot\frac{99}{98}=\frac{1}{A}\cdot\frac{100}{8}.$$

Therefore

$$A<\frac{5\sqrt{2}}{2},$$

and

$$\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\dots\cdot\frac{100}{99}=\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot A<\frac{64}{7}\sqrt{2}<13.$$

We obtain that

$$\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\dots\cdot\frac{100}{99}<13. \tag{7.76}$$

On the other hand

$$\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\dots\cdot\frac{100}{99}=\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot B,$$

where

$$B=\frac{8}{7}\cdot\frac{10}{9}\cdot\dots\cdot\frac{100}{99}>\frac{9}{8}\cdot\frac{11}{10}\cdot\dots\cdot\frac{101}{100}=\frac{1}{B}\cdot\frac{101}{7}.$$

Hence

$$B>\sqrt{\frac{101}{7}},$$

and

$$\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\dots\cdot\frac{100}{99}=\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot B>\frac{16}{5}\sqrt{\frac{101}{7}}>12.$$

Thus, it follows that

$$\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\dots\cdot\frac{100}{99}>12. \tag{7.77}$$

From (7.76) *and* (7.77), *we obtain that*

$$12<\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\dots\cdot\frac{100}{99}<13.$$

Therefore

$$\left[\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\dots\cdot\frac{100}{99}\right]=12.$$

Problem 4. Find the product of all real solutions of the equation

$$\frac{x^{2^{2016}}+x^{2^{2015}}+1}{(x^2-x+1)(x^4-x^2+1)\cdot\dots\cdot(x^{2^{2015}}-x^{2^{2014}}+1)}=20x-16.$$

Solution. *Note that if $k\in\mathbb{N}$, then*

$$(x^{2^k}-x^{2^{k-1}}+1)(x^{2^k}+x^{2^{k-1}}+1)=(x^{2^k}+1)^2-(x^{2^{k-1}})^2=x^{2^{k+1}}+x^{2^k}+1.$$

Thus, it follows that

$$\frac{x^{2^{2016}}+x^{2^{2015}}+1}{(x^2-x+1)(x^4-x^2+1)\cdot\dots\cdot(x^{2^{2015}}-x^{2^{2014}}+1)}=$$

$$=\frac{x^{2^{2015}}+x^{2^{2014}}+1}{(x^2-x+1)(x^4-x^2+1)\cdot\dots\cdot(x^{2^{2014}}-x^{2^{2013}}+1)}=\dots=x^2+x+1.$$

We have obtained that $x^2+x+1=20x-16$. Therefore $x^2-19x+17=0$. This means that the given equation has two real solutions and the product of all solutions is equal to 17.

Problem 5. Let the circle ω intersects the hyperbola $y=\frac{30}{x}$ at points $A_i(x_i,y_i)$, where $i=1,2,3,4$. Find $y_1y_2y_3y_4$.

Solution. *Let the circle*

$$(x-x_0)^2+(y-y_0)^2=R^2,$$

intersects the hyperbola $y = \frac{30}{x}$ *at points* $A_i(x_i, y_i)$*, where* $i = 1,2,3,4$*. Then, the numbers* y_1, y_2, y_3, y_4 *satisfy the equation*

$$\left(\frac{30}{y} - x_0\right)^2 + (y - y_0)^2 = R^2.$$

Now, let us rewrite this equation in the following way

$$y^4 - 2y_0y^3 + (x_0^2 + y_0^2 - R^2)y^2 - 60x_0y + 900 = 0.$$

According to the Vieta's theorem, we have that

$$y_1y_2y_3y_4 = 900.$$

Problem 6. Let the vertices of parabolas $y = ax^2 + bx + c$ and $y = a_1x^2 + b_1x + c_1$ be different. Given that any of those two vertices is on the parabola corresponding to the other one. Find $a + a_1$.

Solution. *Let* $y = ax^2 + b + c = a(x - m)^2 + n$*. Consider the parallel translation (rigid motion) defined by* $x' = x - m$*,* $y' = y - n$*.*

One of the given parabolas will be transformed to the parabola $y = ax^2$ *and the other one will be transformed to the parabola* $y + n = a_1(x + m)^2 + b_1(x + m) + c_1$ *(the equation has the form* $y = a_1x^2 + b_2x + c_2$*).*

Note that any of two vertices of these (new) parabolas is on the parabola corresponding to the other one. Thus, it follows that $c_2 = 0$ *and*

$$-\frac{b_2^2}{4a_1} = a \cdot \left(-\frac{b_2}{2a_1}\right)^2.$$

Hence, we obtain that $a + a_1 = 0$*, as* $b_2 \neq 0$ *(otherwise the vertices of the parabolas coincide).*

Problem 7. Find the integer part of the greatest solution of the equation

$$\frac{1}{[x]} + \frac{1}{\{x\}} = \frac{15}{x},$$

where by $[x]$ we denote the integer part and by $\{x\}$ the fractional part of a real number x.

Solution. *Let us rewrite this equation in the following form*

$$x^2 - 15[x]x + 15[x]^2 = 0.$$

Hence, it follows that either

$$x = \frac{15[x] - \sqrt{165}[x]}{2},$$

or

$$x = \frac{15[x] + \sqrt{165}[x]}{2}.$$

We have that $[x] \leq x < [x] + 1$, *thus*

$$0 \leq [x] < \frac{13 + \sqrt{165}}{2}.$$

Therefore, the integer part of the greatest solution of the given equation is equal to 12 and the greatest solution is $6(15 - \sqrt{165})$.

Problem 8. Given that

$$\begin{cases} x^3 + x^2y + xy^2 + y^3 = -\dfrac{1640}{x^4 + y^4}, \\ y^3 + y^2z + yz^2 + z^3 = \dfrac{255}{y^4 + z^4}, \\ z^3 + z^2x + zx^2 + x^3 = -\dfrac{1649}{z^4 + x^4}. \end{cases}$$

Find the value of the expression

$$\frac{9x + 1895y}{2z}.$$

Solution. *From the given equations, we obtain that*

$$x^4 - y^4 = -\frac{1640(x-y)}{x^4 + y^4},$$

$$y^4 - z^4 = \frac{255(y-z)}{y^4 + z^4},$$

$$z^4 - x^4 = -\frac{1649(z-x)}{z^4 + x^4}.$$

Therefore

$$x^8 - y^8 = -1640(x-y), \tag{7.78}$$

$$y^8 - z^8 = 255(y-z), \tag{7.79}$$

$$z^8 - x^8 = -1649(z - x). \tag{7.80}$$

Summing up equations (7.78), (7.79) *and* (7.80), *we obtain that*

$$9x + 1895y - 1904z = 0.$$

Thus, it follows that

$$\frac{9x + 1895y}{2z} = 952.$$

Problem 9. At most, how many numbers can one choose among the numbers 1, ..., 15, such that if we consider four numbers among the chosen ones, then the sum of any two is not equal to the sum of the two others?

Solution. *Note that from the numbers 1, 2, 3, 5, 8, 13 one cannot choose four numbers, such that the sum of any two is equal to the sum of the two others. Proof by contradiction argument. Assume that we choose numbers a, b, c, d, such that* $a + d = b + c$ *and* $a < b < c < d$, *then* $d \geq b + c$. *This leads to a contradiction.*

Now, let us prove that if we have chosen any seven numbers $x_1 < x_2 < \cdots < x_7$ *from the numbers 1, 2, ..., 15, then it is possible to choose four numbers among the chosen ones, such that the sum of any two is equal to the sum of the two others.*

Consider the numbers $x_i - x_j$, *where* $i > j$, $i, j \in \{1, 2, \ldots, 7\}$. *Their total number is equal to 21. If there are three numbers among the chosen ones pairwise equal, then this ends the proof. If there are two numbers among the chosen ones pairwise equal, then we need to prove the statement for the case* $x_m + x_n = 2x_k$, *where* $m < k < n$.

This means that, in the considered differences, the numbers 1, 2, 3, 4, 5, 6, 7 are present twice. Therefore $1, 8, 15 \in \{x_1, x_2, \ldots, x_7\}$.

On the other hand, in the considered differences the numbers $8, 9, \ldots, 14$ *are present too.*

Without loss of generality, one can assume that $2 \in \{x_1, x_2, \ldots, x_7\}$, *otherwise we can consider the numbers* $16 - x_1$, $16 - x_2$, ..., $16 - x_7$.

We deduce that

$$1, 2, 8, 15 \in \{x_1, x_2, \ldots, x_7\}.$$

In this case, we have that

$$14 \in \{x_1, x_2, \ldots, x_7\},$$

as $8 - 2 = 14 - 8$.

This ends the proof, as we have that $2 + 14 = 1 + 15$.

Hence, the answer is 6.

Problem 10. Let, by one step, from the couple (m, n) is possible to obtain the couple $(m + 2, n + m + 1)$. Given that, by several steps, from the couple $(-2014, -1016)$ is possible to obtain the couple $(2016, p)$. Find p.

Solution. *Note that*

$$m^2 - 4n = (m+2)^2 - 4(n+m+1).$$

Hence, in any step, the value of the expression $m^2 - 4n$ *is constant. Thus, it follows that*

$$(-2014)^2 - 4(-1016) = 2016^2 - 4p.$$

Therefore $p = 999$.

Problem 11. Let real numbers p, q be such that the following inequality

$$|x^3 - 3x^2 - px - q| \leq \frac{3\sqrt{3}+1}{2},$$

holds true for any $x \in [1,4]$. Find $p^2 + (2q+17)^2$.

Solution. *If* $x = 1$, $x = 4$ *or* $x = \sqrt{3}$, *then we obtain that*

$$\frac{-3\sqrt{3}-5}{2} \leq p+q \leq \frac{3\sqrt{3}-3}{2},$$

$$\frac{31-3\sqrt{3}}{2} \leq 4p+q \leq \frac{33+3\sqrt{3}}{2},$$

$$\frac{3\sqrt{3}-19}{2} \leq \sqrt{3}p+q \leq \frac{9\sqrt{3}-17}{2},$$

respectively. We have that

$$p+q \geq \frac{-3\sqrt{3}-5}{2},\ 4p+q \geq \frac{31-3\sqrt{3}}{2},\ \sqrt{3}p+q \leq \frac{9\sqrt{3}-17}{2}. \tag{7.81}$$

Therefore

$$\frac{4-\sqrt{3}}{3}(p+q) \geq \frac{4-\sqrt{3}}{3} \cdot \frac{-3\sqrt{3}-5}{2},$$

and

$$\frac{\sqrt{3}-1}{3}(4p+q) \geq \frac{\sqrt{3}-1}{3} \cdot \frac{31-3\sqrt{3}}{2}.$$

Summing up these inequalities, we deduce that

$$\sqrt{3}p+q \geq \frac{9\sqrt{3}-17}{2}.$$

On the other hand, from (7.81), *we obtain that*

$$\sqrt{3}p+q=\frac{9\sqrt{3}-17}{2}.$$

Hence, it follows that

$$p+q=\frac{-3\sqrt{3}-5}{2},$$

and

$$4p+q=\frac{31-3\sqrt{3}}{2}.$$

Therefore $p=6$, $q=-1.5\sqrt{3}-8.5$. *Thus*

$$p^2+(2q+17)^2=63.$$

Problem 12. Let quadratic trinomial $p(x)$ be, such that the inequality

$$p(x^3)\geq p(3x^2-x+3),$$

holds true for any real value of x. Find the value of the expression

$$\frac{p(2)-p(1)}{p(26)-p(25)}.$$

Solution. *Let*

$$p(x)=a(x-m)^2+n.$$

Note that $a>0$. *Proof by contradiction argument. Assume that* $a<0$, *then let us choose* $x>\max(3,m)$. *Thus, it follows that*

$$x^3>3x^2-x+3.$$

Therefore

$$p(x^3)<p(3x^2-x+3).$$

This leads to a contradiction.

If $x_0=\sqrt[3]{m}$, *then*

$$p(m)=p(x_0^3)\geq p(3x_0^2-x_0+3).$$

We obtain that $3x_0^2-x_0+3=m$. *Hence*

$$x_0^3=3x_0^2-x_0+3.$$

Thus $x_0=3$ *and* $m=27$. *Therefore*

$$\frac{p(2)-p(1)}{p(26)-p(25)} = \frac{-(26^2a-25^2a)}{-(2^2a-1^2a)} = 17.$$

Remark 7.1. If $p(x) = a(x-27)^2+c$ and $a>0$, then the inequality

$$p(x^3) \geq p(3x^2-x+3),$$

holds true for any real value of x.

7.3.14 Problem Set 14

Problem 1. Let the sum of the first 2016 terms of a geometric progression be equal to 2016 and the product of 1000th and 1017th terms be equal to 16. Find the sum of multiplicative inverses of the first 2016 terms.

Solution. *Denote by (b_n) the given geometric progression. We have that*

$$\frac{b_{k+1}}{b_k} = \frac{b_{2017-k}}{b_{2016-k}},\ k=1,2,\ldots,2015.$$

Thus, it follows that

$$b_k \cdot b_{2017-k} = b_{k+1} \cdot b_{2016-k},\ k=1,2,\ldots,2016. \tag{7.82}$$

Note that

$$\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_{2016}} = \frac{b_{2016}}{b_1 b_{2016}}+\frac{b_{2015}}{b_2 b_{2015}}+\cdots+\frac{b_1}{b_{2016} b_1} =$$

$$= \frac{b_{2016}+b_{2015}+\cdots+b_1}{b_{1000}b_{1017}} = \frac{2016}{16} = 126,$$

as according to (7.82), we have that

$$b_1 b_{2016} = b_2 b_{2015} = \cdots = b_{1000} b_{1017} = \cdots = b_{1008} b_{1009}.$$

Therefore

$$\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_{2016}} = 126.$$

Problem 2. Let x, y be real numbers, such that

$$x^3-y^3+6x^2+9y^2+17x-32y+60=0.$$

Find $y-x$.

Solution. *Note that the given equation is equivalent to the following equation*

$$(y-3)^3+5(y-3)=(x+2)^3+5(x+2).$$

If $y-3>x+2$, then we have that $(y-3)^3>(x+2)^3$ and $5(y-3)>5(x+2)$. Thus, it follows that

$$(y-3)^3+5(y-3)>(x+2)^3+5(x+2).$$

In a similar way, if $y-3<x+2$, then we obtain that

$$(y-3)^3+5(y-3)<(x+2)^3+5(x+2).$$

If $y-3=x+2$, then

$$(y-3)^3+5(y-3)=(x+2)^3+5(x+2).$$

Therefore $y-3=x+2$. Hence, we deduce that $y-x=5$.

Problem 3. Let x_0 be the greatest real solution of the following equation

$$\frac{1}{x^4}-\frac{1}{x^3(x+1)}-\frac{1}{x^2(x+1)}-\frac{1}{x(x+1)}-\frac{1}{x+1}=1.$$

Find $(2x_0+1)^2$.

Solution. *Let x_1 be a positive solution of*

$$\frac{1}{x}-\frac{1}{x+1}=1.$$

Hence, we have that

$$\frac{1}{x_1}-\frac{1}{x_1+1}=1.$$

Thus, it follows that

$$\frac{1}{x_1}\left(\frac{1}{x_1}-\frac{1}{x_1+1}\right)-\frac{1}{x_1+1}=1.$$

Therefore

$$\frac{1}{x_1^2}\left(\frac{1}{x_1}-\frac{1}{x_1+1}\right)-\frac{1}{x_1(x_1+1)}-\frac{1}{x_1+1}=1.$$

We deduce that

$$\frac{1}{x_1^4}-\frac{1}{x_1^3(x_1+1)}-\frac{1}{x_1^2(x_1+1)}-\frac{1}{x_1(x_1+1)}-\frac{1}{x_1+1}=1.$$

Hence, x_1 is also a solution of the last equation. We obtain that $x_1 \leq x_0$.

If $x_0 > x_1$, then

$$\frac{1}{x_0} - \frac{1}{x_0+1} < \frac{1}{x_1} - \frac{1}{x_1+1} = 1.$$

Thus, it follows that

$$\frac{1}{x_0}\left(\frac{1}{x_0} - \frac{1}{x_0+1}\right) - \frac{1}{x_0+1} < \frac{1}{x_0} - \frac{1}{x_0+1} = 1.$$

Therefore

$$\frac{1}{x_0^2}\left(\frac{1}{x_0} - \frac{1}{x_0+1}\right) - \frac{1}{x_0(x_0+1)} - \frac{1}{x_0+1} < 1.$$

We deduce that

$$\frac{1}{x_0^4} - \frac{1}{x_0^3(x_0+1)} - \frac{1}{x_0^2(x_0+1)} - \frac{1}{x_0(x_0+1)} - \frac{1}{x_0+1} < 1.$$

This leads to a contradiction. Hence

$$x_0 = x_1 = \frac{\sqrt{5}-1}{2}.$$

We obtain that $(2x_0+1)^2 = 5$.

Problem 4. Let the number 143 586 729 be written on the blackboard. Consider any two neighbour digits of this number. In one step, we deduce by 1 both considered digits. If these neighbour digits are greater than 0, then we continue the steps in the same way. What is the smallest number that one can obtain after several such steps?

Solution. *Note that after several steps one can obtain the number 209.*

$$143586729 \to 33586729 \to \cdots \to 586729 \to \cdots \to 36729 \to \ldots$$

$$\to 3729 \to \cdots \to 429 \to \cdots \to 209.$$

Now, let us prove that one cannot obtain a number less than 209.

The initial number has nine digits, assume that all obtained numbers also have nine digits (but they can start with 0 digits). Note that if from the number $\overline{a_1a_2\ldots a_9}$ we have obtained the number $\overline{b_1b_2\ldots b_9}$, then

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + a_9 = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + b_7 - b_8 + b_9.$$

On the other hand, for the initial number we have that

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + a_9 = 11.$$

Thus, it follows that one cannot obtain a one- or two-digit numbers. Moreover, if we have obtained three-digit number $\overline{abc}$, *then* $a + c - b = 11$. *Hence* $\overline{abc} \geq 209$.

Therefore, the smallest number that one can obtain after several such steps is equal to 209.

Problem 5. Find the sum of all integer solutions of the following inequality

$$\log_2 x \geq \frac{|x-2| + |x+2|}{4}.$$

Solution. *We have that*

$$\sqrt{2}^{k+1} > \sqrt{2}^{k} + 1, \ k = 4, 5, \ldots$$

Thus, it follows that

$$\sqrt{2}^{n} > n, \ n = 5, 6 \ldots$$

Therefore, we obtain that

$$\frac{|n-2| + |n+2|}{4} = \frac{n}{2} > \log_2 n.$$

One can easily verify that from the numbers $n = 1, 2, 3, 4$, *the solutions of the given inequality are the numbers 2, 3, 4.*

Hence, the sum of all integer solutions of the given inequality is equal to 9.

Problem 6. Evaluate the expression

$$\left[\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2^3}\right)\cdot\ldots\cdot\left(1+\frac{1}{2^{2016}}\right)\right],$$

where by $[x]$ we denote the integer part of a real number x.

Solution. *Note that*

$$2 < \left(1+\frac{1}{2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{2^4}\right) < A,$$

where

$$A = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{2^4}\right)\cdot\ldots\cdot\left(1+\frac{1}{2^{2016}}\right).$$

By mathematical induction method, one can prove that $n^2 - 1 < 2^n$, *where* $n \in \mathbb{N}$, $n \geq 5$.

Thus, it follows that

$$A \leq \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \frac{17}{16} \cdot \left(1 + \frac{1}{5^2 - 1}\right)\left(1 + \frac{1}{6^2 - 1}\right) \cdot \dots \cdot \left(1 + \frac{1}{2016^2 - 1}\right) \leq$$

$$\leq \frac{2295}{1024} \cdot \frac{5 \cdot 2016}{4 \cdot 2017} < \frac{2295 \cdot 5}{4096} < 3.$$

Hence, we obtain that $2 < A < 3$.
Therefore $[A] = 2$.

Problem 7. Let x, y, $z \geq 0$ and $x + y + z = 1$. Find the greatest value of the expression

$$\sqrt{4x + (y - z)^2} + \sqrt{4y + (z - x)^2} + \sqrt{4z + (x - y)^2}.$$

Solution. *Note that*

$$\sqrt{4x + (y - z)^2} \leq 2x + y + z, \tag{7.83}$$

as $x(1 - x) \leq xy + xz + yz$. *In a similar way, we obtain that*

$$\sqrt{4y + (z - x)^2} \leq 2y + x + z, \tag{7.84}$$

$$\sqrt{4z + (x - y)^2} \leq 2z + x + y. \tag{7.85}$$

Summing up inequalities (7.83), (7.84), (7.85), *we deduce that*

$$\sqrt{4x + (y - z)^2} + \sqrt{4y + (z - x)^2} + \sqrt{4z + (x - y)^2} \leq 4(x + y + z) = 4.$$

If $x = 1$, $y = z = 0$, *then*

$$\sqrt{4x + (y - z)^2} + \sqrt{4y + (z - x)^2} + \sqrt{4z + (x - y)^2} = 4.$$

Hence, the greatest value of the given expression is equal to 4.

Problem 8. Let $p(x)$ be a polynomial with real coefficients, such that for any x it holds true

$$x^4 + 4x^3 - 8x^2 - 48x - 47 \leq p(x) \leq 2016|x^4 + 4x^3 - 8x^2 - 48x - 47|.$$

Find $p(4)$.

Solution. *We have that* $q(-5) = 118$, $q(-3) = -2$, $q(-2) = 1$, $q(2) = -127$, $q(4) = 145$, *where*

$$q(x) = x^4 + 4x^3 - 8x^2 - 48x - 47.$$

Hence, we obtain that continuous function $q(x)$ *accepts values of the opposite signs at the endpoints the following intervals* $[-5,-3]$, $[-3,-2]$, $[-2,2]$, $[2,4]$. *Therefore, all the four roots of polynomial* $q(x)$ *are real numbers. Moreover,* $x_1 \in (-5,-3)$, $x_2 \in (-3,-2)$, $x_3 \in (-2,2)$ *and* $x_4 \in (2,4)$.

From the following condition

$$q(x) \le p(x) \le 2016|q(x)|,$$

we obtain that

$$0 = q(x_i) \le p(x_i) \le 2016|q(x_i)| = 0, \ i = 1,2,3,4.$$

Therefore $p(x_i) = 0, \ i = 1,2,3,4.$

Let $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$. *We have that*

$$p(x) \ge x^4 + 4x^3 - 8x^2 - 48x - 47.$$

Thus, it follows that $n \ge 4$. *Otherwise, if* $n < 4$, *then taking* $x \to \infty$ *in the following condition*

$$a_0 \cdot x^{n-4} + a_1x^{n-5} + \cdots + a_nx^{-4} \ge 1 + \frac{4}{x} - \frac{8}{x^2} - \frac{48}{x^3} - \frac{47}{x^4},$$

we deduce that $0 \ge 1$. *This leads to a contradiction.*

In a similar way, from the condition $p(x) \le 2016|q(x)|$, *we obtain that* $n \le 4$. *Therefore* $n = 4$. *We have that*

$$p(x) = a_0(x - x_1)(x - x_2)(x - x_3)(x - x_4) = a_0q(x).$$

On the other hand, from the condition $p(x) \ge q(x)$, *it follows that* $(a_0 - 1)q(x) \ge 0$, *for any real value of* x.

Note that $q(2) < 0$ *and* $q(4) > 0$. *Hence* $a_0 = 1$ *and* $p(x) = q(x)$.

We obtain that $p(4) = q(4) = 145$.

Problem 9. Let a, b, c, d be pairwise distinct complex numbers, such that $a + b + c + d \neq 0$. Given that $|a| = |b| = |c| = |d| = \sqrt{10}$ and $|a+b+c+d| = 10$. Find the value of the expression $|abc + abd + acd + bcd|$.

Solution. *Let us denote by* $\overline{z}$ *the complex conjugate of a complex number* z. *We have that*

$$|abc + abd + acd + bcd| = |\overline{abc + abd + acd + bcd}| = |\overline{abc} + \overline{abd} + \overline{acd} + \overline{bcd}| =$$

$$= |\overline{a}\overline{b}\overline{c}+\overline{d}\overline{a}\overline{b}+\overline{a}\overline{d}\overline{c}+\overline{b}\overline{c}\overline{d}| = \left|\frac{10}{a}\cdot\frac{10}{b}\cdot\frac{10}{c}+\frac{10}{a}\cdot\frac{10}{b}\cdot\frac{10}{d}+\frac{10}{a}\cdot\frac{10}{c}\cdot\frac{10}{d}+\frac{10}{b}\cdot\frac{10}{c}\cdot\frac{10}{d}\right| =$$

$$= 1000\frac{|a+b+c+d|}{|abcd|} = 10|a+b+c+d|.$$

Thus, it follows that

$$|abc+abd+acd+bcd| = 100.$$

Problem 10. Let $p(x)$ be a polynomial of degree four with integer coefficients, such that

$$2x^4-x^3+3x^2-36x-38 < p(x) < 2x^4-x^3+3x^2-36x-36,$$

holds true on some pairwise disjoint line segments for which the sum of the lengths is equal to 8. How many real solutions does the equation $p(x) = x^4-3x^3+2x^2-36x-36$ have?

Solution. *We have that none of the following equations has more than four solutions*

$$p(x) = 2x^4-x^3+3x^2-36x-36, \tag{7.86}$$

and

$$p(x) = 2x^4-x^3+3x^2-36x-38. \tag{7.87}$$

Therefore, the set of solutions of the given inequalities (see the assumptions of the problem) is a union of not more than four open intervals (as the endpoints of those intervals are the solutions of (7.86) or (7.87)).

According to the assumptions of the problem, the sum of the lengths of those intervals is greater than 8.

Note that if in the ith open interval the number of integers is equal to k_i, then its length is less than k_i+1.

Therefore, there are at least five integer numbers in this open interval.

Let $x_1 < x_2 < x_3 < x_4 < x_5$ be those integers, according to the assumptions of the problem, we have that

$$q(x_i)-1 < p(x_i) < q(x_i)+1, \ i = 1,2,3,4,5,$$

where

$$q(x) = 2x^4-x^3+3x^2-36x-37.$$

We also have that $p(x_i)$, $q(x_i) \in \mathbb{Z}$, thus

$$p(x_i) = q(x_i), \ i = 1,2,3,4,5.$$

Hence, we obtain that $p(x) = q(x)$ for any real value of x.

We deduce that the following equation

$$p(x) = x^4 - 3x^3 + 2x^2 - 36x - 36,$$

is equivalent to the equation

$$x^4 + 2x^3 + x^2 - 1 = 0.$$

The last equation can be rewritten as

$$(x^2 + x - 1)(x^2 + x + 1) = 0.$$

Therefore, the given equation has two real solutions.

Problem 11. Let sequence (x_n) be, such that $x_1 = -1062000$ and

$$x_{n+1} = x_n + \sqrt{3844 - 4x_n} - 1,\ n = 1, 2, \ldots$$

Find $|x_{1000}|$.

Solution. *Let us consider the following equations*

$$x^2 - 62x + x_n = 0,\ n = 1, 2, \ldots, 1000.$$

If $n = 1$, then we have that the solutions of the equation

$$x^2 - 62x + x_1 = 0,$$

are the numbers $y_1 = 1062$ and $z_1 = -1000$.

By mathematical induction, let us prove that if numbers y_n, z_n $(y_n > z_n)$ are the solutions of the equation

$$x^2 - 62x + x_n = 0,$$

then $y_n = 1063 - n$ and $z_n = -1001 + n$, where $n = 1, 2, \ldots, 1000$.

Basis. *If $n = 1$, then we have that the statement holds true.*

Inductive step. *Assume that the statement holds true for $n = k$, where $1 \leq k \leq 999$, let us prove that it holds true for $n = k + 1$. Note that*

$$(y_k - 1) + (z_k + 1) = y_k + z_k = 62,$$

and

$$(y_k - 1)(z_k + 1) = y_k z_k + (y_k - z_k) - 1 = x_k + \frac{62 + \sqrt{62^2 - 4x_k}}{2} -$$

$$-\frac{62-\sqrt{62^2-4x_k}}{2}-1=x_k+\sqrt{3844-4x_k}-1=x_{k+1}.$$

Thus, it follows that the numbers

$$y_{k+1}=y_k-1=1063-(k+1),$$

and

$$z_{k+1}=z_k+1=-1001+(k+1),$$

are the solutions of the following equation

$$x^2-62x+x_{k+1}=0.$$

This ends the proof of the statement.

We deduce that

$$x_{1000}=y_{1000}\cdot z_{1000}=-63.$$

Therefore $|x_{1000}|=63$.

Problem 12. Find the number of all sets M with real elements, such that
a) M has three elements.
b) If $a\in M$, then $2a^2-1\in M$.

Solution. *Note that if* $a\in M$, *then* $|a|\le 1$.

We proceed the proof by contradiction argument. Assume that $a_0\in M$ *and* $|a_0|>1$.

Without loss of generality, one can assume that a_0 *is the element of* M *that has the greatest absolute value.*

Thus, it follows that

$$2a_0^2-1=a_0^2+a_0^2-1>a_0^2>|a_0|.$$

Hence, we obtain that

$$2a_0^2-1>|a_0|.$$

This leads to a contradiction, as $2a_0^2-1\in M$.

Let $a\in M$, *we have that* $|a|\le 1$. *Therefore, there exists* $\phi\in[0,\pi]$, *such that* $a=\cos\phi$. *Hence* $2a^2-1=\cos 2\phi\in M$, $\cos 4\phi\in M$, $\cos 8\phi\in M$.

According to assumption a), we have that two among the numbers $\cos\phi$, $\cos 2\phi$, $\cos 4\phi$, $\cos 8\phi$ *are equal. Therefore*

$$a\in\left\{1,-\frac{1}{2},\cos\frac{2\pi}{5},\cos\frac{4\pi}{5},-1,\frac{1}{2},\cos\frac{2\pi}{7},\cos\frac{4\pi}{7},\cos\frac{6\pi}{7},\cos\frac{2\pi}{9},\cos\frac{4\pi}{9},\cos\frac{8\pi}{9},\right.$$

$$\cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right\}.$$

Now, let us write all such sets M:

$$\left\{\frac{\sqrt{3}}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{\frac{1}{2}, -\frac{1}{2}, 1\right\}, \left\{-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\frac{1}{2}\right\}, \left\{0, -1, 1\right\}, \left\{-1, 1, -\frac{1}{2}\right\},$$

$$\left\{\cos\frac{\pi}{5}, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}\right\}, \left\{\cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, -\frac{1}{2}\right\}, \left\{\cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, 1\right\},$$

$$\left\{\cos\frac{3\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{2\pi}{5}\right\}, \left\{\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{6\pi}{7}\right\}, \left\{\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}\right\}.$$

Hence, the number of all such sets M is equal to 11.

7.3.15 Problem Set 15

Problem 1. Given that

$$x + \frac{1}{x} = 1.$$

Evaluate the expression

$$x^4 + \frac{1}{x^4} + \frac{4}{x} - \frac{4}{x^2}.$$

Solution. *We have that*

$$x^2 = 1 - \frac{2}{x} + \frac{1}{x^2}.$$

Therefore

$$x^2 - \frac{1}{x^2} = 1 - \frac{2}{x}.$$

Thus, it follows that

$$x^4 - 2 + \frac{1}{x^4} = 1 - \frac{4}{x} + \frac{4}{x^2}.$$

Hence, we obtain that

$$x^4 + \frac{1}{x^4} + \frac{4}{x} - \frac{4}{x^2} = 3.$$

Problem 2. In how many ways can one insert numbers 9, 12, 16, 45 instead of $*$ in the following expression (inserted numbers can be equal)

$$\sqrt{*+\sqrt{*+\sqrt{*+\sqrt{*}}}},$$

such that the obtained number is a rational number?

Solution. *Note that if* a, b, c, d *are positive integers and*

$$\sqrt{a+\sqrt{b+\sqrt{c+\sqrt{d}}}},$$

is a rational number, then the numbers

$$\sqrt{b+\sqrt{c+\sqrt{d}}},\ \sqrt{c+\sqrt{d}},\ \sqrt{d},$$

are rational numbers too.

Hence, either $d = 9$ *or* $d = 16$.

If $d = 9$, *then this leads to a contradiction, as the number*

$$\sqrt{c+\sqrt{9}},$$

is not a rational number.

If $d = 16$, *then either* $c = 12$ *or* $c = 45$.

Thus, it follows that all possible values are:

$d = 16$, $c = 12$, $b = 45$, $a = 9$.

$d = 16$, $c = 12$, $b = 12$, $a = 12$.

$d = 16$, $c = 12$, $b = 12$, $a = 45$.

$d = 16$, $c = 45$, $b = 9$, $a = 12$.

$d = 16$, $c = 45$, $b = 9$, $a = 45$.

Hence, we obtain that the answer is 5.

Problem 3. Let x, y, z be such numbers that

$$(x-y)(y-z)(z-x) = -1.$$

Evaluate the expression

$$\frac{1}{(x-y)^3(y-z)^3} + \frac{1}{(y-z)^3(z-x)^3} + \frac{1}{(z-x)^3(x-y)^3}.$$

Solution. *Let*

$$a = \frac{1}{(x-y)(y-z)},\ b = \frac{1}{(y-z)(z-x)},\ c = \frac{1}{(z-x)(x-y)}.$$

Note that

$$a+b+c = \frac{z-x+x-y+y-z}{(x-y)(y-z)(z-x)} = 0.$$

Thus, according to the following identity

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ac),$$

it follows that

$$a^3+b^3+c^3 = 3abc = \frac{3}{((x-y)(y-z)(z-x))^2} = 3.$$

Hence, we obtain that

$$\frac{1}{(x-y)^3(y-z)^3} + \frac{1}{(y-z)^3(z-x)^3} + \frac{1}{(z-x)^3(x-y)^3} = 3.$$

Problem 4. Let $*$ be a mathematical operation defined on the set of positive integers. Given that for any positive integers a, b it holds true $a*a = 2a$, $a*b = b*a$ and

$$a*(a+b) = a*b + \frac{a^2}{(a,b)},$$

where by (a,b) we denote the greatest common divisor of a and b. Find $24*10$.

Solution. *We have that*

$$24*10 = 10*24 = 10*(10+14) = 10*14 + \frac{10^2}{2} = 10*(10+4)+50 =$$

$$= 10*4 + \frac{10^2}{2} + 50 = 4*10+100 = 4*(4+6)+100 = 4*6+\frac{4^2}{2}+100 =$$

$$= 4*(4+2)+108 = 4*2+\frac{4^2}{2}+108 = 2*4+116 = 2*(2+2)+116 =$$

$$= 2*2+\frac{2^2}{2}+116 = 122.$$

Thus, it follows that

$$24*10 = 122.$$

An example of such mathematical operation $*$ can be $a * b = (a,b) + [a,b]$, where by $[a,b]$ we denote the least common multiple of positive integers a and b.

Problem 5. Let $x^2 + y^2 = x + y + xy$, where $x, y \in \mathbb{R}$. Find the greatest possible value of $x^2 + y^2$.

Solution. *Note that*

$$(x+y)^2 = x + y + 3xy \leq x + y + 3\frac{(x+y)^2}{4}.$$

Thus, it follows that

$$0 \leq x + y \leq 4.$$

Therefore

$$xy \leq \frac{(x+y)^2}{4} \leq 4.$$

We deduce that

$$x^2 + y^2 = x + y + xy \leq 4 + 4 = 8.$$

If $x = y = 2$, then $x^2 + y^2 = 8$. Hence, the greatest value of $x^2 + y^2$ is equal to 8.

Problem 6. Let the distance from the point $M(7, 14)$ to the graph of the function $y = \sqrt[3]{x}$ is equal to d. Find d^2.

Solution. *Note that point $X(x, \sqrt[3]{x})$ is a point on the graph of the function $y = \sqrt[3]{x}$. We have that*

$$MX^2 = (x-7)^2 + (\sqrt[3]{x} - 14)^2.$$

Let us denote $\sqrt[3]{x} = t$, then d^2 is the smallest value of the following function

$$f(t) = t^6 - 14t^3 + t^2 - 28t + 245.$$

Note that

$$f(t) = (t-2)^2((t^2+2t)^2 + 8t^2 + 18t + 25) + 145 \geq 145,$$

and

$$f(2) = 145.$$

Thus, it follows that $d^2 = 145$.

Problem 7. Evaluate the expression

$$\left[\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\cdot\dots\cdot\left(1+\frac{1}{100^2}\right)\right],$$

where by $[x]$ we denote the integer part of a real number x.

Solution. *Note that* $k^2 > k^2 - 1$, $k = 2, \dots, 100$. *Thus, it follows that*

$$A = \left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right) \cdot \dots \cdot \left(1 + \frac{1}{100^2}\right) <$$

$$< 2 \cdot \left(1 + \frac{1}{2^2 - 1}\right) \cdot \dots \cdot \left(1 + \frac{1}{100^2 - 1}\right) =$$

$$= 2 \cdot \frac{2^2}{1 \cdot 3} \cdot \frac{3^2}{2 \cdot 4} \cdot \dots \cdot \frac{100^2}{99 \cdot 101} = \frac{400}{101} < 4.$$

Hence, we obtain that $A < 4$.

Note that $(k+1)^2 - 1 > k^2$, $k = 3, \dots, 100$. *Thus, it follows that*

$$A = \frac{5}{2}\left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{4^2}\right) \cdot \dots \cdot \left(1 + \frac{1}{100^2}\right) >$$

$$> \frac{5}{2}\left(1 + \frac{1}{4^2 - 1}\right)\left(1 + \frac{1}{5^2 - 1}\right) \cdot \dots \cdot \left(1 + \frac{1}{101^2 - 1}\right) =$$

$$= \frac{5}{2} \cdot \frac{4^2}{3 \cdot 5} \cdot \dots \cdot \frac{5^2}{4 \cdot 6} \cdot \dots \cdot \frac{101^2}{100 \cdot 102} = \frac{20 \cdot 101}{6 \cdot 102} > 3.$$

We deduce that $A > 3$.

Thus, it follows that

$$3 < A < 4.$$

Therefore

$$\left[\left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right) \cdot \dots \cdot \left(1 + \frac{1}{100^2}\right)\right] = 3.$$

Problem 8. Let x_0 be the greatest real solution of the equation

$$\frac{1}{(x-1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(2x+1)^2} = \frac{9}{4}.$$

Find $(4x_0 + 5)^2$.

Solution. *The given equation is equivalent to the following equation*

$$\left(\frac{1}{x-1} + \frac{1}{x+2}\right)^2 - \frac{2}{(x-1)(x+2)} + \frac{1}{(2x+1)^2} = \frac{9}{4}.$$

This can be rewritten as

$$\left(\frac{1}{x-1}+\frac{1}{x+2}-\frac{1}{2x+1}\right)^2=\frac{9}{4}.$$

Hence, we obtain that either

$$(x+1)(2x^2-x+4)=0,$$

or

$$x(2x^2+5x-1)=0.$$

Thus, it follows that

$$(4x_0+5)^2=33.$$

Problem 9. Let (a_n) be such sequence that $a_1=1+2^{2016}$ and

$$a_{n+1}=1.25a_n-0.75\sqrt{a_n^2-2^{2018}},\ n=1,2,\dots,1008.$$

Find the remainder of division a_{1009} by 100.

Solution. *Let us consider*

$$x^2-a_nx+2^{2016}=0,\ n=1,2,\dots,1009.$$

By mathematical induction, let us prove that the solutions of this equations are the numbers 2^{n-1} and 2^{2017-n}.

Basis. *If $n=1$, then the solutions of the equation*

$$x^2-(1+2^{2016})x+2^{2016}=0,$$

are the numbers 1 and 2^{2016}.

Inductive step. *Let $k\in\mathbb{N}$, $1\le k\le 1008$ and the statement holds true for $n=k$. Let us prove that the statement holds true for $n=k+1$.*

We have that

$$2^{(k+1)-1}+2^{2017-(k+1)}=1.5\cdot 2^{k-1}+0.5\cdot(2^{k-1}+2^{2017-k})=1.5\cdot\frac{a_k-\sqrt{a_k^2-2018}}{2}+$$

$$+0.5\cdot a_k=1.25a_k-0.75\sqrt{a_k^2-2018}=a_{k+1}.$$

On the other hand

$$2^{(k+1)-1}\cdot 2^{2017-(k+1)}=2^{2016}.$$

This ends the proof of the statement.

Thus, it follows that

$$a_{1009} = 2^{1008} + 2^{1008} = 2^{1009}.$$

Note that

$$2^2(2^{20} - 1) \mid 2^9((2^{20})^{50} - 1) = a_{1009} - 2^9.$$

On the other hand, we have that $100 \mid 2^2(2^{10} + 1)$. *Therefore*

$$100 \mid a_{1009} - 512.$$

Hence, we obtain that a_{1009} *is divisible by 100 with a remainder of 12.*

Problem 10. Let sequence (x_n) be such that

$$x_1 = 25\sqrt{2},$$

$$x_2 = \frac{25\sqrt{2}}{\cos\frac{7\pi}{1200} + \sqrt{3}\sin\frac{7\pi}{1200}},$$

$$\dots$$

$$x_{n+2} = \frac{x_n x_{n+1}}{2\cos\frac{7\pi}{1200}\cdot x_n - x_{n+1}},\ n = 1, 2, \dots, 99.$$

Find x_{101}.

Solution. *Let us prove by mathematical induction that*

$$x_i = \frac{25\sqrt{2}}{2\sin\left(\frac{\pi}{6} + \frac{7\pi}{1200}(i-1)\right)},\ i = 1, 2, \dots, 101.$$

Basis. *If* $i = 1, 2$, *then the statement holds true.*

Inductive step. *If the statement holds true for any* $i \leq k$, *where* $k \in \{2, \dots, 100\}$, *then prove that it holds true for* $i = k + 1$.

We have that

$$x_{k+1} = \frac{x_{k-1}x_k}{2\cos\frac{7\pi}{1200}x_{k-1} - x_k} =$$

$$\frac{25\sqrt{2}}{4\sin\left(\frac{\pi}{6} + \alpha(k-2)\right)\sin\left(\frac{\pi}{6} + \alpha(k-1)\right)} \cdot \frac{1}{\frac{2\cos\alpha}{2\sin\left(\frac{\pi}{6} + \alpha(k-2)\right)} - \frac{1}{2\sin\left(\frac{\pi}{6} + \alpha(k-1)\right)}},$$

where $\alpha = \frac{7\pi}{1200}$. *Hence, we obtain that*

$$x_{k+1} = \frac{25\sqrt{2}}{2\left(2\cos\alpha\sin\left(\frac{\pi}{6}+\alpha(k-1)\right)-\sin\left(\frac{\pi}{6}+\alpha(k-2)\right)\right)} = \frac{25\sqrt{2}}{2\sin\left(\frac{\pi}{6}+\alpha k\right)}.$$

Therefore, the statement holds true for $i = k+1$.

This ends the proof of the statement.

If $i = 101$, *then we have that*

$$x_{101} = \frac{25\sqrt{2}}{2\sin\left(\frac{\pi}{6}+\frac{7\pi}{12}\right)} = 25.$$

Problem 11. Let a, b, c, d be positive numbers, such that $c^2+d^2-a^2-b^2 = \sqrt{3}(cd-ab)$, $b^2+c^2-a^2-d^2 = ad+bc$ and $\sqrt{b^2+d^2}+\sqrt{a^2+c^2} > \sqrt{a^2+d^2+ad} + \sqrt{b^2+c^2-bc}$. Find the value of the following expression

$$\left(\frac{ab+cd+\sqrt{3}ad}{bc}\right)^2.$$

Solution. *Consider a triangle AMB, such that* $MA = a$, $MB = d$, $\angle AMB = 120°$. *Let C, D be such points inside the angle AMB, such that* $MC = c$, $\angle BMC = 30°$ *and* $MD = b$, $\angle AMD = 30°$.

According to the law of cosines, from the triangles AMB, MBC, MCD and MAD, we have that

$$AB^2 = a^2+d^2+ad,$$

$$BC^2 = c^2+d^2-\sqrt{3}cd,$$

$$CD^2 = b^2+c^2-bc,$$

$$AD^2 = a^2+b^2-\sqrt{3}ab.$$

According to the condition of the problem, we have that $AB = CD$, $BC = AD$ *and* $AC+BD > AB+CD$.

Note that points M and C are on the different sides of the line AB. Otherwise, from the condition $AC+BD > AB+CD$ *it follows that the point D is also inside of the triangle AMB. Hence, we obtain that* $AB > CD$. *This leads to a contradiction.*

Therefore, quadrilateral ABCD is a parallelogram.

We deduce that

$$(MCD)-(MAB) = \frac{1}{2}(ABCD) = (MAD)+(MBC).$$

Thus, it follows that

$$\frac{1}{2}ab\sin 30^\circ + \frac{1}{2}cd\sin 30^\circ = \frac{1}{2}bc\sin 60^\circ - \frac{1}{2}ad\sin 60^\circ.$$

Therefore

$$\left(\frac{ab+cd+\sqrt{3}ad}{bc}\right)^2 = \left(\sqrt{3}\right)^2 = 3.$$

Problem 12. Let $p(x)$, $q(x)$, $r(x)$ be quadratic trinomials with real coefficients, such that for any x it holds true

$$|p(x)|+|q(x)| = |r(x)|.$$

Given that $p(1) = q(2) = 0$ and $r(3) = 9$, $r(4) = 29$. Find $r(10)$.

Solution. *Let*

$$p(x) = a_1x^2 + b_1x + c,$$

and

$$q(x) = a_2x^2 + b_2x + c_2,$$

where $a_1 > 0$ and $a_2 > 0$.

The given condition is equivalent to

$$|p(x)q(x)| = \frac{r^2(x)-p^2(x)-q^2(x)}{2}. \tag{7.88}$$

Note that the degree of the polynomial

$$\frac{r^2(x)-p^2(x)-q^2(x)}{2} = Q(x),$$

is not more than 4. On the other hand, from (7.88) it follows that

$$\lim_{x\to\infty}\frac{Q(x)}{x^4} = a_1a_2.$$

Hence, the polynomial $Q(x)$ is of the fourth degree.

Note that $Q(1) = 0$, $Q(2) = 0$ and $Q(x) \geq 0$ for any value of x. Therefore

$$Q(x) = a_1a_2(x-1)^2(x-2)^2.$$

Thus, one of the following cases is possible

$$p(x) = a_1(x-1)(x-2),\ q(x) = a_2(x-1)(x-2)$$

or

$$p(x) = a_1(x-1)^2,\ q(x) = a_2(x-2)^2.$$

In the first case, we obtain that

$$|r(x)| = (a_1 + a_2)|(x-1)(x-2)|.$$

Thus, it follows that

$$2(a_1 + a_2) = 9,\ 6(a_1 + a_2) = 29.$$

This leads to a contradiction.

In the second case, we obtain that

$$|r(x)| = a_1(x-1)^2 + a_2(x-2)^2.$$

Hence, it follows that

$$4a_1 + a_2 = 9,\ 9a_1 + 4a_2 = 29.$$

We deduce that $a_1 = 1$, $a_2 = 5$. *Therefore*

$$r(x) = (x-1)^2 + 5(x-2)^2.$$

Thus, it follows that

$$r(10) = 401.$$

7.4 Calculus

7.4.1 Problem Set 1

Problem 1. Evaluate the expression

$$\lim_{n\to\infty} \frac{4}{n^2}\left(1+\frac{6}{1\cdot 4}\right)\left(1+\frac{8}{2\cdot 5}\right)\cdot\ldots\cdot\left(1+\frac{2n+4}{n\cdot(n+3)}\right).$$

Solution. *Note that*

$$1 + \frac{2k+4}{k(k+3)} = \frac{(k+1)(k+4)}{k(k+3)},$$

therefore

$$\frac{4}{n^2}\left(1+\frac{6}{1\cdot 4}\right)\left(1+\frac{8}{2\cdot 5}\right)\cdot\ldots\cdot\left(1+\frac{2n+4}{n\cdot(n+3)}\right) = \frac{4}{n^2}\cdot\frac{2\cdot 5\cdot 3\cdot 6\cdot\ldots\cdot(n+1)(n+4)}{1\cdot 4\cdot 2\cdot 5\cdot\ldots\cdot n(n+3)} =$$

$$= \frac{4}{n^2} \cdot \frac{(n+1)! \cdot \frac{(n+4)!}{4!}}{n! \cdot \frac{(n+3)!}{3!}} = 1 + \frac{5}{n} + \frac{4}{n^2},$$

thus

$$\lim_{n\to\infty} \frac{4}{n^2}\left(1+\frac{6}{1\cdot 4}\right)\left(1+\frac{8}{2\cdot 5}\right)\cdot\dots\cdot\left(1+\frac{2n+4}{n\cdot(n+3)}\right) = \lim_{n\to\infty}\left(1+\frac{5}{n}+\frac{4}{n^2}\right) = 1.$$

Problem 2. Let $f(x) = (x+1)\left(\frac{1}{2}x+1\right)\left(\frac{1}{3}x+1\right)\cdot\dots\cdot\left(\frac{1}{2014}x+1\right)$. Denote by $f^{(n)}$ the n^{th} derivative of a function f. Find $f^{(2014)}$.

Solution. *Note that*

$$f(x) = \frac{1}{2014!}x^{2014} + a_1 x^{2013} + \dots + a_{2014},$$

hence

$$f^{(2014)} = \frac{1}{2014!} \cdot 2014! = 1.$$

Problem 3. Let $f(x) = (x-7)^{2014}$. Find the solution of the equation

$$f'(x) + f'''(x-2014) + f'''(x+2014) = 0.$$

Solution. *We have*

$$f'(x) + f'''(x-2014) + f'''(x+2014) = 2014(x-7)^{2013} + 2014\cdot 2013\cdot 2012(x-2021)^{2011} +$$

$$+2014\cdot 2013\cdot 2012(x+2007)^{2011}.$$

Note that the function

$$g(x) = 2014(x-7)^{2013} + 2014\cdot 2013\cdot 2012(x-2021)^{2011} + 2014\cdot 2013\cdot 2012(x+2007)^{2011}$$

is increasing and $g(7) = 0$*, thus the unique solution of the equation* $g(x) = 0$ *is* 7.

Problem 4. Evaluate the expression

$$\int_0^2 \log_2(x + \sqrt{x^2-2x+2} - 1)dx.$$

Solution. *Note that*

$$\int_0^2 \log_2(x + \sqrt{x^2-2x+2} - 1)dx = \int_{-1}^1 \log_2(t + \sqrt{t^2+1})dt.$$

The function $F(t) = \log_2(t + \sqrt{t^2+1})$ *is odd, as*

$$F(-t) = \log_2(-t + \sqrt{t^2+1}) = \log_2 \frac{1}{t + \sqrt{t^2+1}} = -F(t),$$

thus

$$\int_{-1}^{1} \log_2(t + \sqrt{t^2+1})dt = 0.$$

Problem 5. Let the sequence (x_n) be such that $\lim_{n\to\infty}(x_n - x_{n-1}) = 0$. Find $\lim_{n\to\infty} \frac{x_n}{n}$.

Solution. *Let* $\varepsilon > 0$ *and a positive integer* n_0 *is such that for* $n \geq n_0$ *we have* $|x_n - x_{n-1}| < \varepsilon$. *Obviously,* $x_n = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{n_0} - x_{n_0-1}) + x_{n_0-1}$, *therefore*

$$\frac{|x_n|}{n} \leq \frac{|x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{n_0} - x_{n_0-1}|}{n} + \frac{|x_{n_0-1}|}{n} < \frac{\varepsilon(n - n_0 + 1)}{n} + \frac{|x_{n_0-1}|}{n}.$$

Let us choose n such that $\frac{|x_{n_0-1}|}{n} < \varepsilon$. *Therefore, for a sufficiently large n we obtain that* $\frac{|x_n|}{n} < 2\varepsilon$. *Hence*

$$\lim_{n\to\infty} \frac{x_n}{n} = 0.$$

Problem 6. Find the minimum value of a natural number a, such that the equation $x^3 + (3-a)x^2 + 3x + 1 = 0$ has three solutions in the set of real numbers.

Solution. *The given equation is equivalent to the following equation*

$$a = \frac{(x+1)^3}{x^2}.$$

Next, construct the graph of the function $a = \frac{(x+1)^3}{x^2}$ *and check that the given equation has three solutions in the set of real numbers iff* $a \in (6,75; +\infty)$.

Problem 7. Let the sequence (x_n) be such that $0 < x_1 < 1$ and $x_{n+1} = x_n(1 - x_n)$. Find $\lim_{n\to\infty}(nx_n)$.

Solution. *Note that*

$$x_2 = x_1(1 - x_1) \leq \frac{1}{4} < \frac{1}{2}, \quad x_3 < \frac{1}{2} \times \frac{1}{2} < \frac{1}{3}, \quad x_4 < \frac{1}{3} \times \frac{2}{3} < \frac{1}{4}, \ldots$$

We have that

$$\frac{1}{x_{n+1}} = \frac{1}{x_n} + \frac{1}{1 - x_n},$$

therefore

$$\frac{1}{x_{n+1}}=\frac{1}{x_n}+\frac{1}{1-x_n}=\frac{1}{x_{n+1}}=\frac{1}{x_{n-1}}+\frac{1}{1-x_{n-1}}+\frac{1}{1-x_n}=\cdots=\frac{1}{x_1}+\frac{1}{1-x_1}+\cdots+\frac{1}{1-x_n}.$$

Hence

$$\frac{1}{x_{n+1}}<\frac{1}{x_1}+\frac{1}{1-x_1}+1+1+1+\frac{1}{2}+\cdots+1+\frac{1}{n-1}<a+n+1+1+\frac{1}{2}+\cdots+\frac{1}{n+1},$$

where

$$a=\frac{1}{x_1}+\frac{1}{1-x_1},$$

thus

$$\frac{1}{a+n+1+1+\frac{1}{2}+\cdots+\frac{1}{n+1}}<x_{n+1}<\frac{1}{n+1}.$$

We obtain that

$$y_n=\frac{1}{\frac{a}{n+1}+1+\frac{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}{n+1}}<(n+1)x_{n+1}<1.$$

We have that

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)\right)=0,$$

then from the Problem 5 we deduce that

$$\lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n}=0,$$

therefore $\lim\limits_{n\to\infty}y_n=1$. *Thus* $\lim\limits_{n\to\infty}(nx_n)=1$.

Problem 8. Find the product of the solutions of the following equation $3^{\frac{x}{2}}-2^{x-1}=1$.

Solution. *Note that numbers* 2 *and* 4 *are solutions for the given equation. Let us now prove that the given equation cannot have three solutions. We proceed by a contradiction argument. Assume that it has three solutions, then the equation* $f'(x)=0$ *will have two solutions, where* $f(x)=3^{\frac{x}{2}}-2^{x-1}$. *On the other hand, the following equation*

$$f'(x)=\frac{1}{2}\left(3^{\frac{x}{2}}\ln 3-2^x\ln 2\right)$$

has only one solution. Thus, the product of the solutions is equal to 8.

Problem 9. Evaluate the expression

$$\lim_{n\to\infty}\left(6\sin\left(\frac{\pi}{3}+\frac{1^2}{n^3}\right)+6\sin\left(\frac{\pi}{3}+\frac{2^2}{n^3}\right)+\cdots+6\sin\left(\frac{\pi}{3}+\frac{n^2}{n^3}\right)-3\sqrt{3}n\right).$$

Solution. *Let $\varepsilon>0$, then there exists $\delta>0$ such that for $0\neq|x|<\delta$ we have that*

$$\left|\frac{\sin(\frac{\pi}{3}+x)-\sin\frac{\pi}{3}}{x}-\frac{1}{2}\right|<\frac{\varepsilon}{12}.$$

Let n_0 be such that for $n>n_0$ we have $\frac{3n+1}{2n^2}<\frac{\varepsilon}{2}$ and $\frac{1}{n}<\delta$.
Hence

$$\left|\sin\left(\frac{\pi}{3}+\frac{1^2}{n^3}\right)-\frac{\sqrt{3}}{2}-\frac{1}{2}\times\frac{1^2}{n^3}\right|<\frac{\varepsilon}{12}\times\frac{1^2}{n^3},\left|\sin\left(\frac{\pi}{3}+\frac{2^2}{n^3}\right)-\frac{\sqrt{3}}{2}-\frac{1}{2}\times\frac{2^2}{n^3}\right|<\frac{\varepsilon}{12}\times\frac{2^2}{n^3},\ldots,$$

$$\left|\sin\left(\frac{\pi}{3}+\frac{n^2}{n^3}\right)-\frac{\sqrt{3}}{2}-\frac{1}{2}\times\frac{n^2}{n^3}\right|<\frac{\varepsilon}{12}\times\frac{n^2}{n^3}.$$

Therefore, by summing up all the inequalities, we deduce that

$$\left|\sin\left(\frac{\pi}{3}+\frac{1^2}{n^3}\right)+\sin\left(\frac{\pi}{3}+\frac{2^2}{n^3}\right)+\cdots+\sin\left(\frac{\pi}{3}+\frac{n^2}{n^3}\right)-\frac{\sqrt{3}}{2}n-\frac{2n^2+3n+1}{12n^2}\right|<\frac{2n^2+3n+1}{72n^2}\varepsilon.$$

Thus

$$\left|6\sin\left(\frac{\pi}{3}+\frac{1^2}{n^3}\right)+6\sin\left(\frac{\pi}{3}+\frac{2^2}{n^3}\right)+\cdots+6\sin\left(\frac{\pi}{3}+\frac{n^2}{n^3}\right)-3\sqrt{3}n-\frac{2n^2+3n+1}{2n^2}\right|<\frac{2n^2+3n+1}{12n^2}\varepsilon.$$

We obtain that

$$\left|6\sin\left(\frac{\pi}{3}+\frac{1^2}{n^3}\right)+6\sin\left(\frac{\pi}{3}+\frac{2^2}{n^3}\right)+\cdots+6\sin\left(\frac{\pi}{3}+\frac{n^2}{n^3}\right)-3\sqrt{3}n-1\right|<$$

$$<\frac{3n+1}{2n^2}+\frac{2n^2+3n+1}{12n^2}\varepsilon<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Finally, we deduce that for $n>n_0$

$$\left|6\sin\left(\frac{\pi}{3}+\frac{1^2}{n^3}\right)+6\sin\left(\frac{\pi}{3}+\frac{2^2}{n^3}\right)+\cdots+6\sin\left(\frac{\pi}{3}+\frac{n^2}{n^3}\right)-3\sqrt{3}n-1\right|<\varepsilon.$$

Hence

$$\lim_{n\to\infty}\left(6\sin\left(\frac{\pi}{3}+\frac{1^2}{n^3}\right)+6\sin\left(\frac{\pi}{3}+\frac{2^2}{n^3}\right)+\cdots+6\sin\left(\frac{\pi}{3}+\frac{n^2}{n^3}\right)-3\sqrt{3}n\right)=1.$$

7.4.2 Problem Set 2

Problem 1. How many integer values does the function $y = 8tgx\cos x$ have?

Solution. *Note that* $y = 8\sin x$ *and* $D(y) = \{x : \cos x \neq 0\}$, *thus* $E(y) = (-8, 8)$. *Hence, the number of integer values of the function y is* 15.

Problem 2. Evaluate the expression

$$\lim_{x\to 0} \frac{3^{tg50x} - 1}{3^x - 1}.$$

Solution. *We have*

$$\lim_{x\to 0} \frac{3^{tg50x} - 1}{3^x - 1} = \lim_{x\to 0} \left(\frac{3^{tg50x} - 1}{tg50x} \cdot \frac{tg50x}{50x} \cdot \frac{tg50}{\frac{3^x-1}{x}} \right) = \ln 3 \cdot 1 \cdot \frac{50}{\ln 3} = 50.$$

Problem 3. Find the greatest value of C in the set of real numbers, such that the inequality $|tgx - tgy| \geq C|x - y|$ holds true for any $x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution. *We have* $|tgx - 0| \geq C|x|$. *Thus, if* $0 < x < \frac{\pi}{2}$, $\frac{tgx}{x} \geq C$. *Therefore*

$$\lim_{x\to 0} \frac{tgx}{x} \geq C,$$

hence $C \leq 1$.

Let us prove that for any $x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, *we have that* $|tgx - tgy| \geq |x - y|$. *Indeed*

$$|tgx - tgy| = \frac{2|\sin(x-y)|}{\cos(x-y) + \cos(x+y)} \geq \frac{2|\sin(x-y)|}{\cos(x-y) + 1} = 2tg\frac{|x-y|}{2} \geq 2 \cdot \frac{|x-y|}{2} = |x - y|.$$

Thus, the greatest possible value of C is equal to 1.

Problem 4. Find the number of the solutions of the equation $8 \cdot 2^x - 6 \cdot 3^x + 5^x - 3 = 0$.

Solution. *Note that* $0, 1, 2$ *are the solutions of the given equation. Consider the function* $f(x) = 8 \cdot 2^x - 6 \cdot 3^x + 5^x$. *If besides those solutions the equation* $f(x) = 3$ *has another solution, then by Rolle's theorem* $f'(x) = 0$, *the equation* $\left(\frac{5}{2}\right)^x \ln 5 - 6\left(\frac{3}{2}\right)^x = 8\ln 2$ *has at least three solutions. Hence, again by Rolle's theorem the equation*

$$\left(\frac{5}{2}\right)^x \ln 5 \cdot \ln\frac{5}{2} - 6\left(\frac{3}{2}\right)^x \ln\frac{5}{2} \cdot \ln 3 = 0$$

or equivalently, the equation

$$\left(\frac{5}{3}\right)^x = \frac{6\ln\dfrac{3}{2}\cdot\ln 3}{\ln\dfrac{5}{2}\cdot\ln 5}$$

has at least two solutions, which leads to a contradiction. Therefore, the given equation has three solutions.

Problem 5. Evaluate the expression

$$\lim_{x\to\pi}\frac{40x\cos\dfrac{x}{2}}{\pi^2-x^2}.$$

Solution. *We have that*

$$\lim_{x\to\pi}\frac{40x\cos\frac{x}{2}}{\pi^2-x^2} = \lim_{t\to 0}\frac{40(\pi+t)\cos\frac{\pi+t}{2}}{\pi^2-(\pi+t)^2} = \lim_{t\to 0}\frac{40(\pi+t)\sin\frac{t}{2}}{t^2+2\pi t} =$$

$$= \lim_{t\to 0}\left(\frac{20(\pi+t)}{t+2\pi}\cdot\frac{\sin\dfrac{t}{2}}{\dfrac{t}{2}}\right) = 10.$$

Problem 6. Let $a_1,\dots,a_n$ be the sides of a convex n-gon, which is in a unit square. Given that there exists $\alpha < 0$ such that $a_1^\alpha+\cdots+a_n^\alpha < 4$. Find all possible values of n.

Solution. *We have that a convex n-gon with the sides $a_1,\dots,a_n$ is in a unit square, thus $a_1+\cdots+a_n < 4$. Consider the function $f(x) = x^\alpha - \alpha x$ on the interval $(0,\infty)$, hence $f'(x) = \alpha x^{\alpha-1} - \alpha$. If $x > 1$, then $f'(x) > 0$. If $0 < x < 1$, then $f'(x) < 0$, thus $f(x) \geq f(1) = 1-\alpha$. We deduce that $a_1^\alpha - \alpha a_1 + \cdots + a_n^\alpha - \alpha a_n \geq n(1-\alpha)$. On the other hand, we have that $a_1^\alpha - \alpha a_1 + \cdots + a_n^\alpha - \alpha a_n < 4-4\alpha$. Hence $n < 4$. For $n = 3$, we may take $a_1 = a_2 = a_3 = \frac{0{,}99}{\cos 15^\circ}$.*

Problem 7. Let (u_n) be a sequence of real numbers, such that $u_1 = 1, u_{n+1} = u_n + \dfrac{1}{u_n}$, $n = 1,2,\dots$ Find $[50u_{100}]$, where by $[x]$ we denote the integer part of a real number x.

Solution. *Let us prove that $14{,}2 < u_{100} < 14{,}22$. We have that*

$$u_{n+1}^2 = u_n^2 + 2 + \frac{1}{u_n^2}.$$

Therefore

$$u_2^2 = u_n^1 + 2 + \frac{1}{u_1^2}, \quad u_3^2 = u_2^2 + 2 + \frac{1}{u_2^2}, \ldots, u_n^2 = u_{n-1}^2 + 2 + \frac{1}{u_{n-1}^2}.$$

Summing up these equations, we obtain that

$$u_n^2 = 2n + \frac{1}{u_2^2} + \cdots + \frac{1}{u_{n-1}^2}, n = 3, 4, \ldots \tag{7.89}$$

As $u_{n-1} > u_{n-2} > \cdots > u_2 = 2$*, thus*

$$\frac{1}{u_2^2} + \cdots + \frac{1}{u_{n-1}^2} < \frac{n}{4}.$$

Hence

$$2n \le u_n^2 < \frac{9}{4}n, n = 2, 3, \ldots \tag{7.90}$$

From (7.89) *and* (7.90)*, it follows that for* $n = 100$ *we deduce that*

$$200 + \frac{4}{9}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99}\right) < u_{100}^2 < 200 + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99}\right).$$

Let us give the upper and lower estimations for the following sum

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99} = \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10}\right) + \left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11}\right) + \cdots +$$

$$\left(\frac{1}{35} + \cdots + \frac{1}{99}\right) = 1{,}675 + \left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11}\right) + \cdots + \left(\frac{1}{35} + \cdots + \frac{1}{99}\right) > 1{,}675 + 0{,}344 +$$

$$\left(\frac{1}{12} + \cdots + \frac{1}{34}\right) + \left(\frac{1}{35} + \cdots + \frac{1}{99}\right) > 2{,}019 + \frac{23^2}{23 \times 23} + \frac{65^2}{65 \times 67} > 3{,}989.$$

Therefore

$$200 + \frac{4}{9}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99}\right) > 201{,}7 > (14{,}2)^2.$$

For $k = 2, 3, \ldots$ *we have that*

$$\frac{1}{k} < \int_{k-1}^{k} \frac{1}{x}\,\mathrm{d}x = \ln k - \ln(k-1).$$

Hence

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99} = \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10}\right) + \left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11}\right) + \cdots +$$

$$\left(\frac{1}{35}+\cdots+\frac{1}{99}\right)=1,675+\left(\frac{1}{7}+\frac{1}{9}+\frac{1}{11}\right)+\cdots+\left(\frac{1}{35}+\cdots+\frac{1}{99}\right)<2,02+$$

$$\left(\frac{1}{12}+\cdots+\frac{1}{99}\right)<2,02+(\ln 12-\ln 11+\cdots+\ln 99-\ln 98)=2,02+\ln 9<4,22.$$

Thus

$$200+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{99}\right)<202,11<(14,22)^2.$$

Therefore $(14,2)^2<u_{100}^2<(14,22)^2$. *Hence* $710<50u_{100}<711$, *thus* $[50u_{100}]=710$.

Problem 8. We call two infinite sequences (a_n) and (b_n) "similar", if $a_n \neq b_n$ only for finite number of values of n (any sequence is considered similar to itself). We call any convergent sequence (x_n) of real numbers "nice", if $x_{n+1}=x_n^2-2$, for any $n=1,2,\dots$ (where x_1 is any number). We have a list of m sequences, such that any "nice" sequence is "similar" to one of the sequences in the list. Find the minimum possible value of m.

Solution. *First, we show that any "nice" convergent sequence (x_n), that is $x_{n+1}=x_n^2-2, n=1,2,\dots$ is "similar" either to the sequence $-1,-1,-1,\dots$ or $2,2,2\dots$. Let us prove that if $x_n\neq 2, n=1,2,\dots,$ then $\lim\limits_{n\to\infty}x_n\neq 2$. We proceed by a contradiction argument, assume $x_n\neq 2, n=1,2,\dots$ and $\lim x_n=2$. Denote by $t_n=x_n-2$, then $x_{n+1}=x_n^2-2$ is equivalent to $t_{n+1}=t_n^2+4t_n$. For large enough n, we have that $|t_{n+1}|=|t_n|\times|t_n+4|>3|t_n|$, $t_n\neq 0, n=1,2,\dots$ Therefore, the sequence $|t_n|$ is divergent, which leads to a contradiction, as $\lim\limits_{n\to\infty}=0$. In the similar way, we can show that if $x_n\neq -1, n=1,2,\dots,$ then $\lim\limits_{n\to\infty}x_n\neq -1$. If for some k, $x_k=2$, then for all $n>k$, $x_n=2$, hence $\lim\limits_{n\to\infty}x_n=2$. In the similar way, if for some k, $x_k=-1$, then for all $n>k$, $x_n=-1$, hence $\lim\limits_{n\to\infty}x_n=-1$. Thus, we have proven that there exists a limit of sequence (x_n) iff there exists k, such that $x_k=2$ or $x_k=-1$. Hence, we deduce that $m=2$.*

Problem 9. Let $f(x)$ be a continuous function on $[0,1]$, such that

$$\int_0^1 f^2(x)dx=\frac{e^2}{2}+\frac{11}{6},$$

$$\int_0^1 f(x)e^x dx=\frac{e^2}{2}+\frac{1}{2}$$

$$\int_0^1 f(x)xdx=\frac{4}{3}.$$

Find $f(0)$.

Solution. *Note that*

$$\int_0^1 (f(x)-x-e^x)^2dx = \int_0^1 f^2(x)dx - 2\int_0^1 xf(x)dx - 2\int_0^1 f(x)e^x dx + \int_0^1 (x+e^x)^2 dx.$$

By a simple, but technical computation, we obtain that

$$\int_0^1 (f(x)-x-e^x)^2dx = -\frac{e^2}{2} - \frac{11}{6} + \frac{1}{3} + 2 + \frac{e^2}{2} - \frac{1}{2} = 0.$$

Therefore

$$(f(x)-x-e^x)^2 = 0.$$

Hence $f(0) = 0 + e^0 = 1$.

7.4.3 Problem Set 3

Problem 1. Consider the function $y = \{x\}\cdot[x]$, where by $\{x\}$ and $[x]$ we denote the rational and integer part of a real number x, respectively. Find the number of real numbers, which do not belong to the domain of the function y.

Solution. *Let a be a real number. We choose a integer n, such that* $0 \le \frac{a}{n} < 1$. *For* $x = n + \frac{a}{n}$, *we have that* $y = \{n+\frac{a}{n}\}\cdot[n+\frac{a}{n}] = \{\frac{a}{n}\}\cdot n = \frac{a}{n}\cdot n = a$. *Therefore, the number of real numbers, which do not belong to the domain of the function y, is equal to* 0.

Problem 2. Evaluate the expression

$$\lim_{x\to 0}\frac{4(\sqrt[20]{1+x}-1)}{\sqrt[15]{1+x}-1}.$$

Solution. *We have that*

$$\lim_{x\to 0}\frac{4(\sqrt[20]{1+x}-1)}{\sqrt[15]{1+x}-1} = \lim_{x\to 0}\frac{4(1+x-1)((\sqrt[15]{1+x})^{14}+\cdots+1)}{(1+x-1)((\sqrt[20]{1+x})^{19}+\cdots+1)} =$$

$$= 4\lim_{x\to 0}\frac{\sqrt[15]{1+x})^{14}+\sqrt[15]{1+x})^{13}+\cdots+1}{\sqrt[20]{1+x})^{19}+\sqrt[20]{1+x})^{18}+\cdots+1} = \frac{4\cdot 15}{20} = 3.$$

Problem 3. Find the greatest value of C in the set of real numbers, such that the inequality $|lnx - lny| \ge C|x-y|$ holds true for any $x, y \in (0,1]$.

Solution. *Let n be a positive integer. We choose* $x = 1 - \frac{1}{n}$ *and* $y = 1$. *We have that*

$$|ln\left(1 - \frac{1}{n}\right)| \geq C \cdot \frac{1}{n},$$

therefore

$$C \leq |ln\left(1 - \frac{1}{n}\right)^n|.$$

Hence

$$\lim_{n\to\infty} C \leq \lim_{n\to\infty} |ln\left(1 - \frac{1}{n}\right)^n|.$$

Thus $C \leq 1$. *Let us prove that if* $x, y \in (0,1]$, *then*

$$|lnx - lny| \geq |x - y|.$$

According to Rolle's theorem, we have that $lnx - lny = \frac{1}{\gamma}(x - y)$, *where either* $x \leq \gamma \leq y$ *or* $y \leq \gamma \leq x$. *Hence, we deduce that* $\gamma \in [0,1]$. *Thus* $|lnx - lny| = \frac{1}{|\gamma|}|x - y| \geq |x - y|$. *Therefore, the greatest value of C is equal to 1.*

Problem 4. Find the greatest integer value of a, such that the equation $\sqrt{900 - x^2} = \sqrt{x - a} + 7\sqrt{10}$ has a solution.

Solution. *Note that if* x_0 *is a solution of the given equation, then* $\sqrt{900 - x_0^2} \geq 7\sqrt{10}$. *Therefore* $|x_0| \leq \sqrt{410}$. *Thus, if* $a \geq 21$, *then the given equation has no solutions, as* $x_0 \geq 21$.

If $a = 20$, *then the function* $f(x) = \sqrt{900 - x^2} - \sqrt{x - 20} - 7\sqrt{10}$ *is continuous on* $[20, 30]$ *and* $f(20) = \sqrt{500} - 7\sqrt{20} > 0$, $f(30) = -\sqrt{10} - 7\sqrt{10} < 0$. *Therefore, according to the mean value theorem, there exists* $c \in (20, 30)$, *such that* $f(c) = 0$. *Hence, the given equation has a solution for* $a = 20$. *Thus, the greatest possible value of a is equal to 20.*

Problem 5. Evaluate the expression

$$\lim_{x\to 0} \frac{\ln(\sin 3x + \cos 3x)}{\ln(\sin x + \cos x)}.$$

Solution. *Note that*

$$\lim_{x\to 0} \frac{\ln(\sin 3x + \cos 3x)}{\ln(\sin x + \cos x)} =$$

$$= \lim_{x\to 0} \left(\frac{\ln(\sin 3x + \cos 3x) - ln1}{\sin 3x + \cos 3x - 1} \cdot \frac{\sin x + \cos x - 1}{\ln(\sin x + \cos x) - ln1} \cdot \frac{\sin 3x + \cos 3x - 1}{\sin x + \cos x - 1} \right).$$

It is known that

$$\lim_{t\to 1}\frac{\ln t - ln 1}{t-1} = 1.$$

We have that

$$\lim_{x\to 0}\frac{\sin 3x+\cos 3x-1}{\sin x+\cos x-1} = \lim_{x\to 0}\frac{\sin\frac{3x}{2}\left(\cos\frac{3x}{2}-\sin\frac{3x}{2}\right)}{\sin\frac{x}{2}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)} =$$

$$= \lim_{x\to 0}\frac{3\cdot\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\left(\cos\frac{3x}{2}-\sin\frac{3x}{2}\right)}{\frac{\sin\frac{x}{2}}{\frac{x}{2}}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)} = 3.$$

Therefore, we obtain that

$$\lim_{x\to 0}\frac{\ln(\sin 3x+\cos 3x)}{\ln(\sin x+\cos x)} = 3.$$

Problem 6. Let f be a non-deceasing and continuous function on $[0,1]$, such that

$$\int_0^1 f(x)dx = 2\int_0^1 xf(x)dx.$$

Given that $f(1) = 10.5$. Find the value of $f(0)+f(0.5)$.

Solution. *Note that*

$$\int_0^1 (f(x)-f(1-x))(1-2x)\,\mathrm{d}x = \int_0^1 f(x)(1-2x)\,\mathrm{d}x - \int_0^1 f(1-x)(1-2x)\,\mathrm{d}x =$$

$$= -\int_1^0 f(t)(2t-1)\,\mathrm{d}(1-t) = \int_0^1 f(t)(1-2t)\,\mathrm{d}t = 0.$$

We have that the function $g(x) = (f(x)-f(1-x))(1-2x)$ is continuous on $[0,1]$ and $g(1-x) = (f(x)-f(1-x))(1-2x) = g(x)$.

If $0\le x\le \frac{1}{2}$, then $g(x) = (f(x)-f(1-x))(1-2x)\le 0$ as $f(x)\le f(1-x)$. Using that $g(1-x) = g(x)$, we deduce that $g(x)\le 0$ for $x\in[0,1]$. We have that $\int_0^1 g(x)\,\mathrm{d}x = 0$, thus $g(x) = 0$, for $x\in[0,1]$. Therefore, $f(x) = f(1-x)$ for $x\in[0,1]$. Hence $f(0) = f(1)$ which means that non-decreasing function f is constant. Thus $f(0)+f(0.5) = 2f(1) = 21$.

Problem 7. Evaluate the expression

$$\frac{3^{\frac{1}{103}}}{3^{\frac{1}{103}}+\sqrt{3}}+\frac{3^{\frac{2}{103}}}{3^{\frac{2}{103}}+\sqrt{3}}+\cdots+\frac{3^{\frac{102}{103}}}{3^{\frac{102}{103}}+\sqrt{3}}.$$

Solution. *Consider*

$$f(x)=\frac{3^x}{3^x+\sqrt{3}}.$$

We need to evaluate the following sum

$$f\left(\frac{1}{103}\right)+f\left(\frac{2}{103}\right)+\cdots+f\left(\frac{102}{103}\right).$$

Note that

$$f(x)+f(1-x)=\frac{3^x}{3^x+\sqrt{3}}+\frac{3^{1-x}}{3^{1-x}+\sqrt{3}}=\frac{3^x}{3^x+\sqrt{3}}+\frac{3}{3+\sqrt{3}\cdot 3^x}=$$

$$=\frac{3^x}{3^x+\sqrt{3}}+\frac{\sqrt{3}}{3^x+\sqrt{3}}=1.$$

Therefore

$$f\left(\frac{1}{103}\right)+f\left(\frac{2}{103}\right)+\cdots+f\left(\frac{102}{103}\right)=\left(f\left(\frac{1}{103}\right)+f\left(\frac{102}{103}\right)\right)+\cdots+$$

$$+\left(f\left(\frac{51}{103}\right)+f\left(\frac{52}{103}\right)\right)=51.$$

Problem 8. Let (u_n) be a sequence of real numbers, such that $u_1=10^9, u_{n+1}=\dfrac{u_n^2+2}{2u_n}$, $n=1,2,\ldots$ Find $[10^{13}(u_{36}-\sqrt{2})]$, where by $[x]$ we denote the integer part of a real number x.

Solution. *Let us prove that* $0<u_{36}-\sqrt{2}<10^{-13}$. *Then* $[10^{13}(u_{36}-\sqrt{2})]=0$. *Note that*

$$u_{n+1}=\frac{u_n}{2}+\frac{1}{u_n}\geq 2\sqrt{\frac{u_n}{2}\cdot\frac{1}{u_n}}=\sqrt{2}.$$

One can easily show that $u_n\in\mathbb{Q}$ *for any* $n=1,2,\ldots$, *hence* $u_n>\sqrt{2}$.

Therefore $u_{36}-\sqrt{2}<10^{-13}$.

Denote by $x_n=u_n-\sqrt{2}$. *We have that* $x_n+\sqrt{2}=u_{n+1}=\dfrac{u_n^2+2}{2u_n}=\dfrac{(x_n+\sqrt{2})^2+2}{2(x_n+\sqrt{2})}$.

Thus $x_{n+1}=\dfrac{x_n^2}{2(x_n+\sqrt{2})}$. *We deduce that* $x_{n+1}<\dfrac{x_n}{2}$ *and* $x_{n+1}<\dfrac{x_n^2}{2\sqrt{2}}$. *Therefore,*

we obtain that

$$x_{n+1} < \frac{x_n}{2} < \frac{x_{n-1}}{2^2} < \cdots < \frac{x_1}{2^n}.$$

Using that $x_1 < 10^9$ *and* $2^{10} > 10^3$, *we deduce that* $x_{31} < \dfrac{10^9}{2^{30}} < 1$.

On the other hand, we have that

$$x_{32} < \frac{1}{2\sqrt{2}}, \ x_{33} < \frac{1}{16\sqrt{2}}, \ x_{34} < \frac{1}{2^{10}\sqrt{2}}, \ x_{35} < \frac{1}{2^{22}\sqrt{2}},$$

$$x_{36} < \frac{1}{2^{46}\sqrt{2}} < \frac{1}{2^{40}\cdot 10} < 10^{-13}.$$

Hence $u_{36} - \sqrt{2} < 10^{-13}$.

Problem 9. Let $f : \mathbb{N} \to \mathbb{Z}$ and $f(1) = 5$. For any positive integers a, b, c given that $a+b+c \mid f(a)+f(b)+f(c)-3abc$. Find $f(9)$.

Solution. *For any positive integers b and c, we have that* $b+c+2 \mid f(1)+f(b+1)+f(c)-3(b+1)c$ *and* $b+c+2 \mid f(2)+f(b)+f(c)-6bc$. *Therefore* $b+c+2 \mid f(b+1)-f(b)+f(1)-f(2)+3bc-3c$. *Hence* $b+c+2 \mid f(b+1)-f(b)+f(1)-f(2)+3(b-1)(b+c+2)-3(b-1)(b+2)$. *Thus* $b+c+2 \mid f(b+1)-f(b)+f(1)-f(2)-3b^2-3b+6$. *As b and c are arbitrary positive integers, from the last inequality we deduce that the following equality holds true* $f(b+1)-f(b)+f(1)-f(2)-3b^2-3b+6=0$ *for any positive integer b. Hence* $(f(b+1)-(b+1)^3)-(f(b)-b^3) = (f(2)-2^3)-(f(1)-1^3)$. *This means that if we denote by* $g(x) = f(x)-x^3$, *then* $g(b+1)-g(b) = g(2)-g(1)$. *Let* $k+l = g(1)$ *and* $2k+l = g(2)$. *By mathematical induction argument, one can deduce that* $g(n) = kn+l$. *Note that* $k = g(2)-g(1) \in \mathbb{Z}$. *Therefore* $f(n) = n^3+kn+l$. *We obtain that* $f(a)+f(b)+f(c)-3abc = a^3+b^3+c^3-3abc+k(a+b+c)+3l$ *and* $a+b+c \mid a^3+b^3+c^3-3abc+k(a+b+c)+3l$. *Using that* $a+b+c \mid a^3+b^3+c^3-3abc+k(a+b+c)+3l = (a+b+c)(a^2+b^2+c^2-ab-bc-ac)+k(a+b+c)+3l$, *we obtain that* $a+b+c \mid 3l$. *Thus* $l=0$. *We have that* $f(1)=5$, *hence* $k=4$ *and* $f(9) = 9^3+4\cdot 9 = 765$.

7.4.4 Problem Set 4

Problem 1. Let M and m be the greatest and smallest values of function $f(x) = \sqrt{x-5}+\sqrt{28-2x}$, respectively. Find the value of $\dfrac{M^{10}}{m^9}$.

Solution. *Note that* $D(f) = [5,14]$ *and* $f'(x) = \dfrac{1}{2\sqrt{x-5}} - \dfrac{1}{\sqrt{28-2x}}$, *thus the function f on* $[5,14]$ *has a single critical point* $x=8$. *We have that* $f(5) = \sqrt{18}$, $f(8) = 3\sqrt{3}$, $f(14) = 3$, *thus* $m=3$ *and* $M = 3\sqrt{3}$. *Therefore* $\dfrac{M^{10}}{m^9} = 729$.

Problem 2. Evaluate the expression

$$\lim_{x\to 0}(\cos x)^{ctgx}.$$

Solution. *We have that*

$$\lim_{x\to 0}(\cos x)^{ctgx} = \lim_{x\to 0}((1+(\cos x-1))^{\frac{1}{\cos x-1}})^{(\cos x-1)ctgx} =$$

$$= \lim_{x\to 0}((1+(\cos x-1))^{\frac{1}{\cos x-1}})^{-tg\frac{x}{2}\cdot\cos x} = e^0 = 1.$$

Problem 3. Let $f(x) = x+\sin x$ and $g(x) = e^{-x} - x^3$. Find the greatest integer solution of the inequality $f(g(f(x))) > f(g(f(\sqrt{2015})))$.

Solution. *Note that $f'(x) = 1+\cos x$, thus the function $f(x)$ is increasing on each of the intervals $[-\pi+2\pi n, \pi+2\pi n]$, $n\in\mathbb{Z}$, therefore it is an increasing function. On the other hand $g'(x) = -e^{-x} - 3x^2 < 0$, therefore the function $g(x)$ is decreasing. Hence, the given inequality is equivalent to the following inequalities: $g(f(x)) > g(f(\sqrt{2015}))$, $f(x) < f(\sqrt{2015})$, $x < \sqrt{2015}$.*

Hence, the greatest integer solution is $x = 44$.

Problem 4. Evaluate the expression

$$\int_{2-\sqrt{3}}^{2+\sqrt{3}} \frac{3^{3-x}-3^{x-1}}{2^{x-1}+2^{3-x}}dx.$$

Solution. *We have that*

$$\int_{2-\sqrt{3}}^{2+\sqrt{3}} \frac{3^{3-x}-3^{x-1}}{2^{x-1}+2^{3-x}}dx = \int_{2-\sqrt{3}}^{2} \frac{3^{3-x}-3^{x-1}}{2^{x-1}+2^{3-x}}dx + \int_{2}^{2+\sqrt{3}} \frac{3^{3-x}-3^{x-1}}{2^{x-1}+2^{3-x}}dx =$$

$$= \int_{2-\sqrt{3}}^{2} \frac{3^{3-x}-3^{x-1}}{2^{x-1}+2^{3-x}}dx + \int_{2}^{2-\sqrt{3}} \frac{3^{3-(4-t)}-3^{4-t-1}}{2^{4-t-1}+2^{3-(4-t)}}d(4-t) =$$

$$= \int_{2-\sqrt{3}}^{2} \frac{3^{3-x}-3^{x-1}}{2^{x-1}+2^{3-x}}dx + \int_{2-\sqrt{3}}^{2} \frac{3^{t-1}-3^{3-t}}{2^{3-t}+2^{t-1}}dt = 0.$$

Problem 5. Find the solution of the following equation in the set of real numbers

$$\left(\frac{19x-157}{6x^2-31x-17}\right)'' = 0.$$

Solution. *We have that*

$$\frac{19x-157}{6x^2-31x-17}=\frac{9}{2x+1}-\frac{4}{3x-17}.$$

Therefore

$$\left(\frac{19x-157}{6x^2-31x-17}\right)''=(-18(2x+1)^{-2}+12(3x-17)^{-2})'=\frac{72}{(2x+1)^3}-\frac{72}{(3x-17)^3}.$$

Hence, the given equation is equivalent to the following equation $(2x+1)^3=(3x-17)^3$. *Thus, we deduce that* $x=18$.

Problem 6. Evaluate the expression

$$\lim_{n\to\infty}\frac{16}{\pi}\left(arctan\frac{3}{1^2+3\cdot 1+1}+\cdots+arctan\frac{3}{n^2+3\cdot n+1}\right).$$

Solution. *We have that*

$$\tan(\arctan(k+3)-\arctan k)=\frac{\tan(\arctan(k+3))-\tan(\arctan k)}{1+\tan(\arctan(k+3))\tan(\arctan k)}=$$

$$=\frac{3}{1+(k+3)k}=\frac{3}{k^2+3k+1}.$$

Therefore, for $k\in\mathbb{N}$ *we have that*

$$0<\arctan(k+3)-\arctan k<\arctan(k+3)<\frac{\pi}{2}.$$

Hence, we obtain that

$$\arctan(k+3)-\arctan k=\arctan\frac{3}{k^2+3k+1}.$$

Thus, we deduce that

$$\frac{16}{\pi}\left(\arctan\frac{3}{1^2+3\cdot 1+1}+\cdots+\arctan\frac{3}{n^2+3\cdot n+1}\right)=$$

$$=\frac{16}{\pi}(\arctan 4-\arctan 1+\arctan 5-\arctan 2+\arctan 6-$$

$$-\arctan 3+\cdots+\arctan(n+3)-\arctan n)=$$

$$= \frac{16}{\pi}(\arctan(n+3)+\arctan(n+2)+\arctan(n+1)-\arctan 1-\arctan 2-\arctan 3) =$$

$$= \frac{16}{\pi}(\arctan(n+3)+\arctan(n+2)+\arctan(n+1)-\frac{\pi}{4}-(\arctan 2+\arctan 3)) =$$

$$= \frac{16}{\pi}(\arctan(n+3)+\arctan(n+2)+\arctan(n+1)-\pi).$$

Hence

$$\lim_{n\to\infty}\frac{16}{\pi}\left(\arctan\frac{3}{1^2+3\cdot 1+1}+\cdots+\arctan\frac{3}{n^2+3\cdot n+1}\right) =$$

$$= \lim_{n\to\infty}\frac{16}{\pi}\left(\arctan(n+3)+\arctan(n+2)+\arctan(n+1)-\pi\right) = \frac{16}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}-\pi\right) = 8.$$

Problem 7. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function. Given that for any x it holds true $f'(x)\cos x+(\cos x-\sin x)f(x)=0$ and $f(0)=e^{\frac{\pi}{3}}$. Find $f\left(\frac{\pi}{3}\right)$.

Solution. *Note that*

$$(f(x)e^x\cos x)' = (f(x)e^x)'\cos x + f(x)e^x(\cos x)' = (f'(x)e^x+f(x)e^x)\cos x - f(x)e^x\sin x =$$

$$= e^x(f'(x)\cos x+(\cos x-\sin x)f(x)) = 0,$$

thus $f(x)e^x\cos x = C$. *We have that* $f(0)=e^{\frac{\pi}{3}}$. *Therefore* $C=f(0)e^0\cos 0=e^{\frac{\pi}{3}}$. *Hence, we deduce that* $f\left(\frac{\pi}{3}\right)=2$.

Problem 8. Let (u_n) be a sequence of real numbers, such that $u_1=1, u_{n+1}=u_n+\frac{1}{u_n^2}$, $n=1,2,\ldots$ Find $[10u_{9000}]$, where by $[x]$ we denote the integer part of a real number x.

Solution. *We have that* $u_{n+1}^3=u_n^3+3+\frac{3}{u_n^3}+\frac{1}{u_n^6}$, *where* $n=1,2,\ldots$ *Therefore*

$$u_2^3=u_1^3+3+\frac{3}{u_1^3}+\frac{1}{u_1^6}, u_3^3=u_2^3+3+\frac{3}{u_2^3}+\frac{1}{u_2^6},\ldots,u_{n+1}^3=u_n^3+3+\frac{3}{u_n^3}+\frac{1}{u_n^6}.$$

Summing up the following equalities, we obtain that

$$u_{n+1}^3=3n+3\left(\frac{1}{u_1^3}+\cdots+\frac{1}{u_n^3}\right)+\left(\frac{1}{u_1^6}+\cdots+\frac{1}{u_n^6}\right).$$

Hence $u_{n+1}^3 > 3n + \frac{3}{u_1^3} = 3(n+1)$ *or* $u_{n+1} > \sqrt[3]{3(n+1)}$, *thus* $u_{9000} > \sqrt[3]{3 \cdot 9000} = 30$.

On the other hand

$$u_{n+1}^3 = 3n + 3\left(\frac{1}{u_1^3} + \cdots + \frac{1}{u_n^3}\right) + \left(\frac{1}{u_1^6} + \cdots + \frac{1}{u_n^6}\right) <$$

$$< 3(n+1) + 1 + 3\left(\frac{1}{3 \cdot 2} + \cdots + \frac{1}{3 \cdot n}\right) + \left(\frac{1}{3^2 \cdot 2^2} + \cdots + \frac{1}{3^2 \cdot n^2}\right) =$$

$$= 3(n+1) + 1 + \left(\frac{1}{2} + \cdots + \frac{1}{n}\right) + \frac{1}{9}\left(\frac{1}{2^2} + \cdots + \frac{1}{n^2}\right).$$

We have that

$$\frac{1}{2} + \cdots + \frac{1}{n} < \int_1^2 \frac{1}{x} dx + \int_2^3 \frac{1}{x} dx + \cdots + \int_{n-1}^n \frac{1}{x} dx = \int_1^n \frac{1}{x} dx = \ln n$$

and

$$\frac{1}{2^2} + \cdots + \frac{1}{n^2} < \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} =$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n} < 1.$$

Therefore

$$u_{n+1}^3 < 3(n+1) + 1 + \ln n + \frac{1}{9}.$$

We obtain that

$$u_{9000}^3 < 27002 + \ln 9000 < 27002 + \ln 2^{25} < 27027 < \left(30 + \frac{1}{100}\right)^3.$$

Hence $u_{9000} < 30.01$, *thus* $300 < 10u_{9000} < 300.1$. *Then* $[10u_{9000}] = 300$.

Problem 9. Let $f : (0, +\infty) \to (0, 1]$, $g : (0, +\infty) \to (0, 1]$ and g be a non-decreasing function. Given that for any positive x, y it holds true $f(x)f(y) = g(x) f(yf(x))$. Find the number of the solutions of the following inequality $f(x) > g(x)$.

Solution. *Let us prove that if* $x \in (0, +\infty)$, *then* $f(x) \leq g(x)$.

Assume there exists $x_1 > 0$, *such that* $f(x_1) > g(x_1) + a$, *where* $a > 0$.

Consider the following sequence $x_{n+1} = x_n f(x_n)$, *where* $n = 1, 2, \ldots$ *Let us show (using the method of mathematical induction) that* $f(x_n) > g(x_n) + na$.

Indeed, if $n = 1$, *we have that* $f(x_1) > g(x_1) + a$.

Let $k \in \mathbb{N}$ *and* $f(x_k) > g(x_k) + ka$. *We have that* $x_{k+1} = x_k f(x_k) \leq x_k$. *Note that* $f(x_k) f(x_k) = g(x_k) f(x_k f(x_k))$, *hence*

$$f(x_{k+1}) = \frac{f^2(x_k)}{g(x_k)} > \frac{(g(x_k)+ka)^2}{g(x_k)} > g(x_k) + 2ka \geq g(x_k) + (k+1)a \geq g(x_{k+1}) + (k+1)a.$$

Therefore $f(x_n) > g(x_n) + na$, where $n \in \mathbb{N}$, this leads to a contradiction, as $1 \geq f(x_n) > g(x_n) + na > na$. This ends the proof of the statement. Thus, the number of the solutions of the following inequality $f(x) > g(x)$ is equal to 0.

7.4.5 Problem Set 5

Problem 1. Let a be a real number. It is called a "special" number, if the function $y = [x] \cdot \{x\}$ accepts the value equal to a at finitely many points. We denote by $\{x\}$ the fractional part of a real number x and by $[x]$ the integer part. Find the number of all "special" numbers.

Solution. *Let a be any real number, one can choose infinitely many n integer numbers, such that $0 \leq \frac{a}{n} < 1$.*

If $x = n + \frac{a}{n}$, we have that

$$y = \left\{n + \frac{a}{n}\right\}\left[n + \frac{a}{n}\right] = \left\{\frac{a}{n}\right\} \cdot n = \frac{a}{n} \cdot n = a.$$

Therefore, function y accepts the value equal to any integer number at infinitely many points. Hence, the number of "special" numbers is equal to 0.

Problem 2. Let us denote by $\{x\}$ the fractional part of a real number x. Evaluate the expression

$$\lim_{n\to\infty}\{(2+\sqrt{3})^n\}.$$

Solution. *Consider the following sequence $a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$. We have that $a_1 = 4$, $a_2 = 14$ and $a_{n+2} = (2+\sqrt{3})^{n+2} + (2-\sqrt{3})^{n+2} = ((2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1})(2+\sqrt{3}+2-\sqrt{3}) - ((2+\sqrt{3})^n + (2-\sqrt{3})^n) = 4a_{n+1} - a_n$. Therefore, by mathematical induction one can show that $a_{n+1} > a_n$, $n = 1, 2, \ldots$ and $a_n \in \mathbb{N}$, $n = 1, 2, \ldots$ Hence, $\{(2+\sqrt{3})^n\} + \{(2-\sqrt{3})^n\} = 1$ and $0 < 2-\sqrt{3} < 1$. Thus, $\{(2-\sqrt{3})^n\} = (2-\sqrt{3})^n$ and*

$$\lim_{n\to\infty}\{(2-\sqrt{3})^n\} = 0.$$

Therefore,

$$\lim_{n\to\infty}\{(2+\sqrt{3})^n\} = 1.$$

Problem 3. Find the smallest positive integer a, such that any $x \in \left[\frac{\pi}{2}, \pi\right]$ is a solution of the following inequality

$$\frac{a+5-(a^2-4)\cos x}{a^2+2a+2.5-0.5\cos 2x} < 1.$$

Solution. *Note that* $a^2+2a+2.5-0.5\cos 2x = (a+1)^2+1.5-0.5\cos 2x > 0$, *thus the given inequality is equivalent to the following inequality* $\cos^2 x-(a^2-4)\cos x+2-a-a^2 < 0$.

If $x \in \left[\frac{\pi}{2}, \pi\right]$, *then* $\cos x \in [-1, 0]$.

Quadratic function $f(t) = t^2-(a^2-4)t+2-a-a^2$ *accepts its greatest value on* $[-1,0]$ *at* $t=-1$ *or* $t=0$. *Therefore, any number that belongs to* $\left[\frac{\pi}{2}, \pi\right]$ *is a solution of the given inequality, iff*

$$\begin{cases} f(-1) < 0 \\ f(0) < 0 \end{cases}$$

$$\begin{cases} -1-a < 0 \\ 2-a-a^2 < 0. \end{cases}$$

Hence, the smallest value of a is equal to 2.

Problem 4. Evaluate the expression

$$\lim_{x\to 0} \frac{24\arcsin(\sqrt{1+x}-1)}{\arcsin(\sqrt[3]{1+x}-1)}.$$

Solution. *We have that*

$$\lim_{x\to 0} \frac{24\arcsin(\sqrt{1+x}-1)}{\arcsin(\sqrt[3]{1+x}-1)} = \lim_{x\to 0}\left(\frac{24\arcsin(\sqrt{1+x}-1)}{\sqrt{1+x}-1} \cdot \frac{\sqrt[3]{1+x}-1}{\arcsin(\sqrt[3]{1+x}-1)} \cdot \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}\right) =$$

$$= 24 \cdot 1 \cdot 1 \cdot \lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1} = 24 \lim_{x\to 0} \frac{(\sqrt[3]{1+x})^2+\sqrt[3]{1+x}+1}{\sqrt{1+x}+1} = 24 \cdot \frac{3}{2} = 36.$$

Problem 5. Find the sum of all such numbers a that for any of them the equation

$$|||||x-5|-1|-1|-1|-1| = x-a$$

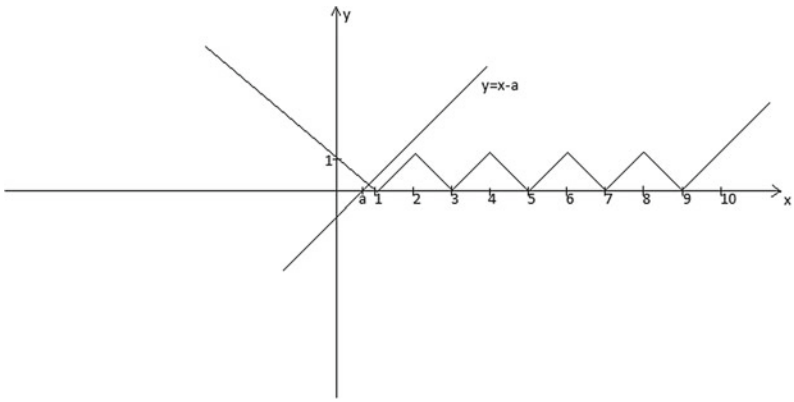
has infinitely many solutions.

Solution. *In the figure below is pictured the graph of the functions*

$$y = |||||x-5|-1|-1|-1|-1|$$

and

$$y = x-a.$$



Thus, the given equality has infinitely many solutions for any of the following values $a=1$, $a=3$, $a=5$, $a=7$, $a=9$. Hence, the sum of all values is equal to 25.

Problem 6. Evaluate the expression

$$\int_{0.5-\sqrt{3}}^{0.5+\sqrt{3}} \frac{\sqrt{3}\cdot 3^x}{3^x+\sqrt{3}}dx.$$

Solution. *Let*

$$\int_{0.5-\sqrt{3}}^{0.5+\sqrt{3}} \frac{\sqrt{3}\cdot 3^x}{3^x+\sqrt{3}}dx = a,$$

we make the following substitution $x=1-t$, thus

$$\int_{0.5+\sqrt{3}}^{0.5-\sqrt{3}} \frac{\sqrt{3}\cdot 3^{1-t}}{3^{1-t}+\sqrt{3}}d(1-t) = a.$$

Hence

$$\int_{0.5-\sqrt{3}}^{0.5+\sqrt{3}} \frac{3}{\sqrt{3}+3^t}dt = a.$$

Therefore

$$2a = \int_{0.5-\sqrt{3}}^{0.5+\sqrt{3}} \frac{\sqrt{3}\cdot 3^x}{\sqrt{3}+3^x}dx + \int_{0.5-\sqrt{3}}^{0.5+\sqrt{3}} \frac{3}{\sqrt{3}+3^x}dx = \int_{0.5-\sqrt{3}}^{0.5+\sqrt{3}} \sqrt{3}dx = \sqrt{3}\cdot 2\sqrt{3} = 6.$$

Problem 7. Given that a is such number that the following inequality

$$\log_a(2+\sqrt{x^2+ax+3})\log_3(x^2+ax+4) - \log_a 2 \le 0$$

has a unique solution. Find a^2.

Solution. *Note that if x_0 is a solution, then $-a-x_0$ is also a solution, as $x_0(x_0+a)=(-x_0-a)(-x_0-a+a)$. Thus, the given inequality has a unique solution, if $x_0=-a-x_0$. Hence $x_0=-\frac{a}{2}$.*

Consider the following function

$$f(x)=\log_a(2+\sqrt{x^2+ax+3})\log_3(x^2+ax+4)-\log_a 2.$$

We have that $f(-\frac{a}{2})\leq 0$. If $f(-\dfrac{a}{2})<0$, then as $f(x)$ is a continuous function and functions x^2+ax+3, x^2+ax+4 accept their minimum values at $x=-\frac{a}{2}$, then $D(f)=(-\infty,+\infty)$.

There exists $\varepsilon>0$, such that for $x\in\left(-\frac{a}{2}-\varepsilon,-\frac{a}{2}+\varepsilon\right)$ it follows $f(x)<0$. This leads to a contradiction. Therefore, $f(-\frac{a}{2})=0$.

$$\log_a\left(2+\sqrt{3-\frac{a^2}{4}}\right)\log_3\left(4-\frac{a^2}{4}\right)-\log_a 2=0.$$

$$\begin{cases}\log_2\left(2+\sqrt{3-\frac{a^2}{4}}\right)\log_3\left(4-\frac{a^2}{4}\right)-1=0,\\ a>0,\ \ a\neq 1.\end{cases}$$

Note that the following function

$$g(x)=\log_2\left(2+\sqrt{3-\frac{x^2}{4}}\right)\log_3\left(4-\frac{x^2}{4}\right)-1$$

is decreasing on $[0,2\sqrt{3}]$ and $g(2\sqrt{2})=0$. Thus, $a=2\sqrt{2}$ is the unique solution of the system above.

Let us verify that for $a=2\sqrt{2}$ the given inequality has a unique solution.

$$\log_{2\sqrt{2}}(2+\sqrt{x^2+2\sqrt{2}x+3})\log_3(x^2+2\sqrt{2}x+4)-\log_{2\sqrt{2}}2\leq 0.$$

$$\log_2(2+\sqrt{(x+\sqrt{2})^2+1})\log_3((x+\sqrt{2})^2+2)\leq 1,$$

the unique solution of the last inequality is $x=-\sqrt{2}$. As, if $x\neq-\sqrt{2}$, then

$$\log_2(2+\sqrt{(x+\sqrt{2})^2+1})\log_3((x+\sqrt{2})^2+2)>\log_2 3\cdot\log_3 2=1.$$

This ends the proof.

Problem 8. Let $u_0=0.001$, $u_{n+1}=u_n(1-u_n)$, $n=0,1,\ldots$. We denote by $[x]$ the integer part of a real number x. Find $[2000u_{1000}]$.

Solution. *We have that* $u_{n+1} = u_n(1-u_n)$, *then*

$$\frac{1}{u_{n+1}} = \frac{1}{u_n(1-u_n)} = \frac{1}{u_n} + \frac{1}{1-u_n}.$$

Therefore

$$\frac{1}{u_1} = \frac{1}{u_0} + \frac{1}{1-u_0},\ \frac{1}{u_2} = \frac{1}{u_1} + \frac{1}{1-u_1}, \dots, \frac{1}{u_{1000}} = \frac{1}{u_{999}} + \frac{1}{1-u_{999}}.$$

Summing up these equations, we obtain

$$\frac{1}{u_{1000}} = \frac{1}{u_0} + \frac{1}{1-u_0} + \cdots + \frac{1}{1-u_{999}} =$$

$$= 10^3 + \frac{1}{1-u_0} + \cdots + \frac{1}{1-u_{999}} > 10^3 + 1 + \cdots + 1 = 2000.$$

Hence $u_{1000} < \dfrac{1}{2000}$.

Here, we have used the following inequality $0 < u_n < 1$, *which can be proved in the following way:*

$$u_{n+1} = u_n(1-u_n) \le \left(\frac{u_n + (1-u_n)}{2}\right)^2 = \frac{1}{4} < 1,\ \ (n = 0, 1, 2, \dots)$$

If $u_n \le 0$, *then using that* $u_n = u_{n-1}(1-u_{n-1})$, *we deduce* $u_{n-1} \le 0$. *In the similar way, one can deduce that* $u_0 \le 0$. *This leads to a contradiction, thus* $u_n > 0$.

Therefore, $0 < 2000u_{1000} < 1$. *Hence,* $[2000u_{1000}] = 0$.

Problem 9. Let $f : (0,+\infty) \to (0,1]$, $g : (0,+\infty) \to (0,1]$. Given that f is non-increasing function, g is non-decreasing and $f(x)f(y) = g(x)f(yf(x))$ holds true for any $x, y > 0$. Find the number of possible values of $\dfrac{f(2015)}{g(2015)}$.

Solution. *Let us show that if* $x \in (0,+\infty)$, *then* $f(x) \le g(x)$.

Assume there exists $x_1 > 0$, *such that* $f(x_1) > g(x_1) + a$, *where* $a > 0$.

Consider the following sequence $x_{n+1} = x_n f(x_n)$, *where* $n = 1, 2, \dots$. *Let us prove (using the mathematical induction) that* $f(x_n) > g(x_n) + na$.

For $n = 1$, *we have that* $f(x_1) > g(x_1) + a$.

Let $k \in \mathbb{N}$ *and* $f(x_k) > g(x_k) + ka$. *We have that* $x_{k+1} = x_k f(x_k) \le x_k$. *Note that* $f(x_k)f(x_k) = g(x_k)f(x_k f(x_k))$, *thus*

$$f(x_{k+1}) = \frac{f^2(x_k)}{g(x_k)} > \frac{(g(x_k)+ka)^2}{g(x_k)} > g(x_k) + 2ka \ge g(x_k) + (k+1)a \ge g(x_{k+1}) + (k+1)a.$$

Hence, $f(x_n) > g(x_n) + na$*, for* $n \in \mathbb{N}$*. This leads to a contradiction, as* $1 \geq f(x_n) > g(x_n) + na > na$*. This ends the proof of the statement.*

Let us now prove that for $x \in (0, \infty)$ *it holds true* $f(x) = g(x)$*.*

Assume there exists $x_0 > 0$*, such that* $f(x_0) \neq g(x_0)$*, then* $f(x_0) < g(x_0)$*. Therefore,*

$$f(x_0) f\left(\frac{x_0}{f(x_0)}\right) = g(x_0) f(x_0),$$

$$f\left(\frac{x_0}{f(x_0)}\right) = g(x_0) > f(x_0).$$

We have that f *is non-increasing function, thus from the last inequality we deduce that* $\frac{x_0}{f(x_0)} < x_0$*. This leads to a contraction.*

Hence, we obtain that $f(x) = g(x)$*, for* $x \in (0, \infty)$*.*

7.4.6 Problem Set 6

Problem 1. Given that

$$f(x) = \frac{1}{x(x+1)} + \frac{2}{(x+1)(x+2)} + \cdots + \frac{999}{(x+999)(x+1000)}$$

and

$$g(x) = \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+999}.$$

Evaluate the following expression $f(-1001) - g(-1001)$.

Solution. *We have that*

$$f(x) = \frac{1}{x} - \frac{1}{x+1} + 2\left(\frac{1}{x+1} - \frac{1}{x+2}\right) + \cdots + 999\left(\frac{1}{x+999} - \frac{1}{x+1000}\right) =$$

$$= \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+999} - \frac{999}{x+1000} = g(x) - \frac{999}{x+1000},$$

thus

$$f(-1001) - g(-1001) = 999.$$

Problem 2. Evaluate the expression

$$\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt[3]{1+3x}}{1 - \cos x}.$$

Solution. *We have that*

$$a-b=\frac{a^6-b^6}{a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5},$$

hence

$$\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt[3]{1+3x}}{1-\cos x}=\frac{1}{6}\lim_{x\to 0}\frac{(1+2x)^3-(1+3x)^2}{1-\cos x}=$$

$$=\frac{1}{6}\lim_{x\to 0}\frac{8x^3+3x^2}{2\sin^2\frac{x}{2}}=\frac{1}{6}\lim_{x\to 0}\frac{2(8x+3)}{\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2}=1.$$

Problem 3. Given that

$$f(x)=\frac{x^2}{x^2-100x+5000}.$$

Evaluate the expression

$$f(1)+f(2)+\cdots+f(100).$$

Solution. *Note that*

$$f(x)=\frac{x^2}{x^2-100x+5000}=\frac{2x^2}{x^2+(x-100)^2},$$

thus

$$f(x)+f(100-x)=\frac{2x^2}{x^2+(x-100)^2}+\frac{2(100-x)^2}{(100-x)^2+x^2}=2.$$

We have that

$$f(1)+f(2)+\cdots+f(100)=(f(1)+f(99))+\cdots+(f(49)+f(51))+f(50)+f(100)=$$

$$=98+1+2=101.$$

Problem 4. Let

$$A=\pi^2\int_0^1\frac{\sin(\pi x)}{1+\sin(\pi x)}dx,$$

and

$$B=\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx.$$

Find the value of the expression $\dfrac{A}{B}$.

Solution. *We have that*
$$A = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx,$$

hence

$$A - B = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x} dx = \int_{\pi}^{0} \frac{t\sin(\pi - t)}{1+\sin(\pi - t)} d(\pi - t) = B.$$

Thus $\frac{A}{B} = 2$.

Problem 5. Given that a is such a number that any number belonging to $[-1,3]$ is a value of the following function

$$f(x) = \frac{a + 2\sin x + 1}{\cos^2 x + a^2 + |a| + 1}.$$

Find a.

Solution. *We have that $D(f) = \mathbb{R}$ and $f(x)$ is continuous function on $\mathbb{R}$. Thus, by mean value theorem, it is sufficient to require that the numbers -1 and 3 are values of $f(x)$ function. Consider the following equation*

$$\frac{a + 2\sin x + 1}{\cos^2 x + a^2 + |a| + 1} = -1.$$

The last equation is equivalent to the following equation

$$\sin^2 x - 2\sin x - a^2 - |a| - a - 3 = 0,$$

or
$$(\sin x - 1)^2 = a^2 + (|a| + a) + 4.$$

The last equation has a solution, only in the case, when $a = 0$.
If $a = 0$, then the following equation

$$\frac{2\sin x + 1}{\cos^2 x + 1} = 3,$$

also has a solution. Therefore, we obtain that the only possible value of a is 0.

Problem 6. Given that the line, defined by the following equation $y = kx + b$, is tangent at two points to the graph of the following function

$$f(x) = x^4 - 6x^3 + 13x^2 - 6x + 1.$$

Find k.

Solution. *Let x_1 and x_2 be the abscissas of the given points (points of tangency). Then, x_1 and x_2 are double roots for the polynomial $f(x) - kx - b$. Therefore*

$$f(x) - kx - b = (x - x_1)^2 (x - x_2)^2.$$

We have that $x_1 + x_2 = 3$, $x_1^2 + x_2^2 + 4x_1x_3 = 13$, $k + 6 = 2x_1x_2^2 + 2x_2x_1^2$. Thus $k = 6$.

Problem 7. Let a be a real number, such that the equation

$$|2x + a| - |x - a| + |x - 2a| = -x^2 - ax - 1.25a^2 + 5a - 4$$

has only one real solution. Find a.

Solution. *We have that*

$$x^2 + ax + 1.25a^2 - 5a + 4 + |2x + a| - |x - a| + |x - 2a| = 0.$$

Denote by $g(x) = |2x + a| - |x - a| + |x - 2a|$. Note that the functions $f(x) = x^2 + ax + 1.25a^2 - 5a + 4$ and $f(x) + g(x)$ are increasing on $\left[-\frac{a}{2}, +\infty\right)$ and are decreasing on $\left(-\infty, -\frac{a}{2}\right]$. Indeed,

$$g(x) = \begin{cases} kx + b, \text{ if } x \geq -\frac{a}{2}, \text{where } k \geq 0, \\ lx + m, \text{ if } x < -\frac{a}{2}, \text{where } l \leq 0. \end{cases}$$

On the other hand, we have that

$$\lim_{x \to \infty} (f(x) + g(x)) = +\infty.$$

Hence, if the equation $f(x) + g(x)$ has only one root, it should be $x = -\frac{a}{2}$. We deduce that $a^2 - 5a + 4 + |a| = 0$, thus $a = 2$.

For $a = 2$, we obtain that

$$x^2 + 2x - 1 + |2x + 2| - |x - 2| + |x - 4| = 0.$$

The unique root of the last equation is -1.

Problem 8. Let (u_n) be an increasing sequence of positive integers, such that

$$u_1^3 + \cdots + u_n^3 = (u_1 + \cdots + u_n)^2, \ n = 1, 2, \ldots$$

Find the number of possible values of u_{2015}.

Solution. *Note that*

$$(u_1+\cdots+u_n)^2=u_n^2+2u_n(u_1+\cdots+u_{n-1})+(u_1+\cdots+u_{n-1})^2=u_n^2+2u_n(u_1+\cdots+u_{n-1})+$$

$$+u_{n-1}^2+2u_{n-1}(u_1+\cdots+u_{n-2})+(u_1+\cdots+u_{n-2})^2=\cdots=u_n^2+2u_n(u_1+\cdots+u_{n-1})+$$

$$+u_{n-1}^2+2u_{n-1}(u_1+\cdots+u_{n-2})+\cdots+u_2^2+2u_2u_1+u_1^2.$$

Therefore, given equation one can write in the following way

$$0=2u_n\Big(\frac{u_n(u_n-1)}{2}-(u_1+\cdots+u_{n-1})\Big)+2u_{n-1}\Big(\frac{u_{n-1}(u_{n-1}-1)}{2}-(u_1+\cdots+u_{n-2})\Big)+\dots$$

$$+2u_2\Big(\frac{u_2(u_2-1)}{2}-u_1\Big)+u_1^2(u_1-1). \tag{7.91}$$

As $u_1>u_{i-1}$, then $u_i\geq u_{i-1}+1$. Therefore, for $i=2,\dots,n$ we have that

$$\frac{u_i(u_i-1)}{2}=1+2+\cdots+(u_i-1)\geq u_1+\cdots+u_{i-1}.$$

From (7.91), we obtain that

$$u_1-1=0,\ \frac{u_2(u_2-1)}{2}-u_1=0,\dots,\ \frac{u_n(u_n-1)}{2}-(u_1+\cdots+u_{n-1})=0.$$

Hence, we deduce that $u_1=1,\ u_i=u_{i-1}+1$, for $i=2,\dots,n$. Therefore, $u_i=i$ for any $i=1,\dots,n$.

Problem 9. Let $p(x)$ and $q(x)$ be polynomials with real coefficients. Given that $q(x)$ is increasing polynomial function, $p(x)$ is a polynomial of even degree and that the equation $p(x)-q(2x)=0$ has two real roots. Let $f:\mathbb{R}\to\mathbb{R}$ and $p(f(x+y))=q(f(x)+f(y))$, for any real numbers x,y. Find the number of all such functions f.

Solution. *We have that $q(x)$ is increasing, hence $q'(x)$ does not accept negative values. Therefore, $q'(x)$ is a polynomial of even degree. Thus, $q(x)$ is a polynomial of odd degree.*

Take $y=0$, then

$$p(f(x))=q(f(x)+f(0)). \tag{7.92}$$

As the polynomials $p(x)$ and $q(x)$ are of different degrees, then the polynomial $p(x)-q(x+f(0))$ is not equal to 0. By (7.92), we deduce that $E(f)$ is a finite set.

Let $E(f)=\{a_1,\dots,a_n\}$, where $a_1<\cdots<a_n$ and $f(x_i)=a_i$, for $i=1,\dots,n$.

If $n\geq 2$, then

$$p(f(2x_1))=q(f(x_1)+f(x_1))=q(2a_1)<q(a_1+a_2)=p(f(x_1+x_2))<$$

$$q(2a_2)=p(f(2x_2))<\cdots<q(2a_n)=p(f(2x_n)).$$

This leads to a contradiction, as $|E(p(f(x)))| \leq |E(f)| = n$.

Hence, $n = 1, f(x) = c$, $p(c) = q(2c)$. *We obtain that* c *is a root of* $p(x) - q(2x) = 0$. *Therefore,* c *has two values* c_1 *and* c_2. *Obviously, for any of the functions* $f(x) = c_1$ *and* $f(x) = c_2$*, the following equality holds true* $p(f(x+y)) = q(f(x) + f(y))$ *for any* x, y.

7.4.7 Problem Set 7

Problem 1. Given that

$$\lim_{n\to\infty} \left(\frac{(n+1)^4}{n^4+1}\right)^n = e^a.$$

Find a.

Solution. *We have that*

$$\lim_{n\to\infty} \left(\frac{(n+1)^4}{n^4+1}\right)^n = \lim_{n\to\infty} \left(\left(1 + \frac{1}{\dfrac{n^4+1}{4n^3+6n^2+4n}}\right)^{\dfrac{n^4+1}{4n^3+6n^2+4n}}\right)^{\dfrac{4n^4+6n^3+4n^2}{n^4+1}}$$

$$= e^{\lim\limits_{n\to\infty} \dfrac{4+\dfrac{6}{n}+\dfrac{4}{n^2}}{1+\dfrac{1}{n^4}}} = e^4.$$

Therefore, we obtain that $a = 4$.

Problem 2. Find the number of elements of the range of the following function

$$f(x) = \arctan x + \arctan \frac{1-x}{1+x}.$$

Solution. *We have that*

$$\tan\left(\arctan x + \arctan \frac{1-x}{1+x}\right) = \frac{x + \dfrac{1-x}{1+x}}{1 - x \cdot \dfrac{1-x}{1+x}} = 1.$$

If $x < -1$, $\arctan x \in \left(-\frac{\pi}{2}, 0\right)$, $\arctan\frac{1-x}{1+x} \in \left(-\frac{\pi}{2}, 0\right)$, *then*

$$\arctan x + \arctan\frac{1-x}{1+x} = -\frac{3\pi}{4}.$$

If $-1 < x \leq 1$, $\pi > \arctan x + \arctan\frac{1-x}{1+x} > -\frac{\pi}{2} + 0$, *then*

$$\arctan x + \arctan\frac{1-x}{1+x} = \frac{\pi}{4}.$$

In a similar way, we deduce that if $x > 1$, *then*

$$\arctan x + \arctan\frac{1-x}{1+x} = \frac{\pi}{4}.$$

Therefore, the number of elements of the range of function f *is equal to 2.*

Problem 3. Let $y = 3x - 5$ be the equation of the tangent line at point x_0 of function $f(x)$. Find the value of the first derivative of the function $\frac{f(x)}{x} + 6f(x) + \frac{5}{x} - 2x + 7$ at point x_0.

Solution. *According to the assumptions of the problem, we have that* $f'(x_0) = 3$, $f(x_0) - f'(x_0)x_0 = -5$. *Note that*

$$\left(\frac{f(x)}{x} + 6f(x) + \frac{5}{x} - 2x + 7\right)' = \frac{f'(x)x - f(x)}{x^2} + 6f'(x) - \frac{5}{x^2} - 2.$$

We obtain that

$$\frac{f'(x_0)x_0 - f(x_0)}{x_0^2} + 6f'(x_0) - \frac{5}{x_0^2} - 2 = \frac{3}{x_0} - \frac{3x_0 - 5}{x_0^2} + 18 - \frac{5}{x_0^2} - 2 = 16.$$

Problem 4. Let f be a continuous function defined on $[0,1]$. Given that the following numbers $\int_0^1 (f(x))^{2014}dx$, $\int_0^1 (f(x))^{2015}dx$, $\int_0^1 (f(x))^{2016}dx$ make an arithmetic progression. Find the value of the following expression

$$\int_0^1 (f(x))^2 + (1 - f(x))^2 dx.$$

Solution. *We have that*

$$\int_0^1 (f(x))^{2014}dx + \int_0^1 (f(x))^{2016}dx = 2\int_0^1 (f(x))^{2015}dx.$$

Thus, we deduce that

$$\int_0^1 (f(x))^{2014}(1-f(x))^2 dx = 0.$$

On the other hand, the function $(f(x))^{2014}(1-f(x))^2$ *is continuous. Hence,* $(f(x))^{2014}(1-f(x))^2 = 0$. *Therefore, for any real number* x *we have obtained that* $f(x)(1-f(x)) = 0$. *Thus, it follows that*

$$\int_0^1 (f(x))^2 + (1-f(x))^2 dx = \int_0^1 1dx = 1.$$

Problem 5. Let

$$f(x) = \frac{a+\sin x - \cos x}{a+\sin x + \cos x}.$$

How many values of a are there, such that $f(-x) = -f(x)$, for any $x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$?

Solution. *According to the assumptions of the problem, we have that* $f\left(-\frac{\pi}{4}\right) = -f\left(\frac{\pi}{4}\right)$. *Thus, we obtain that* $\frac{a-\sqrt{2}}{a} = -\frac{a}{a+\sqrt{2}}$. *Hence,* $a = 1$ *or* $a = -1$.

If $a = -1$, *then the function*

$$f(x) = \frac{-1+\sin x - \cos x}{-1+\sin x + \cos x}$$

is not defined at point $x = 0$. *Thus, in the case, when* $a = -1$ *the assumtion of the problem does not hold true.*

If $a = 1$, *then the function*

$$f(x) = \frac{1+\sin x - \cos x}{1+\sin x + \cos x} = \frac{2\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)}{2\cos\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)}$$

is defined for any point $x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ *and* $f(x) = \tan\frac{x}{2}$. *Therefore,* $f(-x) = -f(x)$.

Problem 6. Let

$$f(x) = \frac{6x+21+28\sqrt{3x+2}}{9x+18-12\sqrt{3x+2}}.$$

Find the value of the following expression $f(f(1)) + f(f(2)) + \cdots + f(f(40))$.

Solution. *Note that the following*

$$y = \frac{6x+21+28\sqrt{3x+2}}{9x+18-12\sqrt{3x+2}}$$

can be rewritten, as

$$3y+2=\frac{(2\sqrt{3x+2}+5)^2}{(\sqrt{3x+2}-2)^2}.$$

Let $D(F)=\left(\frac{2}{3},\infty\right)$, $F:x\to y$ *and*

$$\sqrt{3y+2}=\frac{2\sqrt{3x+2}+5}{\sqrt{3x+2}-2}. \tag{7.93}$$

Note that if $x>\frac{2}{3}$, *then*

$$\frac{2\sqrt{3x+2}+5}{\sqrt{3x+2}-2}>2.$$

Hence, $y>\frac{2}{3}$. *Therefore,* $E(F)=\left(\frac{2}{3},\infty\right)$.

Let us now find the value of the function F^{-1}. *We have that*

$$\sqrt{3x+2}=\frac{2\sqrt{3y+2}+5}{\sqrt{3y+2}-2}.$$

Thus, $F^{-1}=F$, *as*

$$\sqrt{3y+2}=\frac{2\sqrt{3x+2}+5}{\sqrt{3x+2}-2}.$$

Note that if $x>\frac{2}{3}$, *then*

$$y=\frac{1}{3}\left(\left(\frac{2\sqrt{3x+2}+5}{\sqrt{3x+2}-2}\right)^2-2\right)=f(x).$$

Therefore, if $x>\frac{2}{3}$, *then* $f(x)=F(x)$.

Hence, we deduce that $f(f(1))+f(f(2))+\cdots+f(f(40))=F(F(1))+F(F(2))+\cdots+F(F(40))=1+\cdots+40=820$.

Problem 7. Find the value of the following expression

$$\lim_{n\to\infty}\left(16n-\frac{64}{\pi}\cdot\arctan\frac{1^2+3\cdot1-2}{1^2+3\cdot1+4}-\cdots-\frac{64}{\pi}\cdot\arctan\frac{n^2+3\cdot n-2}{n^2+3\cdot n+4}\right).$$

Solution. *If* $x>0$, *then* $\arctan x+\arctan\frac{1-x}{1+x}\in[-\frac{\pi}{2},\pi]$. *On the other hand, we have that*

$$\tan(\arctan x + \arctan \frac{1-x}{1+x}) = \frac{x + \dfrac{1-x}{1+x}}{1 - x \cdot \dfrac{1-x}{1+x}} = 1.$$

Hence, we deduce that

$$\arctan x + \arctan \frac{1-x}{1+x} = \frac{\pi}{4}. \tag{7.94}$$

If $k \in \mathbb{N}$, then from (7.94) it follows that

$$\arctan \frac{3}{k^2+3k+1} = \frac{\pi}{4} - \arctan \frac{k^2+3k-2}{k^2+3k+4}.$$

Therefore,

$$\lim_{n\to\infty}\left(16n - \frac{64}{\pi}\cdot\arctan\frac{1^2+3\cdot 1-2}{1^2+3\cdot 1+4} - \cdots - \frac{64}{\pi}\cdot\arctan\frac{n^2+3\cdot n-2}{n^2+3\cdot n+4}\right) =$$

$$= \frac{64}{\pi}\lim_{n\to\infty}\left(\arctan\frac{3}{1^2+3\cdot 1+1} + \cdots + \arctan\frac{3}{n^2+3\cdot n+1}\right).$$

We have that

$$\tan(\arctan(k+3) - \arctan k) = \frac{\tan(\arctan(k+3)) - \tan(\arctan k)}{1 + \tan(\arctan(k+3))\tan(\arctan k)} =$$

$$= \frac{3}{1+(k+3)k} = \frac{3}{k^2+3k+1}.$$

Therefore, for $k \in \mathbb{N}$, it follows that

$$0 < \arctan(k+3) - \arctan k < \arctan(k+3) < \frac{\pi}{2}.$$

We obtain that

$$\arctan(k+3) - \arctan k = \arctan\frac{3}{k^2+3k+1}.$$

Thus, we deduce that

$$\frac{64}{\pi}\left(\arctan\frac{3}{1^2+3\cdot 1+1} + \cdots + \arctan\frac{3}{n^2+3\cdot n+1}\right) =$$

$$= \frac{64}{\pi}(\arctan 4 - \arctan 1 + \arctan 5 - \arctan 2 + \arctan 6 - \arctan 3 + \cdots + \arctan(n+3) - \arctan n)$$

$$= \frac{64}{\pi}(\arctan(n+3) + \arctan(n+2) + \arctan(n+1) - \arctan 1 - \arctan 2 - \arctan 3) =$$

$$= \frac{64}{\pi}(\arctan(n+3) + \arctan(n+2) + \arctan(n+1) - \frac{\pi}{4} - (\arctan 2 + \arctan 3)) =$$

$$= \frac{64}{\pi}(\arctan(n+3) + \arctan(n+2) + \arctan(n+1) - \pi).$$

Hence

$$\lim_{n\to\infty} \frac{64}{\pi}\left(\arctan\frac{3}{1^2+3\cdot 1+1} + \cdots + \arctan\frac{3}{n^2+3\cdot n+1}\right) =$$

$$= \lim_{n\to\infty} \frac{64}{\pi}\left(\arctan(n+3) + \arctan(n+2) + \arctan(n+1) - \pi\right) = \frac{64}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} - \pi\right) = 32.$$

Problem 8. Find the sum of all the possible integer values of a, such that for any of them the following equation

$$a^2 + 10|x-1| + \sqrt{3x^2 - 6x + 4} = 17a - 3|2x - 3a| - 20$$

has at least one solution.

Solution. *Let us rewrite the given equation in the following way*

$$10|x-1| + \sqrt{3x^2 - 6x + 4} + 3|2x - 3a| = 17a - a^2 - 20.$$

Consider the following function

$$f(x) = 10|x-1| + \sqrt{3x^2 - 6x + 4} + 3|2x - 3a|.$$

Note that if $x \geq 1$, then the function $f(x) = 10x - 10 + \sqrt{3(x-1)^2 + 1} \pm (6x - 9a)$ is increasing. On the other hand, $f(x)$ is decreasing on $(-\infty, 1]$. Hence, $f(x)$ accepts its minimum value at $x = 1$. Therefore, $f(x) = 17a - a^2 - 20$ has a root, iff

$$f(1) \leq 17a - a^2 - 20.$$

Thus, it follows that

$$1 + 3|3a - 2| \leq 17a - a^2 - 20,$$

or equivalently

$$a^2 + 21 + 3|3a - 2| - 17a \leq 0.$$

We obtain that

$$\begin{cases} a \geq \dfrac{2}{3} \\ a^2 - 8a + 15 \leq 0, \end{cases}$$

or

$$\begin{cases} a < \dfrac{2}{3} \\ a^2 - 26a + 27 \leq 0. \end{cases}$$

Hence, we deduce that $a \in \{3, 4, 5\}$.

Problem 9. Find the number of all polynomials $p(x)$ with integer coefficients, such that the inequality $x^2 \leq p(x) \leq x^4 + 1$ holds true for any x.

Solution. *Note that* $p(x) \equiv 0$ *does not satisfy.*

Let for the polynomial $p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ *it holds true* $x^2 \leq p(x) \leq x^4 + 1$, *for any* x, *where* $a_i \in \mathbb{Z}$, $i = 0, 1, \ldots, n$ *and* $a_0 \neq 0$. *Let us prove that* $n \leq 4$. *We proceed by contradiction argument. Assume that* $n \geq 5$, *then if* $x > 0$ *it follows that*

$$\frac{1}{x^{n-2}} \leq a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n} \leq \frac{1}{x^{n-4}} + \frac{1}{x^n}.$$

Thus, when $x \to \infty$, *we obtain that* $0 \leq a_0 \leq 0$, *that is* $a_0 = 0$. *This leads to a contradiction.*

Note that $n \leq 1$ *is not possible, as the set of the solutions of the following inequality* $x^2 - p(x) \leq 0$ *is not the set of real numbers.*

Note that $n = 3$ *is not possible, as the polynomial* $p(x) - x^2$ *can accept negative value.*

Therefore, we deduce that $n = 2$ *or* $n = 4$.

If $n = 2$, *then* $p(x) = a_0 x^2 + a_1 x + a_2$.

We have that $x^2 \leq a_0 x^2 + a_1 x + a_2 \leq x^4 + 1$, *if* $x = 0$, *then* $a_2 = 0$ *or* $a_2 = 1$.

If $a_2 = 0$, *then* $x^2 \leq a_0 x^2 + a_1 x$. *If the solution is real, then* $a_1 = 0$ *and* $a_0 \geq 1$.

Note that $a_0 = 1$ *and* $a_0 = 2$ *satisfy, but* $a_0 \geq 3$ *does not satisfy.*

Hence, we obtain that $p(x) = x^2$ *or* $p(x) = 2x^2$.

If $a_2 = 1$, *then*

$$(a_0 - 1)x^2 + a_1 x + 1 \geq 0, \tag{7.95}$$

and

$$x^4 - a_0 x^2 - a_1 x \geq 0, \tag{7.96}$$

the solutions are real.

Therefore, $a_1 = 0$, *from* (7.95) *we deduce that* $a_0 \geq 1$ *and from* (7.96) *we deduce that* $a_0 \leq 0$. *This leads to a contradiction.*

If $n = 4$, *then* $p(x) = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4$, $a_4 = 0$ *or* $a_4 = 1$.

From the following condition $x^2 \leq a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 \leq x^4 + 1$, *it follows that* $a_0 = 1$, $a_1 = 0$. *Thus,*

$$x^2 \le x^4 + a_2x^2 + a_3x + a_4 \le x^4 + 1, \;\; x \in \mathbb{R}.$$

If $a_4 = 0$*, then* $0 \le x(x^3 + (a_2 - 1)x + a_3)$*,* $x \in \mathbb{R}$*, we obtain that* $a_3 = 0$ *and* $a_2 \ge 1$*. In this case, the solution of the following* $a_2x^2 \le 1$ *cannot be the set of real numbers.*

If $a_4 = 1$*, then* $a_2x^2 + a_3x \le 0$*,* $x \in \mathbb{R}$*, thus* $a_3 = 0$ *and* $a_2 \le 0$*. Therefore,*

$$x^4 + (a_2 - 1)x^2 + 1 \le 0, \;\; x \in \mathbb{R}.$$

Hence, $a_2 = 0$ *or* $a_2 = -1$*. We deduce that* $p(x) = x^4 + 1$ *or* $p(x) = x^4 - x^2 + 1$*. Finally, we obtain that the number of such* $p(x)$ *polynomials is equal to 4.*

7.4.8 Problem Set 8

Problem 1. Evaluate the expression

$$\lim_{x\to 0} \frac{27\pi(\sqrt[3]{10x+27}-3)}{\sin(\pi x)}.$$

Solution. *We have that*

$$\lim_{x\to 0} \frac{27\pi(\sqrt[3]{10x+27}-3)}{\sin(\pi x)} = \lim_{x\to 0} \frac{270\pi x}{\sin \pi x((\sqrt[3]{10x+27})^2 + 3\sqrt[2]{10x+27}+9)} =$$

$$\lim_{x\to 0} \frac{270}{\dfrac{\sin \pi x}{\pi x} \cdot ((\sqrt[3]{10x+27})^2 + 3\sqrt[2]{10x+27}+9)} = \frac{270}{1 \cdot 27} = 10$$

Problem 2. At how many points does the function

$$f(x) = \cos x + \cos(\sqrt{2}x)$$

accept the value $f(0)$?

Solution. *Let us solve* $f(x) = f(0)$ *equation.*

$$\cos x + \cos \sqrt{2}x = 2.$$

We have that $\cos \alpha \le 1$*, thus*

$$\begin{cases} \cos x = 1, \\ \cos \sqrt{2}x = 1. \end{cases}$$

Hence,

$$\begin{cases} x = 2\pi n, \\ \sqrt{2}x = 2\pi m, \end{cases}$$

where $m, n \in \mathbb{Z}$.

If $n \neq 0$, *we obtain that* $\sqrt{2} = \frac{m}{n}$. *This leads to a contradiction.*

Therefore, $n = 0$. *Hence, it follows that* $x = 0$ *and* $m = 0$. *This means that function* $f(x)$ *accepts the value* $f(0)$ *only at one point.*

Problem 3. Evaluate the expression

$$\lim_{x \to 0} \frac{\ln(1 + \tan(50x))}{\sin x}.$$

Solution. *We have that*

$$\frac{\ln(1 + \tan(50x))}{\sin x} = \frac{\ln(1 + \tan(50x))}{\tan(50x)} \cdot \frac{\tan(50x)}{50x} \cdot 50 \cdot \frac{x}{\sin x}.$$

Thus, it follows that

$$\lim_{x \to 0} \frac{\ln(1 + \tan(50x))}{\sin x} = 1 \cdot 1 \cdot 50 \cdot 1 = 50.$$

Problem 4. Find the sum of all integer numbers belonging to the range of the function

$$f(x) = \sqrt{16 - x} + \sqrt{9 + x}.$$

Solution. *Note that*

$$f^2(x) = 25 + 2\sqrt{(16 - x)(9 + x)}.$$

Therefore,

$$25 \leq f^2(x) \leq 25 + (16 - x) + (9 + x) = 50.$$

Hence, $5 \leq f^2(x) \leq 5\sqrt{2}$. *On the other hand, we have that* $f(-9) = 5$ *and* $f(3.5) = 5\sqrt{2}$. *We have that function* $f(x)$ *is defined and continuous on* $[-9, 16]$. *Its minimum value is equal to 5, and its maximum value is equal to* $5\sqrt{2}$. *Thus,* $E(f) = [5, 5\sqrt{2}]$. *Integers that belong to this interval are 5, 6, 7. Hence, their sum is equal to 18.*

Problem 5. Evaluate the expression

$$\lim_{x \to 0} \frac{\sin(2\sin(3\sin 4x))}{\arcsin(\arcsin(\arcsin x))}.$$

Solution. *We have that*

$$\frac{\sin(2\sin(3\sin 4x))}{\arcsin(\arcsin(\arcsin x))} = \frac{\sin(2\sin(3\sin 4x))}{2\sin(3\sin 4x)} \cdot \frac{2\sin(3\sin 4x)}{\sin(3\sin 4x)} \cdot$$

$$\frac{\sin(3\sin 4x)}{3\sin 4x} \cdot \frac{3\sin 4x}{4x} \cdot \frac{4x}{\arcsin x} \cdot \frac{\arcsin x}{\arcsin(\arcsin x)} \cdot \frac{\arcsin(\arcsin x)}{\arcsin(\arcsin(\arcsin))}.$$

Thus

$$\lim_{x\to 0} \frac{\sin(2\sin(3\sin 4x))}{\arcsin(\arcsin(\arcsin x))} = 1\cdot 2\cdot 1\cdot 3\cdot 4\cdot 1\cdot 1 = 24.$$

Problem 6. Find the value of the following sum

$$\int_0^1 (x^3+1)^5\,\mathrm{d}x + \int_1^{32} \sqrt[3]{\sqrt[5]{x}-1}\,\mathrm{d}x.$$

Solution. *Let us consider the following function* $f(x) = (x^3+1)^5$*, where* $D(f) = [0,1]$. *The inverse function is the function*

$$y = \sqrt[3]{\sqrt[5]{x}-1},$$

as

$$(\sqrt[5]{y} = x^3+1, \sqrt[3]{\sqrt[5]{y-1}} = x),$$

where $D(y) = [1,32]$. *Therefore,*

$$\int_0^1 (x^3+1)^5 dx + \int_1^{32} \sqrt[3]{\sqrt[5]{x}-1} dx = \int_0^1 f(x)dx + \int_1^{32} f^{-1}(x)dx =$$

$$= \int_0^1 f(x)dx + \int_1^{32} f^{-1}(y)dy = 1\cdot 32 = 32.$$

Problem 7. Let f be a continuous function defined on $[0,1]$. Given that the numbers $\int_0^1 f^{2014}(x)\,\mathrm{d}x$, $\int_0^1 f^{2015}(x)\,\mathrm{d}x$, $\int_0^1 f^{2016}(x)\,\mathrm{d}x$ make a geometric progression. Find the value of the expression

$$\frac{f(0)+100f(0.5)+200f(1)}{f(0.25)}.$$

Solution. *We have that*

$$\left(\int_0^1 f^{2015}(x)dx\right)^2 = \int_0^1 f^{2014}(x)dx \cdot \int_0^1 f^{2016}(x)dx.$$

Note that there exists $x_0 \in [0,1]$, *such that* $f(x_0) \neq 0$.

Indeed, if $f(x) \equiv 0$, *then* $\int_0^1 f^{2014}(x)dx = 0$. *This leads to a contradiction.*

Hence, from the condition $f(x_0) \neq 0$ *and that* $f(x)$ *is continuous we obtain that*

$$\int_0^1 f^{2014}(x)dx \neq 0.$$

Note that

$$\int_0^1 \left(f^{1007}(x) \cdot \frac{\int_0^1 f^{2015}(x)dx}{\int_0^1 f^{2014}(x)dx} - f^{1008}(x)\right)^2 dx =$$

$$= \frac{(\int_0^1 f^{2015}(x)dx)^2}{\int_0^1 f^{2014}(x)dx} - 2 \cdot \frac{(\int_0^1 f^{2015}(x)dx)^2}{\int_0^1 f^{2014}(x)dx} + \int_0^1 f^{2016}(x)dx = 0.$$

Therefore,

$$f^{1007}(x)\frac{\int_0^1 f^{2015}(x)dx}{\int_0^1 f^{2014}(x)dx} \equiv f^{1008}(x).$$

It follows that

$$f(x) \equiv C \neq 0,$$

where

$$C = \frac{\int_0^1 f^{2015}(x)dx}{\int_0^1 f^{2014}(x)dx}.$$

Hence,

$$\frac{f(0) + 100f(0.5) + 200f(1)}{f(0.25)} = 301.$$

Problem 8. Find the smallest value of the function

$$f(x) = \frac{3 - 2x + 3x^2}{\sqrt[3]{(x^3 - 9x^2 + 3x - 3)^2}}.$$

Solution. *Note that*

$$\left(\frac{1-x}{\sqrt[3]{-x^3+9x^2-3x+3}}\right)^3 + \left(\frac{1-x}{\sqrt[3]{-x^3+9x^2-3x+3}}\right)^3 + \left(\frac{1+x}{\sqrt[3]{-x^3+9x^2-3x+3}}\right)^3 = 1.$$

On the other hand, if $a^3 + b^3 + c^3 = 1$, *then* $a^2 + b^2 + c^2 \geq 1$. *Therefore,*

$$\left(\frac{1-x}{\sqrt[3]{-x^3+9x^2-3x+3}}\right)^2 + \left(\frac{1-x}{\sqrt[3]{-x^3+9x^2-3x+3}}\right)^2 + \left(\frac{1+x}{\sqrt[3]{-x^3+9x^2-3x+3}}\right)^2 \geq 1.$$

We obtain that

$$f(x) = \frac{3x^2 - 2x + 3}{\sqrt[3]{(x^3 - 9x^2 + 3x - 3)}^2} \geq 1.$$

On the other hand, $f(1) = \frac{4}{\sqrt[3]{64}} = 1$, *thus the minimum value of function* $f(x)$ *is equal to* 1.

Problem 9. Let $f : \mathbb{R} \to \mathbb{R}$. Given that

$$\lim_{x\to 0}\left(3\frac{f(4x)}{x} - 5\frac{f(2x)}{x} + 2\frac{f(x)}{x}\right) = 4,$$

and

$$\lim_{x\to 0} f(x) = 0.$$

Find the value of

$$\lim_{x\to 0}\frac{f(x)}{x}.$$

Solution. *Let us denote* $\frac{f(x)}{x} - 1 = g(x)$. *Hence, it follows that*

$$3\frac{f(4x)}{x} - 5\frac{f(2x)}{x} + 2\frac{f(x)}{x} = 12g(4x) - 10g(2x) + 2g(x) + 4.$$

Therefore,

$$\lim_{x\to 0}(6g(4x) - 5g(2x) + g(x)) = 0.$$

Let $3g(2x) - g(x) = \varphi(x)$, *then* $6g(4x) - 5g(2x) + g(x) = 2\varphi(2x) - \varphi(x)$. *We have that*

$$\lim_{x\to 0}(2\varphi(2x) - \varphi(x)) = 0.$$

On the other hand,

$$\varphi(x) = 3\left(\frac{f(3x)}{2x} - 1\right) - \left(\frac{f(x)}{x} - 1\right) = 1,5\frac{f(2x)}{x} - \frac{f(x)}{x} - 2.$$

We deduce that

$$\lim_{x\to 0}(x\varphi(x)) = 0.$$

Let us prove that

$$\lim_{x\to 0}\varphi(x) = 0.$$

Let $\varepsilon > 0$ *be an arbitrary number, choose* δ *such that*

$$|2\varphi(2x) - \varphi(x)| < \frac{\varepsilon}{2},$$

when $0 < |x| < \delta$. *Hence,*

$$|\varphi(2x)| < \frac{1}{2}|\varphi(x)| + \frac{\varepsilon}{4}, 0 < |x| < \delta.$$

Thus, if $0 < |x| < \delta$, *then*

$$\left|\varphi(x)\right| < \frac{1}{2}\left|\varphi\left(\frac{x}{2}\right)\right| + \frac{\varepsilon}{2^2},$$

$$\cdots$$

$$\frac{1}{2^{n-1}}\left|\varphi\left(\frac{x}{2^{n-1}}\right)\right| < \frac{1}{2^n}\left|\varphi\left(\frac{x}{2^n}\right)\right| + \frac{\varepsilon}{2^{n+1}}.$$

Summing up these inequalities, we obtain that

$$\left|\varepsilon(x)\right| < \frac{1}{2^n}\left|\varepsilon\left(\frac{x}{2^n}\right)\right| + \frac{\varepsilon}{2} = \frac{1}{|x|}\cdot\left|\frac{x}{2^n}\cdot\varepsilon\left(\frac{x}{2^n}\right)\right| + \frac{\varepsilon}{2}.$$

Fox fixed x, we choose n, such that

$$\frac{1}{|x|}\cdot\left|\frac{x}{2^n}\cdot\varphi\left(\frac{x}{2^n}\right)\right| < \frac{\varepsilon}{2}.$$

Therefore, for any $\varepsilon > 0$, *there exists* δ, *such that* $0 < |x| < \delta, |\varphi(x)| < \varepsilon$. *We have obtained that*

$$\lim_{x\to 0}\varphi(x) = 0.$$

Thus, it follows that

$$\lim_{x\to 0} 3(g(2x) - g(x)) = 0,$$

and

$$\lim_{x\to 0} xg(x) = 0.$$

In a similar way, we deduce that

$$\lim_{x\to 0} g(x) = 0.$$

Hence,

$$\lim_{x\to 0}\frac{f(x)}{x}=1.$$

7.4.9 Problem Set 9

Problem 1. Evaluate the expression

$$\lim_{x\to\frac{\pi}{2}}(\sin(5x))^{\tan x}.$$

Solution. *Denote* $x=\frac{\pi}{2}+t$, *we have that*

$$(\sin 5x)^{\tan x}=(\cos 5t)^{-ctant}=(1+\cos 5t-1)^{\frac{1}{\cos 5t-1}\cdot\frac{1-\cos 5t}{\sin t}\cos t}=$$

$$=\left((1+(\cos 5t-1))^{\frac{1}{\cos 5t-1}}\right)^{\frac{2\sin^2\frac{5t}{2}}{(\frac{5t}{2})^2}\cdot\frac{25}{4}\cdot t\cos t\cdot\frac{t}{\sin t}}.$$

Therefore,

$$\lim_{x\to\frac{\pi}{2}}(\sin 5x)^{\tan x}=\lim_{t\to 0}\left((1+(\cos 5t-1))^{\frac{1}{\cos 5t-1}}\right)^{12.5\cdot\left(\frac{\sin\frac{5t}{2}}{\frac{5t}{2}}\right)^2\cdot\frac{t}{\sin t}\cdot t\cos t}=$$

$$=e^{\lim\limits_{t\to 0}\left(12.5\cdot\frac{2\sin^2\frac{5t}{2}}{(\frac{5t}{2})^2}\cdot t\cos t\cdot\frac{t}{\sin t}\right)}=e^0=1.$$

Problem 2. Find the smallest solution of the inequality

$$\frac{2^x+3^x+4^x}{5^x+6^x}\le\frac{29}{61}.$$

Solution. *Note that the function*

$$f(x)=\frac{2^x+3^x+4^x}{5^x+6^x}=\left(\left(\frac{1}{2}\right)^x+\left(\frac{3}{4}\right)^x+1\right)\cdot\frac{1}{\left(\frac{5}{4}\right)^x+\left(\frac{3}{2}\right)^x},$$

is a product of two decreasing functions, such that both accept only positive values. Thus, it follows that $f(x)$ *is a decreasing function and* $f(2)=\dfrac{29}{61}$*. Therefore, one needs to solve the following inequality* $f(x)\le f(2)$.

As $f(x)$ *is a decreasing function, hence the set of the solutions of this inequality is* $[2,+\infty)$ *and the smallest solution is equal to 2.*

Problem 3. Given that

$$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{2}{n}\right)^{n+2}\cdot\dots\cdot\left(1+\frac{10}{n}\right)^{n+10}\right)=e^a.$$

Find a.

Solution. *Note that*

$$\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^{n+k}=\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{k}}\right)^{\frac{n}{k}\cdot k\cdot\left(1+\frac{k}{n}\right)}=e^k,$$

where $k=1,2,\dots,10$. *Thus, it follows that*

$$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{2}{n}\right)^{n+2}\cdot\dots\cdot\left(1+\frac{10}{n}\right)^{n+10}\right)=e^1\cdot e^2\cdot\dots\cdot e^{10}=e^{55}.$$

Therefore, $a=55$.

Problem 4. Find the number of all possible values of a, such that for any a the function

$$f(x)=\ln(\sqrt{x^2+1}+ax),$$

is an odd function.

Solution. *Let* $f(x)$ *be an odd function.*

If $a\geq 0$, *then* $f(x)$ *is defined on* $[0,+\infty)$. *Therefore,* $D(f)=(-\infty,\infty)$.

If $a<0$, *then* $f(x)$ *is defined on* $(-\infty,0]$. *Therefore,* $D(f)=(-\infty,\infty)$.

We have that $f(-x)=-f(x)$, *for any* x. *Thus, it follows that*

$$(\sqrt{x^2+1}-ax)(\sqrt{x^2+1}+ax)=1,$$

for $x\in\mathbb{R}$. *We deduce that*

$$x^2(1-a^2)=0,$$

for any x. *Hence,* $a=1$ *or* $a=-1$.

One can easily verify that functions $f(x)=\ln(\sqrt{x^2+1}+x)$ *and* $f(x)=\ln(\sqrt{x^2+1}-x)$ *are odd functions. Therefore, the number of all possible values of* a *is equal to 2.*

Problem 5. Find the values of the function

$$f(x)=\cos x+\cos(\sqrt{2}x),$$

such that the function accepts any of those values at finite number of points.

Solution. *Let* $n\in\mathbb{N}$, *we have that*

$$f(2\pi n) = \cos 2\pi n + \cos(2\sqrt{2}\pi n) = 1 + \cos(2\sqrt{2}\pi n).$$

According to Kronecker's theorem, there exists a sequence of positive integers $n_1 < n_2 < \cdots < n_k < \ldots$, *such that*

$$\lim_{k\to+\infty} f(2\pi n_k) = 2.$$

In a similar way, there exists a sequence of positive integers $m_1 < m_2 < \cdots < m_k < \ldots$, *such that*

$$\lim_{k\to+\infty} f(\pi(2m_k+1)) = -2.$$

Note that if $a \in (-2,2)$ *and* A *is an arbitrary number, then there exists a number* $x > A$, *such that* $f(x) = a$.

Let us choose a positive integer l, *such that* $f(\pi(2m_l+1)) < a$ *and* $\pi(2m_l+1) > A$.

Now, let us choose a positive integer S, *such that* $f(2\pi n_s) > a$ *and* $2\pi n_s > \pi(2m_l+1)$.

Given that $f(x)$ *is continuous function, thus equation* $f(x) = a$ *has a solution belonging to* $[\pi(2m_l+1), 2\pi n_s]$. *This ends the proof of the statement.*

Therefore, if $a \in (-2,2)$, *then* $f(x)$ *accepts the value equal to* a *at infinitely many points.*

On the other hand, $E(f) \subseteq [-2,2]$ *and* $f(x)$ *does not accept the value* -2 *and accepts the value 2, when* $\cos x = 1$ *and* $\cos\sqrt{2}x = 1$. *Thus, it follows that* $x = 0$.

Hence, from all values of function $f(x)$, *only the value 2 is accepted at finite number of points.*

Problem 6. Let function $f(x)$ be defined on $\mathbb{R}$ and be non-decreasing. Given that for any x, it holds true

$$f(x^2 - 23x + 144) \geq f^2(x) - 23f(x) + 144.$$

Find $f(2015)$.

Solution. *Note that* $x^2 - 23x + 144 \geq x$, *thus it follows that*

$$f(x^2 - 23x + 144) \leq f(x).$$

On the other hand,

$$f(x^2 - 23x + 144) \geq f^2(x) - 23f(x) + 144.$$

Hence, we deduce that

$$f(x) \geq f^2(x) - 23f(x) + 144.$$

Therefore,

$$(f(x)-12)^2 \leq 0.$$

We obtain that $f(x) = 12$*, for any* x*. Note that this function satisfies the assumptions of the problem. Thus,* $f(2015) = 12$*.*

Problem 7. Find all possible positive values of a, such that the equation

$$x^2|x+a| = 3a+14,$$

has three solutions.

Solution. *If* $a > 0$*, then using the graph of function* $f(x) = |x^3 + ax^2|$*, we deduce that equation* $f(x) = 3a+14$ *has three solutions, if and only if*

$$f\left(-\frac{2a}{3}\right) = 3a+14. \tag{7.97}$$

Hence, one needs to solve equation (7.102). We have that

$$\frac{8a^3}{27} - 28 - 6a = 0.$$

Therefore,

$$\left(\frac{2a}{3} - 3 - 1\right)\left(\frac{4a^2}{9} + 9 + 1 + 2a + \frac{2a}{3} - 3\right) = 0.$$

Thus, it follows that $a = 6$*.*

Problem 8. Let function $f(x)$ be defined on $\mathbb{R}$ and be infinitely differentiable. Given that $f(0) = f(101) \neq f(1)$ and for any x it holds true

$$f'''(x) + 3f''(x)f'(x) + (f'(x))^3 = 0.$$

Find the value of the following expression

$$\frac{|f(1)| + |f(2)| + \cdots + |f(100)|}{|f(1)| + |f(2)| + \cdots + |f(50)|}.$$

Solution. *Consider the following function*

$$g(x) = e^{f(x)}.$$

Note that

$$g'(x) = f'(x)e^{f(x)},$$

and

$$g''(x) = f''(x)e^{f(x)} + (f'(x))^2 e^{f(x)},$$

and

$$g'''(x) = f'''(x)e^{f(x)} + 3f''(x)f'(x)e^{f(x)} + (f'(x))^3 e^{f(x)} = 0.$$

Therefore, $g(x) = ax^2 + bx + c$ *and* $f(x) = \ln(ax^2 + bx + c)$.

We have that $f(0) = f(101)$, *thus the line* $x = 50.5$ *is the axis of symmetry of function* $y = ax^2 + bx + c$. *Hence, it follows that* $f(1) = f(100)$, $f(2) = f(99)$, $\ldots$, $f(50) = f(51)$. *We obtain that*

$$\frac{|f(1)| + |f(2)| + \cdots + |f(100)|}{|f(1)| + |f(2)| + \cdots + |f(50)|} = 2.$$

Problem 9. Let $f : (0, +\infty) \to \mathbb{R}$ and $f(2) \neq 0$. Given that for any positive numbers x, y it holds true

$$xf(x) - yf(y) = xy^2 f\left(\frac{x}{y}\right).$$

Find $\dfrac{4f(8)}{f(2)}$.

Solution. *Let us prove the following properties.*

P1. $f(1) = 0$.

Indeed, if in the assumption of the problem we take $x = 1$, $y = 1$, *then it follows that* $f(1) = 0$.

P2. *If* $x > 0$, *then* $f\left(\frac{1}{x}\right) = -\frac{f(x)}{x}$.

Indeed, if in the assumption of the problem we take $x = 1$, $y = x$, *then* $f\left(\frac{1}{x}\right) = -\frac{f(x)}{x}$.

P3. *If* $x > 0$, $y > 0$, *then*

$$\frac{f(x)}{x^2} - \frac{f(y)}{y^2} = \frac{1}{yx^2} f\left(\frac{x}{y}\right).$$

Indeed, if in the assumption of the problem we substitute x *by* $\frac{1}{y}$ *and* y *by* $\frac{1}{x}$, *then*

$$\frac{1}{y} f\left(\frac{1}{y}\right) - \frac{1}{x} f\left(\frac{1}{x}\right) = \frac{1}{x^2 y} f\left(\frac{x}{y}\right).$$

According to property P2, it follows that

$$\frac{f(x)}{x^2} - \frac{f(y)}{y^2} = \frac{1}{x^2 y} f\left(\frac{x}{y}\right).$$

P4. $f(x) = c\left(x^2 - \frac{1}{x}\right)$, $c \neq 0$.

Indeed, according to property P3, it follows that

$$\frac{f(x)}{x^2} - \frac{f(y)}{y^2} = \frac{1}{x^2 y} f\left(\frac{x}{y}\right) = \frac{xf(x) - yf(y)}{x^3 y^3}.$$

Thus, we deduce that

$$\frac{f(x)}{\dfrac{x^3-1}{x}} = \frac{f(y)}{\dfrac{y^3-1}{y}},$$

where $x > 0$, $y > 0$, $x \neq 1$, $y \neq 1$. *Hence,*

$$f(x) = c \cdot \frac{x^3-1}{x},$$

where $x \neq 1$, $x > 0$.

According to property P1, it follows that

$$f(x) = c \cdot \frac{x^3-1}{x},$$

where $x > 0$. *Therefore, we obtain that*

$$\frac{4f(8)}{f(2)} = 73.$$

7.4.10 Problem Set 10

Problem 1. Let

$$x_n = \left(\frac{[n+\sin n]}{n+\sin n}\right)^{\dfrac{n+\sin n}{\{n+\sin n\}}}.$$

Evaluate the following expression

$$e \lim_{n\to\infty} x_n,$$

where $[x]$ is the integer part and $\{x\}$ is the fractional part of a real number x.

Solution. *Note that*

$$x_n = \left(1 - \frac{1}{\frac{n+\sin n}{\{n+\sin n\}}}\right)^{\frac{n+\sin n}{\{n+\sin n\}}}$$

and

$$\frac{n+\sin n}{\{n+\sin n\}} > n + \sin n \geq n-1.$$

Thus, it follows that

$$\lim_{n\to\infty} = \frac{1}{e}.$$

Therefore,

$$e \lim_{n\to\infty} x_n = 1.$$

Problem 2. Find all possible values of a, such that for any of them the function

$$f(x) = \ln(\sqrt{x^2+1} + ax)$$

is even.

Solution. *We have that*

$$f(a) = \ln(\sqrt{a^2+1} + a^2).$$

If $f(x)$ is an even function, then $f(-a) = f(a)$. Hence,

$$\sqrt{a^2+1} - a^2 = \sqrt{a^2+1} + a^2.$$

Thus, it follows that $a = 0$.

Note that if $f(x) = \ln\sqrt{x^2+1}$, then $f(-x) = \ln\sqrt{(-x)^2+1} = f(x)$. Therefore, $f(x)$ is an even function.

Problem 3. Evaluate the expression

$$\lim_{x\to+\infty} (\sin(\sin(\sin\sqrt{x^2+1})) - \sin(\sin(\sin x))).$$

Solution. *We have that $|\sin x| \leq |x|$. Hence,*

$$|\sin x - \sin y| = \left|2\sin\frac{x-y}{2}\cdot\cos\frac{x+y}{2}\right| \leq 2\left|\sin\frac{x-y}{2}\right| \leq |x-y|.$$

Thus, it follows that

$$|\sin x - \sin y| \leq |x - y|. \tag{7.98}$$

According to (7.98)*, we obtain that*

$$|\sin(\sin(\sin\sqrt{x^2+1})) - \sin(\sin(\sin x))| \leq |\sin(\sin\sqrt{x^2+1}) - \sin(\sin x)| \leq$$

$$\leq |\sin\sqrt{x^2+1} - \sin x| \leq |\sqrt{x^2+1} - x| = \frac{1}{\sqrt{x^2+1}+x}.$$

Hence, we deduce that

$$\lim_{x\to+\infty}(\sin(\sin(\sin\sqrt{x^2+1})) - \sin(\sin(\sin x))) = 0.$$

Problem 4. Given that point $M(x_0, y_0)$ is a centre of symmetry of the graph of function

$$y = \sqrt[3]{x+1} + \sqrt[3]{x+2} + \sqrt[3]{x+3} + 100.$$

Find $x_0 + y_0$.

Solution. *Let* (x, y) *be a point on the graph of function y, then* $(2x_0 - x, 2y_0 - y)$ *is also a point on the graph of function y. Hence, for any value of x, we have that*

$$\sqrt[3]{2x_0 - x + 1} + \sqrt[3]{2x_0 - x + 2} + \sqrt[3]{2x_0 - x + 3} + 100 =$$

$$= 2y_0 - \sqrt[3]{x+1} - \sqrt[3]{x+2} - \sqrt[3]{x+3} - 100,$$

$$\sqrt[3]{2x_0 - x + 1} + \sqrt[3]{2x_0 - x + 2} + \sqrt[3]{2x_0 - x + 3} + \sqrt[3]{x+1} + \sqrt[3]{x+2} + \sqrt[3]{x+3} =$$

$$= 2\sqrt[3]{x_0+1} + 2\sqrt[3]{x_0+2} + 2\sqrt[3]{x_0+3}.$$

If $x = x_0 + 1$*, then*

$$\sqrt[3]{x_0} + \sqrt[3]{x_0+4} = \sqrt[3]{x_0+1} + \sqrt[3]{x_0+3}.$$

Therefore,

$$x_0 + 3\sqrt[3]{x_0(x_0+4)}(\sqrt[3]{x_0} + \sqrt[3]{x_0+4}) + x_0 + 4 = x_0 + 1 +$$

$$+3\sqrt[3]{(x_0+1)(x_0+3)}(\sqrt[3]{x_0+1} + \sqrt[3]{x_0+4}) + x_0 + 3.$$

Hence,

$$\sqrt[3]{x_0} + \sqrt[3]{x_0+4} = 0.$$

We obtain that $x_0 = -2$*. It follows that* $y_0 = 100$*, thus* $x_0 + y_0 = 98$*.*

One can easily verify that point $(-2, 100)$ *is a centre of symmetry of function y.*

Problem 5. Evaluate the expression

$$\int_0^{\frac{\pi}{2}} \frac{2\sin x - \cos x - \cos 3x}{\sin x + \cos x}\, dx.$$

Solution. *Let*

$$A = \int_0^{\frac{\pi}{2}} \left(1 - \frac{2\cos^3 x}{\sin x + \cos x}\right) dx.$$

Therefore,

$$A = \int_{\frac{\pi}{2}}^{0} \left(1 - \frac{2\cos^3\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\right) d\left(\frac{\pi}{2} - x\right) =$$

$$= \int_{\frac{\pi}{2}}^{0} \left(1 - \frac{2\sin^3 x}{\sin x + \cos x}\right) d\left(\frac{\pi}{2} - x\right) = \int_0^{\frac{\pi}{2}} \left(1 - \frac{2\sin^3 x}{\sin x + \cos x}\right) dx.$$

Thus, it follows that

$$2A = \int_0^{\frac{\pi}{2}} (2 - 2(\cos^2 x - \cos x \sin x + \sin^2 x))\, dx = \int_0^{\frac{\pi}{2}} \sin 2x\, dx =$$

$$= \left(-\frac{\cos 2x}{2}\right)\Big|_0^{\frac{\pi}{2}} = 1.$$

Hence, we obtain that

$$\int_0^{\frac{\pi}{2}} \frac{2\sin x - \cos x - \cos 3x}{\sin x + \cos x}\, dx = 2A = 1.$$

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Given that function g is monotonic on $\mathbb{R}$ and for any x, y it holds true

$$f(x) - f(y) \leq |x - y||g(x) - g(y)|.$$

Find the number of possible values of $f(2015)$, if $f(1) = 1$.

Solution. *We have that*

$$f(y) - f(x) \leq |y - x||g(y) - g(x)|.$$

Hence, for any real numbers x, y it holds true

$$|f(x)-f(y)| \leq |x-y||g(x)-g(y)|. \tag{7.99}$$

Let $n > 1$ is a positive integer. We have that

$$|f(2015)-f(1)| \leq \left|f(2015)-f\left(2015-\frac{2014}{n}\right)\right|+$$

$$+\left|f\left(2015-\frac{2014}{n}\right)-f\left(2015-2\cdot\frac{2014}{n}\right)\right|+\cdots+\left|f\left(2015-(n-1)\frac{2014}{n}\right)-f(1)\right|.$$

Thus, according to (7.121), it follows that

$$|f(2015)-f(1)| \leq \frac{2014}{n}\left(\left|g(2015)-g\left(2015-\frac{2014}{n}\right)\right|+\left|g\left(2015-\frac{2014}{n}\right)-\right.\right.$$

$$\left.\left.-g\left(2015-2\cdot\frac{2014}{n}\right)\right|+\cdots+\left|g\left(2015-(n-1)\frac{2014}{n}\right)-g(1)\right|\right)=\frac{2014}{n}|g(2015)-g(1)|,$$

as function g is monotonic on $(-\infty,+\infty)$.

Therefore, for any positive integer n, it holds true

$$|f(2015)-f(1)| \leq \frac{2014|g(2015)-g(1)|}{n}.$$

We obtain that $f(2015)=f(1)=1$.

Problem 7. Given that the domain of the function

$$f(x)=\sqrt{x^3-ax^2+(a+b)x-1-b}+\sqrt[4]{-x^3+(c+15)x^2-(15c+26)x+26c}$$

consists of three points. Find the possible smallest value of $a+b+c$.

Solution. *Let*

$$p(x)=x^3-ax^2+(a+b)x-1-b,$$

and

$$q(x)=-x^3+(c+15)x^2-(15c+26)x+26.$$

Hence, we have that

$$f(x)=\sqrt{p(x)}+\sqrt[4]{q(x)}.$$

Let $D(f)=\{x_1,x_2,x_3\}$ and $x_1<x_2<x_3$.

Let us prove the following properties:

***P1.** x_i is a root of polynomial $p(x)q(x)$, where $i=1,2,3$.*

Proof by contradiction argument. Assume that x_i is not a root of $p(x)q(x)$, then $p(x_i)>0$ and $q(x_i)>0$. Thus, there exists $\delta>0$, such that $x \in (x_i-\delta, x_i+\delta)$, it

follows that $p(x) > 0$ *and* $q(x) > 0$. *In this case, function* $f(x)$ *is defined on* $(x_i - \delta, x_i + \delta)$. *This leads to a contradiction.*

P2. x_i *is a double root of polynomial* $p(x)q(x)$, *where* $i = 1,2,3$.

Proof by contradiction argument. Assume that $p(x_i) = 0$ *and* $q(x_i) > 0$, *then* x_i *is a double root of polynomial* $p(x)$, *otherwise there exists* $\delta > 0$, *such that function* $f(x)$ *is defined on* $(x_i - \delta, x_i]$ *or* $[x_i, x_i + \delta)$. *This leads to a contradiction.*

P3. x_1, x_2, x_3 *are the only roots of polynomial* $p(x)q(x)$.

The degree of polynomial is 6. Hence, according to P2 we have that P3 holds true.

Note that $1,2,13$ *are roots of polynomial* $p(x)q(x)$. *Thus,* $x_1 = 1$, $x_2 = 2$, $x_3 = 13$.

Therefore, either

$$f(x) = \sqrt{(x-1)(x-2)(x-13)} + \sqrt[4]{-(x-1)(x-2)(x-13)},$$

or

$$f(x) = \sqrt{(x-1)^2(x-2)} + \sqrt[4]{-(x-2)(x-13)^2},$$

or

$$f(x) = \sqrt{(x-1)^2(x-13)} + \sqrt[4]{-(x-2)^2(x-13)}.$$

Hence, either $a = 16$, $b = 25$, $c = 1$ *or* $a = 4$, $b = 1$, $c = 13$ *or* $a = 15$, $b = 12$, $c = 2$. *Thus, the possible smallest value of* $a+b+c$ *is equal to 18.*

Problem 8. Find the smallest value of function $f(x) = 4^{\sin x} + 4^{\cos x}$ on $[0, \frac{\pi}{2}]$.

Solution. *Let us consider the function* $g(x) = 4^{\sin x} + 4^{\cos x} - 5$ *on* $[0, \frac{\pi}{4}]$. *We have that*

$$g'(x) = 4^{\cos x} \cdot \cos x \ln 4 (4^{\sin x - \cos x} - \tan x) \geq 0,$$

as $4^{\sin x - \cos x} - \tan x \geq 0$.

Consider the function $F(x) = \sin x - \cos x - \log_4 \tan x$ *on* $[0, \frac{\pi}{4}]$. *Hence,*

$$F'(x) = \cos x + \sin x - \frac{1}{\sin x \cos x \ln 4} = \frac{1}{\sin x \cos x}\Big(\frac{\sqrt{2}}{2}\sin(x+\frac{\pi}{4})\sin 2x - \frac{1}{\ln 4}\Big) \leq$$

$$\leq \frac{1}{\sin x \cos x}\Big(\frac{\sqrt{2}}{2} - \frac{1}{\ln 4}\Big) < 0.$$

Thus, it follows that $F(x) \geq F(\pi/4) = 0$. *Hence,* $4^{\sin x - \cos x} - \tan x \geq 0$.

We have that $1024 < 2.7^7 < e^7$ *and* $\ln 4 < \frac{7}{5} < \sqrt{2}$.

Therefore, $g'(x) \geq 0$. *Hence,* $g(x) \geq g(0) = 0$.

If $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$, *then* $g\left(\frac{\pi}{2}-x\right) \geq 0$. *Now, using that* $g(x) = g\left(\frac{\pi}{2}-x\right)$, *it follows that* $g(x) = g\left(\frac{\pi}{2}-x\right)$. *Hence,* $g(x) \geq 0$.

Thus, the smallest value of $f(x)$ *on* $[0, \frac{\pi}{2}]$ *is equal to 5.*

Problem 9. Find the sum of the cubes of the roots of the following equation

$$2^{x+3} - 2 \cdot 3^{x+1} + 5^x = 3.$$

Solution. *Let us at first prove the following lemma:*

Lemma 7.13. *Let* $n > 1$, $n \in \mathbb{N}$, $0 < a_1 < \cdots < a_n$ *and* $c_1, \ldots, c_n$ *be nonzero real numbers. Prove that the number of roots of the following equation*

$$c_1 \cdot a_1^x + \cdots + c_n \cdot a_n^x = 0$$

is not more than the number of negative elements of sequence $c_1c_2, c_2c_3, \ldots, c_{n-1}c_n$.

Proof. *Proof by mathematical induction.*

Basis: *Let us show that the statement holds true for* $n = 1$.

If $c_1c_2 < 0$, *then the equation* $c_1 \cdot a_1^x + c_2 \cdot a_2^x = 0$ *has only one root, that is* $x = \log_{\frac{a_2}{a_1}}\left(-\frac{c_1}{c_2}\right)$.

If $c_1c_2 > 0$, *then the equation* $c_1 \cdot a_1^x + c_2 \cdot a_2^x = 0$ *does not have any roots.*

This ends the proof of the statement for the case $n = 1$.

Inductive step: *Let us show that if the statement holds true for* $n = k$, $k \in \mathbb{N}$, *then it holds true also for* $n = k+1$.

We proceed by contradiction argument. Assume that the number of roots of $c_1 \cdot a_1^x + \cdots + c_{k+1} \cdot a_{k+1}^x = 0$ *is more than the number of negative elements of sequence* $c_1c_2, c_2c_3, \ldots, c_kc_{k+1}$. *Consider the following function*

$$f(x) = c_1 \cdot \left(\frac{a_1}{a_{k+1}}\right)^x + \cdots + c_k \cdot \left(\frac{a_k}{a_{k+1}}\right)^x + c_{k+1}.$$

Note that the number of 0 values of the function

$$f'(x) = c_1 \ln \frac{a_1}{a_{k+1}} \cdot \left(\frac{a_1}{a_{k+1}}\right)^x + \cdots + c_k \ln \frac{a_k}{a_{k+1}} \cdot \left(\frac{a_k}{a_{k+1}}\right)^x$$

is more than the number of negative elements of sequence $c_1c_2, c_2c_3, \ldots, c_{k-1}c_k$. *On the other hand, according to the assumption made their number is not more than the number of the negative elements of the following sequence*

$$\left(c_1 \ln \frac{a_1}{a_{k+1}}\right) \cdot \left(c_2 \ln \frac{a_2}{a_{k+1}}\right), \ldots, \left(c_{k-1} \ln \frac{a_{k-1}}{a_{k+1}}\right) \cdot \left(c_k \ln \frac{a_k}{a_{k+1}}\right),$$

which is equal to the number of the negative elements of sequence $c_1c_2, \ldots, c_{k-1}c_k$, *as* $\ln \frac{a_i}{a_{k+1}} < 0$, $i = 1, \ldots, k$. *This leads to a contradiction and ends the proof. Since both the basis and the inductive step have been proved, by mathematical induction, the statement holds true for all* $n \in \mathbb{N}$.

According to the lemma the equation $-3 \cdot 1^x + 8 \cdot 2^x - 6 \cdot 3^x + 5^x = 0$ *has not more than three roots. Note that 0,1,2 are roots of this equation. Therefore,* $0^3 + 1^3 + 2^3 = 9$.

7.4.11 Problem Set 11

Problem 1. Given that

$$\lim_{x \to \frac{\pi}{2}} (1 + \cos(13x))^{\tan x} = e^a.$$

Find a.

Solution. *We have that*

$$e^a = \lim_{x \to \frac{\pi}{2}} (1 + \cos(13x))^{\tan x} = \lim_{t \to 0} (1 + \cos(6.5\pi + 13t))^{-\cot t} =$$

$$= \lim_{t \to 0} \left((1 - \sin 13t)^{-\frac{-1}{\sin 13t}}\right)^{\frac{\sin 13t}{\sin t} \cdot \cos t} = e^{\lim\limits_{t \to 0}\left(\frac{\sin 13t}{13t} \cdot \frac{t}{\sin t} \cdot 13 \cos t\right)} = e^{13}.$$

Thus, it follows that $a = 13$.

Problem 2. Find the smallest value of a, such that the function

$$f(x) = \ln(x^{10} + 1) - a \ln(|x| + 1)$$

is upper bounded.

Solution. *If* $a \geq 10$, *then*

$$f(x) = \ln(x^{10} + 1) - a \ln(|x| + 1) \leq \ln(x^{10} + 1) - 10 \ln(|x| + 1) \leq$$

$$\leq \ln \frac{(x^{10} + 1)}{(|x| + 1)^{10}} \leq \ln 1 = 0.$$

Therefore, $f(x) \leq 0$.

If $a < 10$ *and* $x \neq 0$, *then*

$$f(x) = \ln(x^{10}+1) - a\ln(|x|+1) = \ln\frac{(x^{10}+1)}{(|x|+1)^a} = \ln\frac{\left(1+\frac{1}{x^{10}}\right)}{|x|^{a-10}\left(1+\frac{1}{|x|}\right)^a}.$$

Thus, it follows that

$$\lim_{x\to\infty} f(x) = +\infty.$$

Hence, the smallest value of a, such that $f(x)$ is upper bounded, is equal to 10.

Problem 3. Find the smallest value of the function

$$f(x) = \left(\frac{1}{\sqrt{1-x^2}}\right)^3 + \left(\frac{1}{1-\sqrt{1-x^2}}\right)^3.$$

Solution. *We have that*

$$f(x) = \left(\frac{1}{\sqrt{1-x^2}(1-\sqrt{1-x^2})}\right)^3 - 3\cdot\frac{1}{\sqrt{1-x^2}}\cdot\frac{1}{1-\sqrt{1-x^2}}\cdot\frac{1}{\sqrt{1-x^2}(1-\sqrt{1-x^2})} =$$

$$= \left(\frac{1}{\sqrt{1-x^2}(1-\sqrt{1-x^2})}\right)^2\left(\frac{1}{\sqrt{1-x^2}(1-\sqrt{1-x^2})} - 3\right) \geq 4^2\cdot 1 = 16,$$

as

$$0 \leq \sqrt{1-x^2}\cdot(1-\sqrt{1-x^2}) \leq \frac{1}{4}.$$

On the other hand,

$$f\left(\frac{\sqrt{3}}{2}\right) = 16.$$

Hence, the smallest value of function $f(x)$ is equal to 16.

Problem 4. Find the positive value of a, such that the equation

$$x^2 - 3ax + 4a|x-3a| + 12.5a^2 - 1250 = 0.$$

has a unique solution.

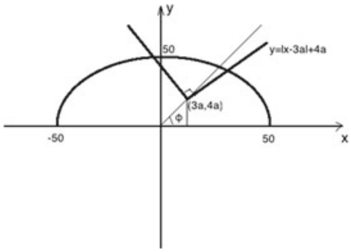
Solution. *We have that*

$$x^2 + (|x-3a|+4a)^2 = 50^2.$$

Thus, if $a > 0$, then the number of the solutions of the given equation is equal to the number of the solutions of the following system

$$\begin{cases} y = \sqrt{50^2 - x^2}, \\ y = |x-3a| + 4a. \end{cases}$$

Let us consider the graphs of the functions $y = \sqrt{50^2 - x^2}$ *and* $y = |x - 3a| + 4a$*, where* $a > 0$*:*



We have that

$$\phi = \arctan \frac{4}{3} > \frac{\pi}{4}.$$

Therefore, the graph of these functions intersects at one point, if the point $(3a, 4a)$ *belongs to the graph of the function* $y = \sqrt{50^2 - x^2}$*. Thus, it follows that* $a = 10$*.*

Problem 5. Solve the equation

$$\sqrt[6]{x} + \sqrt[3]{4\sqrt{x} - 5} = \sqrt{6\sqrt[3]{x} + 1}.$$

Solution. *The given equation is equivalent to the following equation,*

$$\sqrt{6 + \frac{1}{\sqrt[3]{x}}} - \sqrt[3]{4 - \frac{5}{\sqrt{x}}} = 1. \tag{7.100}$$

Note that function

$$f(x) = \sqrt{6 + \frac{1}{\sqrt[3]{x}}} - \sqrt[3]{4 - \frac{5}{\sqrt{x}}},$$

is decreasing and

$$f(64) = \frac{5}{2} - \frac{3}{2} = 1.$$

Therefore, the unique solution of (7.100) *is* $x = 16$*.*

Problem 6. Evaluate the expression

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos 3x}{\sin x + \cos x} dx.$$

Solution. *At first, let us prove that*

$$\int_0^{\frac{\pi}{2}} \frac{2\sin x - \cos x - \cos 3x}{\sin x + \cos x} dx = 1.$$

Solution. *Let*

$$A = \int_0^{\frac{\pi}{2}} \left(1 - \frac{2\cos^3 x}{\sin x + \cos x}\right) dx.$$

Therefore,

$$A = \int_{\frac{\pi}{2}}^0 \left(1 - \frac{2\cos^3\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\right) d\left(\frac{\pi}{2} - x\right) =$$
$$= \int_{\frac{\pi}{2}}^0 \left(1 - \frac{2\sin^3 x}{\sin x + \cos x}\right) d\left(\frac{\pi}{2} - x\right) = \int_0^{\frac{\pi}{2}} \left(1 - \frac{2\sin^3 x}{\sin x + \cos x}\right) dx.$$

Thus, it follows that

$$2A = \int_0^{\frac{\pi}{2}} (2 - 2(\cos^2 x - \cos x \sin x + \sin^2 x))\, dx = \int_0^{\frac{\pi}{2}} \sin 2x\, dx =$$
$$= \left(-\frac{\cos 2x}{2}\right)\Big|_0^{\frac{\pi}{2}} = 1.$$

Hence, we obtain that

$$\int_0^{\frac{\pi}{2}} \frac{2\sin x - \cos x - \cos 3x}{\sin x + \cos x}\, dx = 2A = 1.$$

On the other hand,

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{\sin x + \cos x} dx = -\ln(\sin x + \cos x)\,|_0^{\frac{\pi}{2}} = 0.$$

Thus, it follows that

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos 3x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{2\sin x - \cos x - \cos 3x}{\sin x + \cos x} dx - \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{\sin x + \cos x} dx = 1.$$

Problem 7. Find the greatest value of a, such that the sequence

$$x_1 = a, \;\; x_{n+1} = 24x_n - x_n^3, n = 1, 2, \ldots$$

is bounded.

Solution. *Note that*

$$x_{n+1} + x_n = x_n(5 - x_n)(5 + x_n), \quad n = 1, 2, \ldots \tag{7.101}$$

Hence, if $a = 5$, $x_1 = 5$, $x_2 = -5$, $x_3 = 5$, $x_4 = -5, \ldots$

Therefore, the sequence $x_n = 5 \cdot (-1)^{n-1}$ *is bounded.*

If $a > 5$, *then denote* $a = 5 + h$, *where* $x_1 = 5 + h$ *and* $h > 0$. *By mathematical induction, let us prove that if* $n \in \mathbb{N}$, *then*

$$x_{2n-1} \geq 5 + (2n - 1)h.$$

Basis. *If* $n = 1$, *then* $x_1 = 5 + h \geq 5 + (2 \cdot 1 - 1)h$.

Inductive step. *Assume that for* $k \in \mathbb{N}$, *it holds true* $x_{2k-1} \geq 5 + (2k - 1)h$. *Prove that* $x_{2k+1} \geq 5 + (2k + 1)h$.

We have that

$$x_{2k} = -x_{2k-1} - x_{2k-1}(x_{2k-1} - 5)(x_{2k-1} + 5) \leq -5 - (2k - 1)h - h = -5 - 2kh.$$

Thus, it follows that

$$x_{2k+1} = -x_{2k} - x_{2k}(-x_{2k} - 5)(-x_{2k} + 5) \geq 5 + 2kh + h = 5 + (2k + 1)h.$$

Therefore, for any $n \in \mathbb{N}$, *it holds true*

$$x_{2n-1} \geq 5 + (2n - 1)h.$$

Hence, if $a > 5$, *then sequence* (x_n) *is not bounded.*

Thus, the greatest required value of a is equal to 5.

Problem 8. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Given that $f(1) = 1$, function g is monotonic on $\mathbb{R}$ and that for any numbers x, y it holds true the following inequality

$$f(x) - f(y) \leq |\arctan x - \arctan y||g(x) - g(y)|.$$

Find the value of $f(1) + 2f(2) + \cdots + 40f(40)$.

Solution. *Let us prove that if* $x_0 > 1$, *then* $f(x_0) = f(1)$.

Note that

$$f(y) - f(x) \leq |\arctan y - \arctan x||g(y) - g(x)|.$$

Thus, it follows that for any x, y *it holds true*

$$|f(x) - f(y)| \leq |\arctan x - \arctan y||g(x) - g(y)|.$$

Let $n > 1$ and $n \in \mathbb{N}$. We have that

$$|f(x_0) - f(1)| = \left| f(x_0) - f\left(x_0 - \frac{x_0 - 1}{n}\right) + f\left(x_0 - \frac{x_0 - 1}{n}\right) - f\left(x_0 - 2 \cdot \frac{x_0 - 1}{n}\right) + \right.$$
$$\left. + f\left(x_0 - (n-1) \cdot \frac{x_0 - 1}{n}\right) - f\left(x_0 - n \cdot \frac{x_0 - 1}{n}\right) \right| \leq \left| f(x_0) - f\left(x_0 - \frac{x_0 - 1}{n}\right) \right| +$$
$$+ \left| f\left(x_0 - (n-1) \cdot \frac{x_0 - 1}{n}\right) - f\left(x_0 - n \cdot \frac{x_0 - 1}{n}\right) \right| \leq$$
$$\leq \left| \arctan x_0 - \arctan\left(x_0 - \frac{x_0 - 1}{n}\right) \right| \cdot$$

$$\cdot \left| g\left(x_0\right) - g\left(x_0 - \frac{x_0 - 1}{n}\right) \right| + \cdots + \left| \arctan\left(x_0 - (n-1) \cdot \frac{x_0 - 1}{n}\right) - \right.$$

$$\left. - \arctan\left(x_0 - n \cdot \frac{x_0 - 1}{n}\right) \right| \cdot \left| g\left(x_0 - (n-1) \cdot \frac{x_0 - 1}{n}\right) - g\left(x_0 - n \cdot \frac{x_0 - 1}{n}\right) \right|$$

$$\leq \left| \frac{x_0 - 1}{n} \right| \left(\left| g(x_0) - g\left(x_0 - \frac{x_0 - 1}{n}\right) \right| + \cdots + \left| g\left(x_0 - (n-1) \cdot \frac{x_0 - 1}{n}\right) - g\left(x_0 - n \cdot \frac{x_0 - 1}{n}\right) \right| \right) =$$

$$= \frac{|x_0 - 1|}{n} |g(x_0) - g(1)|,$$

as, for $1 \leq a < b$, it holds true

$$|\arctan a - \arctan b| \leq \tan(\arctan b - \arctan a) = \frac{b - a}{1 + ab} \leq b - a = |a - b|,$$

and the function g is monotonic.

We have obtained that for $n > 1$ and $n \in \mathbb{N}$, it holds true

$$|f(x_0) - f(1)| \leq \frac{|x_0 - 1||g(x_0) - g(1)|}{n}.$$

Thus, it follows that $f(x_0) = f(1)$.

Therefore,

$$f(1) = f(2) = \cdots = f(40).$$

Hence,

$$f(1) + 2f(2) + \cdots + 40f(40) = 1 + 2 + \cdots + 40 = 820.$$

Problem 9. Let $p(x)$, $q(x)$ and $r(x)$ be polynomials with real coefficients, such that

$$p(q(x)) + p(r(x)) = x^2,$$

for any x. Given that $q(1) + r(1) = 1$. Find the greatest possible value of the degree of polynomial $q(x) + r(x)$.

Solution. *Let*

$$p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

where $a_i \in \mathbb{Z}$, $i = 0, 1, \ldots, n$, $a_0 \neq 0$ and $n \in \mathbb{N}$.

Let the degrees of polynomials $q(x)$ and $r(x)$ be m and k, respectively.

Consider the following cases:

If $m \neq k$, then the degree of polynomial $p(q(x)) + p(r(x))$ is $n \cdot \max(m, k)$. This leads to a contradiction.

Therefore, $m = k \in \mathbb{N}$. Let

$$q(x) = b_0x^k + \cdots + b_k,$$

and

$$r(x) = c_0x^k + \cdots + c_k,$$

where $b_0, \ldots, b_k, c_0, \ldots, c_k \in \mathbb{R}$ and $b_0 \neq 0$, $c_0 \neq 0$.

Note that

$$a_0b_0^n + a_0c_0^n = 0.$$

Hence, n is an odd number, $b_0 = -c_0$.

We have that

$$a_0(q(x)+r(x))(q^{n-1}(x) - \cdots + r^{n-1}(x)) + a_1q^{n-1}(x) + a_1r^{n-1}(x) + \cdots + 2a_n = x^2.$$

Note that the degree of the polynomial

$$q^{n-1}(x) - q^{n-2}(x)r(x) + \cdots + r^{n-1}(x)$$

is equal to $(n-1)k$. Therefore, $q(x) + r(x) = c$.

From the condition, $q(1) + r(1) = 1$, follows that $c = 1$. Hence, $q(x) + r(x) = 1$. We obtain that the degree of the polynomial $q(x) + r(x)$ is equal to 1.

If $n = 2$, then $\max(k, m) = 1$.

If $n = 1$, then

$$q(x) + r(x) = \frac{x^2 - 2a_1}{a_0}.$$

Therefore, the answer is 2. For example, $p(x) = x$ and $q(x) + r(x) = x^2$.

7.4.12 Problem Set 12

Problem 1. Evaluate the expression

$$\lim_{x\to 0} \frac{|x|^{x+2}}{\sqrt{1+x^2} - 1}.$$

Solution. *We have that*

$$\frac{|x|^{x+2}}{\sqrt{1+x^2}-1} = \frac{|x|^{x+2}\cdot(\sqrt{1+x^2}+1)}{x^2} = |x|^x(\sqrt{1+x^2}+1).$$

Thus, it follows that

$$\lim_{x\to 0}\frac{|x|^{x+2}}{\sqrt{1+x^2}-1} = \lim_{x\to 0}(|x|^x\cdot(\sqrt{1+x^2}+1)) = \lim_{x\to 0}|x|^x\cdot\lim_{x\to 0}(\sqrt{1+x^2}+1) = 1\cdot 2 = 2.$$

Therefore

$$\lim_{x\to 0}\frac{|x|^{x+2}}{\sqrt{1+x^2}-1} = 2.$$

Problem 2. Find the number of real numbers a, such that the equation

$$|\ln(x+\sqrt{x^2+1})| = a^2-1,$$

has a unique solution.

Solution. *Note that if x_0 is a solution of the given equation, then $-x_0$ is also a solution. We have that*

$$|\ln(-x_0+\sqrt{x_0^2+1})| = \left|\ln\frac{1}{x_0+\sqrt{x_0^2+1}}\right| = |-\ln(x_0+\sqrt{x_0^2+1})| = |\ln(x_0+\sqrt{x_0^2+1})| = a.$$

Hence, if the given equation has a unique solution, then $x=0$.

Therefore

$$|\ln 1| = a^2-1.$$

Thus, either $a=1$ or $a=-1$.

If $a^2=1$, we have that

$$|\ln(x+\sqrt{x^2+1})| = 0,\ \ x+\sqrt{x^2+1} = 1.$$

We obtain that

$$x^2+1 = (1-x)^2.$$

Hence, it follows that $x=0$.

Therefore, the number of real numbers a satisfying the assumptions of the problem is equal to 2.

Problem 3. Evaluate the expression

$$\lim_{x\to 0}\frac{\sin\tan x-\tan\sin x}{x^3}.$$

Solution. *Let us at first calculate*

$$\lim_{x\to 0}\frac{\sin x - x}{x^3},$$

and

$$\lim_{x\to 0}\frac{\tan x - x}{x^3}.$$

We have that

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\cos x - 1}{3x^2} = \lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{3\cdot 4\cdot\left(\frac{x}{2}\right)^2} = -\frac{1}{6}.$$

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\frac{1}{\cos^2 x} - 1}{3x^2} = \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{3x^2\cos^2 x} =$$

$$= \lim_{x\to 0}\left(\frac{2\sin^2\frac{x}{2}}{12\cdot\left(\frac{x}{2}\right)^2}\cdot\frac{1+\cos x}{\cos^2 x}\right) = \frac{1}{3}.$$

Note that

$$\lim_{x\to 0}\frac{\sin(\tan x) - \tan(\sin x)}{x^3} = \lim_{x\to 0}\left(\frac{\sin(\tan x) - \tan x}{x^3} + \frac{\tan x - \sin x}{x^3} - \frac{\tan(\sin x) - \sin x}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin(\tan x) - \tan x}{(\tan x)^3}\cdot\left(\frac{\tan x}{x}\right)^3 + \frac{\sin x\cdot 2\sin^2\frac{x}{2}}{x\cdot 4\cdot\left(\frac{x}{2}\right)^2\cos x} - \frac{\tan(\sin x) - \sin x}{(\sin x)^3}\cdot\left(\frac{\sin x}{x}\right)^3\right) =$$

$$= -\frac{1}{6} + \frac{1}{2} - \frac{1}{3} = 0.$$

Problem 4. Find the number of roots of the equation

$$\sin x = \frac{x^3 - 3x^2 + 5x}{4 - 3x}.$$

Solution. *We have that*

$$|\sin x| = |x|\cdot\frac{x^2 - 3x + 5}{|4 - 3x|}.$$

Note that

$$\frac{x^2-3x+5}{|4-3x|} \geq 1.$$

Note that if $x \neq 0$, then $|sinx| < |x|$.

Therefore, the unique root of the given equation is $x = 0$.

Problem 5. Find the number of functions $f : \mathbb{R} \to \mathbb{R}$, such that (for each of them) holds true $f(x) - f(y) > \sqrt{x-y}$ for any x, y satisfying the condition $x > y$.

Solution. *Let us prove that there does not exist such function.*

Proof by contradiction argument. Assume that there exists such function, then for a positive integer $n > 1$, we deduce that

$$f(1) - f(0) = f(1) - f\left(\frac{n-1}{n}\right) + f\left(\frac{n-1}{n}\right) - f\left(\frac{n-2}{n}\right) + \cdots + f\left(\frac{1}{n}\right) - f(0) >$$

$$> \sqrt{\frac{1}{n}} + \sqrt{\frac{1}{n}} + \cdots + \sqrt{\frac{1}{n}} = \sqrt{n}.$$

Therefore, for any positive integer n we have that

$$f(1) - f(0) > \sqrt{n}.$$

This leads to a contradiction.

Hence, the number of such functions is equal to 0.

Problem 6. Find the number of real numbers a, such that the equation

$$x^2 - 2ax + 4a|x-a| + 2a^2 - 16 = 0,$$

has three solutions.

Solution. *Let us rewrite the given equation in the following way*

$$|x-a|^2 + 4a|x-a| + a^2 - 16 = 0. \tag{7.102}$$

Hence, if the given equation has three solutions, then 0 is one of the roots of the equation

$$y^2 + 4ay + a^2 - 16 = 0.$$

Therefore, either $a = 4$ or $a = -4$.

If $a = 4$, then (7.102) has a unique solution.

If $a = -4$, then (7.102) has three solutions.

Thus, it follows that the number of such real numbers a is equal to 1.

Problem 7. Let sequence (x_n) be such that $x_0 = x_{k+1} = 0, \quad x_i > 0$ and

$$x_{i+1} > \frac{\sqrt{6}+\sqrt{2}}{2} x_i - x_{i-1}, \quad i = 1, 2, \ldots, k.$$

Find the possible smallest value of k.

Solution. *Note that*

$$\frac{\sqrt{6}+\sqrt{2}}{2} = 2\cos\frac{\pi}{12}.$$

Thus, it follows that

$$x_{i+1} > 2x_i \cos\frac{\pi}{12} - x_{i-1}, \quad i = 1, 2, \ldots, k.$$

If $k \leq 11$, *then we have that*

$$x_2 \sin\frac{\pi}{12} > 2x_1 \sin\frac{\pi}{12}\cos\frac{\pi}{12}.$$

$$x_3 \sin\frac{2\pi}{12} > 2x_2 \sin\frac{2\pi}{12}\cos\frac{\pi}{12} - x_1 \sin\frac{2\pi}{12}.$$

$$\ldots$$

$$x_{k+1} \sin\frac{k\pi}{12} \geq 2x_k \sin\frac{k\pi}{12}\cos\frac{\pi}{12} - x_{k-1}\sin\frac{k\pi}{12}.$$

Summing up these inequalities, we obtain that

$$0 > x_k \sin\frac{\pi(k+1)}{12}.$$

This leads to a contradiction.

If $k = 12$, *then the sequence*

$$x_i = \sin\left(\frac{\pi}{13} i\right)$$

satisfies the assumptions of the problem, as

$$\sin\left(\frac{\pi}{13}(i+1)\right) = 2\cos\left(\frac{\pi}{13}\right)\sin\left(\frac{\pi}{13} i\right) - \sin\left(\frac{\pi}{13}(i-1)\right).$$

Hence, the smallest possible value of k *is equal to 12.*

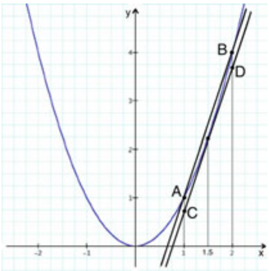
Problem 8. Let real number a, b be, such that the inequality

$$|x^2 - ax - b| \leq \frac{1}{8},$$

holds true for any $x \in [1,2]$. Find the value of $20a + 16b$.

Solution. *Let us construct the graph of the function $y = x^2$ (see the figure below).*

Note that the equation of tangent line to the graph of function $y = x^2$ at abscissa $x_0 = 1.5$ is $y = 3x - 2.25$.



Denote by C and D the intersection points of this tangent line with line $x = 1$ and line $x = 2$. Consider the points $A(1,1)$ and $B(2,4)$. We have that

$$AC = 1 - (3 - 2.25) = 4 - (6 - 2.25) = BD.$$

Thus, it follows that $ABDC$ is a parallelogram. Denote by M and N the midpoints of line segments AC and BD. Denote by M_1 the symmetric point to point M with respect to point A and by N_1 the symmetric point to point N with respect to point B.

We have that

$$MA = M_1A = NB = N_1B = \frac{1}{8}.$$

Therefore, according to the given condition line $y = ax + b$ intersects line segments MM_1, NN_1 and from the condition

$$|1.5^2 - 1.5a - b| \leq \frac{1}{8},$$

we obtain that line $y = ax + b$ coincides with line MN. Hence $a = 3$, $b = -\frac{17}{8}$.

We deduce that

$$20a + 16b = 60 - 34 = 26.$$

Problem 9. Let function $f : (0,\pi) \to \mathbb{R}$ be increasing on $(0,\pi)$, has a second-order derivative and the equation

$$(f(x))^2 + (f'(x))^2 = 1$$

holds true for any $x \in (0, \pi)$. Find the number of all such functions f.

Solution. *We have that*

$$((f(x))^2 + (f'(x))^2)' = 0.$$

Thus, it follows that

$$2f'(x)(f(x) + f''(x)) = 0.$$

We obtain that

$$f(x) + f''(x) = 0.$$

Consider the function

$$F(x) = f'(x)\sin x - f(x)\cos x$$

on interval $(0, \pi)$. *Note that*

$$F'(x) = f''(x)\sin x + f'(x)\cos x - f'(x)\cos x + f(x)\sin x = (f''(x) + f(x))\sin x = 0.$$

Therefore

$$f'(x)\sin x - f(x)\cos x = c.$$

Hence

$$f'(x) = \frac{f(x)\cos x + c}{\sin x}.$$

On the other hand, it is given that

$$(f(x))^2 + (f'(x))^2 = 1.$$

Thus, it follows that

$$(f(x))^2 + \left(\frac{f(x)\cos x + c}{\sin x}\right)^2 = 1.$$

We deduce that

$$f(x) = \cos(x - \phi_1),$$

or

$$f(x) = \cos(x - \phi_2),$$

where

$$\cos\phi_1 = -c, \ \ \sin\phi_1 = \sqrt{1-c^2}, \ \ \cos\phi_2 = -c, \ \ \sin\phi_2 = -\sqrt{1-c^2}.$$

We have that function $f(x)$ is increasing on $(0,\pi)$. Thus,

$$\phi_i = \pi + 2\pi k,\ \ k \in \mathbb{Z},$$

where $f(x) = -\cos x$.

One can easily verify that $f(x) = -\cos x$ satisfies the assumptions of the problem. Therefore, the number of all such functions is equal to 1.

Problem 10. Let function $f:[0,1] \to \mathbb{R}$ be non-decreasing on $[0,1]$ and continuous. Given that

$$\int_0^1 \arcsin x(f(x)-f(\sin x))(f(x)-f(\sin(\sin x)))dx+$$

$$+\int_0^1 \sin x(f(\sin(\sin x))-f(x))(f(\sin(\sin x))-f(\sin x))dx =$$

$$= \int_0^1 x(f(x)-f(\sin x))(f(\sin x)-f(\sin(\sin x)))dx.$$

Find the number of possible values of $f(1)-f(0)$.

Solution. *Note that if $x \in [0,1]$, then we have that*

$$0 \le \sin x \le x \le \arcsin x,\ \ 0 \le \sin(\sin x) \le \sin x \le x,$$

and

$$f(\sin(\sin x)) \le f(\sin x) \le f(x).$$

Thus, it follows that

$$\arcsin x(f(x)-f(\sin x))(f(x)-f(\sin(\sin x))) \ge$$

$$\ge x(f(x)-f(\sin x))(f(\sin x)-f(\sin(\sin x))),$$

and

$$\sin x(f(\sin(\sin x))-f(x))(f(\sin(\sin x))-f(\sin x)) \ge 0.$$

We deduce that

$$\int_0^1 \arcsin x(f(x)-f(\sin x))(f(x)-f(\sin(\sin x)))dx \ge$$

$$\int_0^1 x(f(x)-f(\sin x))(f(\sin x)-f(\sin(\sin x)dx,$$

and

$$\int_0^1 \sin x(f(\sin(\sin x)) - f(x))(f(\sin(\sin x)) - f(\sin x))dx \geq 0.$$

According to the initial assumption of the problem, we obtain that the last two inequalities are equalities.

Hence, for any point $x \in [0,1]$ it holds true

$$f(x) = f(\sin x) = f(\sin(\sin x)).$$

Let $x \in (0,1]$. Consider the sequence

$$x_1 = x, \ \ x_n = \sin x_{n-1}, \ \ n = 2,3,\ldots$$

Note that $x_n \in (0,1]$ and

$$x_n = \sin x_{n-1} \leq x_{n-1}, \ \ n = 2,3,\ldots$$

Therefore, (x_n) is a convergent sequence. On the other hand,

$$f(x_n) = f(\sin x_{n-1}) = f(x_{n-1}), \ \ n = 2,3,\ldots$$

Let

$$\lim_{n\to\infty} x_n = c.$$

Thus, it follows that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sin x_{n-1}.$$

We obtain that $c = \sin c$ and $c \geq 0$. That is $c = 0$.

Hence $f(x) = f(x_n)$. We deduce that

$$f(x) = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(0).$$

Therefore $f(1) - f(0) = 0$.

Problem 11. Let $x_1 = a$, $x_{n+1} = 2016x_n^3 - 2015x_n$, $n = 1,2,\ldots$ Find the number of real numbers a belonging to $[-1,0)$, such that for any a the elements of sequence (x_n) are negative.

Solution. *Note that $a = -1$ and $x_n = -1$, $n = 1,2,\ldots$*

Assume that there exists $a \in (-1,0)$, such that $x_n < 0$, $n = 1,2,\ldots$

We have that

$$x_{n+1} - x_n = 2016x_n(x_n - 1)(x_n + 1), n = 1,2,\ldots$$

Therefore $0 < x_2 - x_1$. *Thus, it follows that* $x_2 \in (-1, 0)$. *Hence* $0 < x_3 - x_2$. *By mathematical induction, we obtain that*

$$-1 < x_1 < x_2 < \cdots < x_n < \cdots.$$

Hence, (x_n) *is a convergent sequence.*

Let

$$\lim_{n\to\infty} x_n = c.$$

Then, we have that

$$c = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} (2016x_n^3 - 2015x_n) = 2016c^3 - 2016c.$$

We deduce that either $c = -1$ *or* $c = 0$ *or* $c = 1$.

We have that $x_n \in (-1, 0)$, $n = 1, 2, \ldots$ *and*

$$x_1 < x_2 < \cdots < x_n < \cdots$$

Therefore $c = 0$.

Then, there exists $k \in \mathbb{N}$, *such that*

$$0 > x_k > -\sqrt{\frac{2015}{2016}}.$$

Thus, it follows that

$$x_{k+1} = x_k(2016x_k^2 - 2015) > 0.$$

This leads to a contradiction.

Hence, only in the case, when $a = -1$ *all the elements of sequence* (x_n) *are negative. Therefore, the answer is 1.*

Problem 12. We call a number C "sympathetic", if there exists a function $f : (0, +\infty) \to (0, +\infty)$, such that it is continuous on $[2016, +\infty)$ and the inequality

$$\frac{f(x + f(x))}{f(x)} \leq C,$$

holds true for any positive x. Find the smallest "sympathetic" number.

Solution. *Assume that c is a "sympathetic" number. We have that*

$$c \geq \frac{f(1 + f(1))}{f(1)},$$

and

$$f(1+f(1))>0,\ f(1)>0.$$

Therefore $c>0$.

Let us prove that $c\geq 1$.

Proof by contradiction argument. Assume that $0<c<1$. *Consider the following sequence*

$$x_1=2016,\ \ x_{n+1}=x_n+f(x_n), n=1,2,\ldots$$

Note that this sequence is an increasing sequence.

On the other hand,

$$f(x_{n+1})=f(x_n+f(x_n))\leq cf(x_n). \tag{7.103}$$

By mathematical induction, from (7.103)*, we obtain that*

$$f(x_n)\leq c^{n-1}f(x_1), n=1,2,\ldots$$

We have that

$$x_n=x_{n-1}+f(x_{n-1})=x_{n-2}+f(x_{n-2})+f(x_{n-1})=\cdots=x_1+f(x_1)+\cdots+f(x_{n-1})\leq$$

$$\leq x_1+f(x_1)(1+c+\cdots+c^{n-2})<x_1+f(x_1)(1+c+\cdots+c^{n-2}+\ldots)=x_1+\frac{f(x_1)}{1-c}.$$

Therefore

$$x_n\in\left[2016, 2016+\frac{f(2016)}{1-c}\right], n=1,2,\ldots$$

According to the assumption of the problem, we have that function f *is continuous on* $\left[2016,2016+\frac{f(2016)}{1-c}\right]$. *Hence, function* f *has a minimum value. Assume that this minimum value is equal to* m.

Thus, it follows that

$$f(x_n)\geq m>0, n=1,2,\ldots \tag{7.104}$$

On the other hand, sequence (x_n) *is convergent. Hence*

$$\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}(x_{n+1}-x_n)=0.$$

This leads to a contradiction, see (7.104). *We deduce that* $c\geq 1$.

Note that $c=1$ *is a "sympathetic" number, as for the function* $f(x)=\dfrac{1}{1+x}$ *it holds true*

$$\frac{f(x+f(x))}{f(x)} < 1,$$

and it satisfies all the other assumptions.

Thus, the smallest "sympathetic" number is 1.

7.4.13 Problem Set 13

Problem 1. Given that

$$\lim_{x\to\infty}\left(\frac{x^3-1}{x+5} - ax^2 - bx - c\right) = 0.$$

Find $a+b+c$.

Solution. *We have that*

$$\lim_{x\to\infty}\left(\frac{x^3-1}{x+5} - ax^2 - bx - c\right) = 0.$$

Thus, it follows that

$$\lim_{x\to\infty}\left(\frac{1}{x^2}\cdot\left(\frac{x^3-1}{x+5} - ax^2 - bx - c\right)\right) = 0.$$

Therefore

$$a = \lim_{x\to\infty}\left(\frac{x^3-1}{x^3+5x^2} - \frac{b}{x} - \frac{c}{x^2}\right) = 1.$$

We obtain that

$$\lim_{x\to\infty}\left(\frac{x^3-1}{x+5} - x^2 - bx - c\right) = 0.$$

Hence

$$\lim_{x\to\infty}\left(\frac{-5x^2-1}{x+5} - bx - c\right) = 0.$$

Thus, it follows that

$$b = \lim_{x\to\infty}\left(\frac{-5x^2-1}{x^2+5x} - \frac{c}{x}\right) = -5.$$

Therefore

$$c = \lim_{x\to\infty}\frac{25x-1}{x+5} = 25.$$

We deduce that $a+b+c = 21$.

Problem 2. Find the product of all numbers belonging to the domain of function

$$f(x) = \sqrt{-x^2 + 11x - 24} + \sqrt{\log_2(\cos \pi x)}.$$

Solution. *Note that the domain of function $f(x)$ is the set of solutions of the following system*

$$\begin{cases} -x^2 + 11x - 24 \geq 0, \\ \log_2(\cos \pi x) \geq 0, \end{cases}$$

or equivalently

$$\begin{cases} 3 \leq x \leq 8, \\ \cos \pi x = 1. \end{cases}$$

Therefore

$$x \in \{4, 6, 8\}.$$

Hence, the product of all numbers belonging to the domain of function $f(x)$ is equal to 192.

Problem 3. Evaluate the expression

$$\int_0^{\pi} \ln \frac{2 - \cos x}{2 + \cos x} dx.$$

Solution. *We have that*

$$\int_0^{\pi} \ln \frac{2-\cos x}{2+\cos x} dx = \int_0^{\frac{\pi}{2}} \ln \frac{2-\cos x}{2+\cos x} dx + \int_{\frac{\pi}{2}}^{\pi} \ln \frac{2-\cos x}{2+\cos x} dx = \int_0^{\frac{\pi}{2}} \ln \frac{2-\cos x}{2+\cos x} dx +$$

$$+ \int_{\frac{\pi}{2}}^{0} \ln \frac{2-\cos(\pi - x)}{2+\cos(\pi - x)} d(\pi - x) = \int_0^{\frac{\pi}{2}} \ln \frac{2-\cos x}{2+\cos x} dx + \int_{\frac{\pi}{2}}^{0} \ln \frac{2-\cos x}{2+\cos x} dx =$$

$$= \int_0^{\frac{\pi}{2}} \ln \frac{2-\cos x}{2+\cos x} dx - \int_0^{\frac{\pi}{2}} \ln \frac{2-\cos x}{2+\cos x} dx = 0.$$

Problem 4. Evaluate the expression

$$\lim_{x \to 0} \frac{\sin(\tan x) - \ln(1+x)}{\sin^2 \dfrac{x}{10}}.$$

Solution. *According to L'Hospital's rule, we have that*

$$\lim_{x\to 0}\frac{\sin x - x}{x^2} = \lim_{x\to 0}\frac{\cos x - 1}{2x} = \lim_{x\to 0}\frac{-x\sin^2\frac{x}{2}}{4\left(\frac{x}{2}\right)^2} = 0,$$

$$\lim_{x\to 0}\frac{\tan x - x}{x^2} = \lim_{x\to 0}\frac{\frac{1}{\cos^2 x} - 1}{2x} = 0,$$

and

$$\lim_{x\to 0}\frac{\ln(1+x) - x}{x^2} = \lim_{x\to 0}\frac{\frac{1}{1+x} - 1}{2x} = -\frac{1}{2}.$$

Thus, it follows that

$$\lim_{x\to 0}\left(\frac{\sin(\tan x) - \tan x}{\sin^2\frac{x}{10}} + \frac{\tan x - x}{\sin^2\frac{x}{10}} - \frac{\ln(1+x) - x}{\sin^2\frac{x}{10}}\right) =$$

$$= \lim_{x\to 0}\left(\frac{\sin(\tan x) - \tan x}{(\tan x)^2}\cdot\left(\frac{\tan x}{x}\right)^2\cdot 100\cdot\left(\frac{\frac{x}{10}}{\sin\frac{x}{10}}\right)^2 + \frac{\tan x - x}{x^2}\cdot 100\cdot\left(\frac{\frac{x}{10}}{\sin\frac{x}{10}}\right)^2 -\right.$$

$$\left. - \frac{\ln(1+x) - x}{x^2}\cdot 100\cdot\left(\frac{\frac{x}{10}}{\sin\frac{x}{10}}\right)^2\right) = 0 + 0 - \left(-\frac{1}{2}\right)\cdot 100\cdot 1 = 50.$$

Therefore

$$\lim_{x\to 0}\frac{\sin(\tan x) - \ln(1+x)}{\sin^2\frac{x}{10}} = 50.$$

Problem 5. Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable function. Given that

$$\int_0^1 f(x)dx = f(1) - \frac{2\sqrt{3}}{3}.$$

Find the smallest possible value of the expression

$$\int_0^1 (f'(x))^2 dx.$$

Solution. *According to the Cauchy–Schwarz inequality, we have that*

$$\int_0^1 (f'(x))^2 dx \cdot \int_0^1 x^2 dx \geq \left(\int_0^1 f'(x) x dx \right)^2 = \left(\int_0^1 x df(x) \right)^2 =$$

$$= \left(xf(x) \mid_0^1 - \int_0^1 f(x) dx \right)^2 = \left(f(1) - \int_0^1 f(x) dx \right)^2 = \frac{4}{3}.$$

Thus, it follows that

$$\int_0^1 (f'(x))^2 dx \geq 4.$$

If $f(x) = \sqrt{3}x^2$, *then*

$$\int_0^1 \sqrt{3}x^2 dx = \frac{\sqrt{3}}{3} = \sqrt{3} - \frac{2\sqrt{3}}{3}.$$

On the other hand

$$\int_0^1 (f'(x))^2 dx = 4.$$

Therefore, the smallest possible value of the expression

$$\int_0^1 (f'(x))^2 dx,$$

is equal to 4.

Problem 6. Given that point $M(x_0, y_0)$ is the centre of symmetry of the graph of function

$$f(x) = x + 301 + \log_2 \frac{x^3 - 13x^2 + 54x - 72}{x^3 + x^2 - 2x}.$$

Find $x_0 + y_0$.

Solution. *Note that*

$$f(x) = x + 301 + \log_2 \frac{(x-3)(x-4)(x-6)}{x(x-1)(x+2)}.$$

Therefore

$$D(f) = (-\infty, -2) \cup (0,1) \cup (3,4) \cup (6, +\infty).$$

We deduce that $x_0 = 2$.

We have that

$$f(4-x) = 305 - x + \log_2 \frac{(1-x)(-x)(-x-2)}{(4-x)(3-x)(6-x)} = 305 - x +$$

$$+\log_2 \frac{x(x-1)(x+2)}{(x-3)(x-4)(x-6)} = 305 - x - \log_2 \frac{(x-3)(x-4)(x-6)}{x(x-1)(x+2)} = 305 - (f(x) - 301).$$

Thus, it follows that

$$\frac{f(x)+f(4-x)}{2} = 303.$$

We obtain that $y_0 = 303$. *Hence* $x_0 + y_0 = 305$.

Problem 7. Let x_1, x_2 be, respectively, the smallest and greatest solutions of the equation

$$\left(\frac{4}{3}\right)^{\cos x} = \sin x,$$

in the interval $[0, 2\pi]$. Find $\dfrac{3x_2}{x_1}$.

Solution. *If* $0 \le x < \dfrac{\pi}{2}$, *then* $\cos x > 0$. *Thus, it follows that*

$$\left(\frac{4}{3}\right)^{\cos x} > \left(\frac{4}{3}\right)^{0} \ge \sin x.$$

Therefore

$$\left(\frac{4}{3}\right)^{\cos x} > \sin x.$$

On the other hand

$$\left(\frac{4}{3}\right)^{\cos\frac{\pi}{2}} = 1 = \sin\frac{\pi}{2}.$$

We obtain that $x_1 = \dfrac{\pi}{2}$.

Now, let us prove that the equation

$$\left(\frac{4}{3}\right)^{\cos x} = \sin x,$$

cannot have three roots in the interval $\left[\dfrac{\pi}{2}, \pi\right)$. *This equation is equivalent to the following equation*

$$\cos x - \log_{\frac{4}{3}} \sin x = 0.$$

Let $\dfrac{\pi}{2} \le t_1 < t_2 < t_3 < \pi$ *and* $f(t_i) = 0$, *where* $i = 1, 2, 3$ *and*

$$f(x) = \cos x - \log_{\frac{4}{3}} \sin x.$$

Then, there exist numbers c_1, c_2, such that $t_1 < c_1 < t_2 < c_2 < t_3$ and

$$f'(c_1) = f'(c_2) = 0. \tag{7.105}$$

Note that the function

$$g(x) = f'(x) = -\sin x - \frac{\cos x}{\sin x \ln \frac{4}{3}},$$

is increasing on $\left[\frac{\pi}{2}, \pi\right)$*, as*

$$g'(x) = -\cos x + \frac{1}{\ln \frac{4}{3} \sin^2 x} > 0,$$

for $x \in \left[\frac{\pi}{2}, \pi\right)$.

Therefore, this leads to a contradiction (see (7.105)).

Note that

$$\left(\frac{4}{3}\right)^{\cos \frac{2\pi}{3}} = \sin \frac{2\pi}{3},$$

and if $\pi \leq x \leq 2\pi$*, then*

$$\left(\frac{4}{3}\right)^{\cos x} > 0 \geq \sin x.$$

Thus, it follows that $x_2 = \dfrac{2\pi}{3}$*. Hence, we obtain that* $\dfrac{3x_2}{x_1} = 4$.

Problem 8. Evaluate the expression

$$\frac{100}{\pi} \cdot \lim_{n\to\infty} \left(\arctan \frac{1}{2} + \arctan \frac{2}{11} + \cdots + \arctan \frac{2n+2}{n^4 + 4n^3 + 7n^2 + 6n + 4}\right).$$

Solution. *Note that if* $m \in \mathbb{N}$*, then*

$$-\frac{\pi}{2} < \arctan \frac{1}{m+2} - \arctan \frac{1}{m+1} < \arctan \frac{1}{m} - 2\arctan \frac{1}{m+1} +$$

$$+ \arctan \frac{1}{m+2} < \arctan \frac{1}{m} - \arctan \frac{1}{m+1} < \frac{\pi}{2}.$$

On the other hand

$$\tan\left(\arctan \frac{1}{m} - 2\arctan \frac{1}{m+1} + \arctan \frac{1}{m+2}\right) =$$

$$= \frac{\tan\left(\arctan\frac{1}{m} - \arctan\frac{1}{m+1}\right) - \tan\left(\arctan\frac{1}{m+1} - \arctan\frac{1}{m+2}\right)}{1 + \tan\left(\arctan\frac{1}{m} - \arctan\frac{1}{m+1}\right)\cdot\tan\left(\arctan\frac{1}{m+1} - \arctan\frac{1}{m+2}\right)} =$$

$$= \frac{\frac{1}{m^2+m+1} - \frac{1}{m^2+3m+3}}{1 + \frac{1}{m^2+m+1}\cdot\frac{1}{m^2+3m+3}} = \frac{2m+2}{m^4+4m^3+7m^2+6m+4}.$$

Therefore

$$\arctan\frac{2m+2}{m^4+4m^3+7m^2+6m+4} = \arctan\frac{1}{m} - 2\arctan\frac{1}{m+1} + \arctan\frac{1}{m+2},$$

and

$$\frac{100}{\pi}\lim_{n\to\infty}\left(\arctan\frac{1}{2} + \arctan\frac{2}{11} + \cdots + \frac{2n+2}{n^4+4n^3+7n^2+6n+4}\right) =$$

$$= \frac{100}{\pi}\lim_{n\to\infty}\left(\arctan 1 - \arctan\frac{1}{n+1} + \arctan\frac{1}{n+2}\right) = 25.$$

Problem 9. Given that $\frac{2\pi}{5}$ is the period of function

$$f(x) = a\sin^5 x + 10\sin^3 x + b\sin x.$$

Find ab.

Solution. *We have that*

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4},$$

and

$$\sin^5 x = \frac{\sin 5x - 5\sin 3x + 10\sin x}{16}.$$

Therefore

$$f(x) = A\sin 5x + B\sin 3x + C\sin x,$$

where $A = \frac{a}{16}$, $B = -\frac{5a}{16} - 2.5$, $C = \frac{5}{8}a + 7.5 + b$.

According to the assumption of the problem, $\frac{2\pi}{5}$ *is the period of function* $f(x)$. *Thus, it follows that* $\frac{2\pi}{5}$ *is the period of function*

$$g(x) = f'(x) = 5A\cos 5x + 3B\cos 3x + C\cos x.$$

Hence $g\left(x+\dfrac{2\pi}{5}\right) = g(x)$, *if* $x \in \mathbb{R}$.

If $x = 0$, *then we obtain that*

$$3B\cos\frac{6\pi}{5} + C\cos\frac{2\pi}{5} = 3B + C.$$

For $x = 0$, *from the condition*

$$f\left(x+\frac{2\pi}{5}\right) = f(x),$$

we deduce that

$$B\sin\frac{6\pi}{5} + C\sin\frac{2\pi}{5} = 0.$$

Therefore

$$C = \frac{-B\sin\dfrac{6\pi}{5}}{\sin\dfrac{2\pi}{5}}.$$

Thus, it follows that

$$3B\cos\frac{6\pi}{5} - B\frac{\sin\dfrac{6\pi}{5}\cos\dfrac{2\pi}{5}}{\sin\dfrac{2\pi}{5}} = 3B - B\frac{\sin\dfrac{6\pi}{5}}{\sin\dfrac{2\pi}{5}}.$$

If $B \neq 0$, *then*

$$3\sin\frac{2\pi}{5}\cos\frac{6\pi}{5} - \sin\frac{6\pi}{5}\cos\frac{2\pi}{5} = 3\sin\frac{2\pi}{5} - \sin\frac{6\pi}{5}.$$

Hence

$$3\sin\frac{2\pi}{5}\left(1-\cos\frac{6\pi}{5}\right) = \sin\frac{6\pi}{5}\left(1-\cos\frac{2\pi}{5}\right).$$

This leads to a contradiction, as

$$\sin\frac{2\pi}{5}\left(1-\cos\frac{6\pi}{5}\right) > 0,$$

and

$$\sin\frac{6\pi}{5}\left(1-\cos\frac{2\pi}{5}\right) < 0.$$

We deduce that $B = 0$ and $C = 0$. Therefore $a = -8$ and $b = -2.5$. Thus, it follows that $ab = 20$.

Problem 10. Find the difference of the greatest and smallest values of the function

$$f(x) = \frac{20}{\pi}\left(\arcsin x + \arcsin\left(x\sqrt{1-x^2} - \sqrt{3}x^2 + \frac{\sqrt{3}}{2}\right)\right).$$

Solution. *Let $x = \sin t$, where $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, then $\arcsin x = t$ and*

$$x\sqrt{1-x^2} - \sqrt{3}x^2 + \frac{\sqrt{3}}{2} = \sin t\cos t + \frac{\sqrt{3}}{2}(1 - 2\sin^2 t) = \sin\left(2t + \frac{\pi}{3}\right).$$

Note that

$$\arcsin\left(x\sqrt{1-x^2} - \sqrt{3}x^2 + \frac{\sqrt{3}}{2}\right) =$$

$$= \arcsin\left(\sin\left(2t + \frac{\pi}{3}\right)\right) = \begin{cases} -\frac{4\pi}{3} - 2t, \textit{if } -\frac{\pi}{2} \leq t < -\frac{5\pi}{12}, \\ 2t + \frac{\pi}{3}, \textit{if } -\frac{5\pi}{12} \leq t \leq \frac{\pi}{12}, \\ \frac{2\pi}{3} - 2t, \textit{if } \frac{\pi}{12} < t \leq \frac{\pi}{2}. \end{cases}$$

Thus, it follows that

$$\arcsin x + \arcsin\left(x\sqrt{1-x^2} - \sqrt{3}x^2 + \frac{\sqrt{3}}{2}\right) = \begin{cases} -\frac{4\pi}{3} - t, \textit{if } -\frac{\pi}{2} \leq t < -\frac{5\pi}{12}, \\ 3t + \frac{\pi}{3}, \textit{if } -\frac{5\pi}{12} \leq t \leq \frac{\pi}{12}, \\ \frac{2\pi}{3} - t, \textit{if } \frac{\pi}{12} < t \leq \frac{\pi}{2}. \end{cases}$$

Hence, the greatest and smallest values of function $f(x)$ are equal to $\frac{35}{3}$ and $-\frac{55}{3}$. Therefore, the difference of the greatest and smallest values of function $f(x)$ is equal to 30.

Problem 11. Let function $f : (0, \pi) \to \mathbb{R}$ and the inequality

$$f(x)\sin y - f(y)\sin x \leq \sqrt[3]{(x-y)^4},$$

holds true for any numbers x, y. Given that $f\left(\frac{\pi}{2}\right) = 100\sqrt{3}$. Find $f\left(\frac{2\pi}{3}\right)$.

Solution. *We have that*

$$f(x)\sin y - f(y)\sin x \leq \sqrt[3]{(x-y)^4},$$

where $x, y \in (0, \pi)$. Thus, it follows that

$$f(y)\sin x - f(x)\sin y \le \sqrt[3]{(y-x)^4}.$$

Therefore

$$|f(x)\sin y - f(y)\sin x| \le \sqrt[3]{(x-y)^4}.$$

Hence, if $x, y \in (0, \pi)$, then

$$\left|\frac{f(x)}{\sin x} - \frac{f(y)}{\sin y}\right| \le \frac{1}{\sin x \cdot \sin y} \cdot \sqrt[3]{(x-y)^4}. \tag{7.106}$$

Note that if $x, y \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, then $\sin x, \sin y \in \left[\frac{\sqrt{3}}{2}, 1\right]$. We obtain that

$$\frac{1}{\sin x \sin y} \le \frac{4}{3}. \tag{7.107}$$

Thus, if $x, y \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, then from (7.106) and (7.107) we obtain that

$$\left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| \le \frac{4}{3}\sqrt[3]{(x-y)^4}. \tag{7.108}$$

Let us choose a positive integer $n > 1$. According to (7.108), we have that

$$\left|\frac{f\left(\frac{2\pi}{3}\right)}{\sin\frac{2\pi}{3}} - \frac{f\left(\frac{\pi}{2}\right)}{\sin\frac{\pi}{2}}\right| = \left|\frac{f\left(\frac{2\pi}{3}\right)}{\sin\frac{2\pi}{3}} - \frac{f\left(\frac{\pi}{2}+\frac{\pi}{6n}(n-1)\right)}{\sin\left(\frac{\pi}{2}+\frac{\pi}{6n}(n-1)\right)} + \frac{f\left(\frac{\pi}{2}+\frac{\pi}{6n}(n-1)\right)}{\sin\left(\frac{\pi}{2}+\frac{\pi}{6n}(n-1)\right)} - \right.$$

$$\left. - \frac{f\left(\frac{\pi}{2}+\frac{\pi}{6n}(n-2)\right)}{\sin\left(\frac{\pi}{2}+\frac{\pi}{6n}(n-2)\right)} + \cdots + \frac{f\left(\frac{\pi}{2}+\frac{\pi}{6n}\right)}{\sin\left(\frac{\pi}{2}+\frac{\pi}{6n}\right)} - \frac{f\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)}\right| \le$$

$$\le \left|\frac{f\left(\frac{2\pi}{3}\right)}{\sin\frac{2\pi}{3}} - \frac{f\left(\frac{\pi}{2}+\frac{\pi}{6n}(n-1)\right)}{\sin\left(\frac{\pi}{2}+\frac{\pi}{6n}(n-1)\right)}\right| + \cdots + \left|\frac{f\left(\frac{\pi}{2}+\frac{\pi}{6n}\right)}{\sin\left(\frac{\pi}{2}+\frac{\pi}{6n}\right)} - \frac{f\left(\frac{\pi}{2}\right)}{\sin\frac{\pi}{2}}\right| \le$$

$$\le n \cdot \frac{4}{3}\sqrt[3]{\left(\frac{\pi}{6n}\right)^4} = \frac{2\pi}{9} \cdot \sqrt[3]{\frac{\pi}{6}} \cdot \frac{1}{\sqrt[3]{n}}.$$

Thus, it follows that

$$\lim_{n\to\infty}\left|\frac{f\left(\frac{2\pi}{3}\right)}{\frac{\sqrt{3}}{2}}-f\left(\frac{\pi}{2}\right)\right|\le\lim_{n\to\infty}\left(\frac{2\pi}{9}\cdot\sqrt[3]{\frac{\pi}{6}}\cdot\frac{1}{\sqrt[3]{n}}\right)=0.$$

Therefore

$$f\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}f\left(\frac{\pi}{2}\right)=150.$$

Problem 12. Let C be the smallest real number, such that the inequality

$$a^{12}+(ab)^6+(abc)^4+(abcd)^3\le C(a^{12}+b^{12}+c^{12}+d^{12}),$$

holds true for any real numbers a, b, c, d. Find $[100C]$, where we denote by $[x]$ the integer part of a real number x.

Solution. *Without loss of generality, one can assume that* $a\ge 0$, $b\ge 0$, $c\ge 0$, $d\ge 0$.

Let x, y, z *be any positive numbers, according to Cauchy's inequality we have that*

$$a^{12}+(ab)^6+(abc)^4+(abcd)^3=a^{12}+\frac{1}{x^6}(xa\cdot b)^6+\frac{1}{x^4y^8}(xya\cdot yb\cdot c)^4+$$

$$\frac{1}{x^3y^6z^9}(xyza\cdot yzb\cdot zc\cdot d)^3\le a^{12}+\frac{1}{2x^6}(x^{12}a^{12}+b^{12})+\frac{1}{3x^4y^8}(x^{12}y^{12}a^{12}+y^{12}b^{12}+c^{12})+$$

$$\frac{1}{4x^3y^6z^9}(x^{12}y^{12}z^{12}a^{12}+y^{12}z^{12}b^{12}+z^{12}c^{12}+d^{12})=A(a^{12}+b^{12}+c^{12}+d^{12}).$$

One can choose x, y, z, *such as*

$$1+\frac{x^6}{2}+\frac{x^8y^4}{3}+\frac{x^9y^6z^3}{4}=\frac{1}{2x^6}+\frac{y^4}{3x^4}+\frac{y^6z^3}{4x^3}=\frac{1}{3x^4y^8}+\frac{z^3}{4x^3y^6}=\frac{1}{4x^3y^6z^9}=A.$$

That is

$$x^{12}=1-\frac{1}{A},\ \ y^{12}=1-\frac{1}{2\sqrt{A(A-1)}},\ \ z^{12}=1-\frac{1}{3\sqrt[3]{A(\sqrt{A(A-1)}-0.5)^2}},$$

and

$$\frac{256}{27}A\left(3\sqrt[3]{A(\sqrt{A(A-1)}-0.5)^2}-1\right)^3=1. \tag{7.109}$$

Let us consider the function

$$f(t)=\frac{256}{27}t\left(3\sqrt[3]{t(\sqrt{t(t-1)}-0.5)^2}-1\right)^3-1,$$

on the interval $[1.42, 1.43]$. *Note that*

$$f(1.42) = \frac{256}{27} \cdot 1.42\left(3\sqrt[3]{1.42(\sqrt{1.42 \cdot 0.42} - 0.5)^2} - 1\right)^3 - 1 <$$

$$< \frac{256}{27} \cdot 1.42(3\sqrt[3]{1.42 \cdot 0.2723^2} - 1)^3 - 1 < \frac{256}{27} \cdot 1.42(3 \cdot 0.4723 - 1)^3 - 1 < 0,$$

and

$$f(1.43) = \frac{256}{27} \cdot 1.42\left(3\sqrt[3]{1.43(\sqrt{1.43 \cdot 0.43} - 0.5)^2} - 1\right)^3 - 1 >$$

$$> \frac{256}{27} \cdot 1.43(3\sqrt[3]{1.43 \cdot 0.08} - 1)^3 - 1 > \frac{256}{27} \cdot 1.43(3 \cdot 0.48 - 1)^3 - 1 > 0.$$

Therefore, there exists a number A*, such that* $1.42 < A < 1.43$ *and* $f(A) = 0$. *As* (7.109) *holds true, hence* $x > 0$, $y > 0$, $z > 0$, $(A > 1.42)$ *and*

$$a^{12} + (ab)^6 + (abc)^4 + (abcd)^3 \le A(a^{12} + b^{12} + c^{12} + d^{12}) \le 1.43(a^{12} + b^{12} + c^{12} + d^{12}).$$

Therefore, we have that $C < 1.43$.

Now, if we choose real numbers a, b, c, d *such that* $xa = b$, $yb = c$, $d = zc$, *then we obtain that*

$$A(a^{12} + b^{12} + c^{12} + d^{12}) \le C(a^{12} + b^{12} + c^{12} + d^{12})$$

Thus, it follows that $1.42 < A \le C$. *Hence, we deduce that* $[100C] = 142$.

7.4.14 Problem Set 14

Problem 1. Let

$$f(x) = \log_2 \frac{2^x + 1}{2^x - 1}.$$

Evaluate the expression

$$f(f(1)) + f(f(2)) + \cdots + f(f(40)).$$

Solution. *Note that if* $x > 0$, *then*

$$f(f(x)) = \log_2 \frac{2^{\log_2 \frac{2^x+1}{2^x-1}} + 1}{2^{\log_2 \frac{2^x+1}{2^x-1}} - 1} = \log_2 \frac{2 \cdot 2^x}{2} = x.$$

Thus, it follows that

$$f(f(1))+f(f(2))+\cdots+f(f(40))=1+2+\cdots+40=820.$$

Problem 2. Let a, b, c, d be real numbers. Given that sequence

$$x_n=\sqrt[3]{an^3+bn^2+cn+d}-3n-1,$$

is convergent. Find a.

Solution. *We have that*

$$x_n=\sqrt[3]{an^3+bn^2+cn+d}-3n-1=$$

$$=\frac{an^3+bn^2+cn+d-(3n+1)^3}{(\sqrt[3]{an^3+bn^2+cn+d})^2+\sqrt[3]{an^3+bn^2+cn+d}(3n+1)+(3n+1)^2}=$$

$$=\frac{(a-27)n+b-27+\dfrac{c-9}{n}+\dfrac{d-1}{n^2}}{\left(\sqrt[3]{a+\dfrac{b}{n}+\dfrac{c}{n^2}+\dfrac{c}{n^3}}\right)^2+\sqrt[3]{a+\dfrac{b}{n}+\dfrac{c}{n^2}+\dfrac{c}{n^3}}\left(3+\dfrac{1}{n}\right)+\left(3+\dfrac{1}{n}\right)^2}.$$

On the other hand

$$\lim_{n\to\infty}\left(\left(\sqrt[3]{a+\frac{b}{n}+\frac{c}{n^2}+\frac{c}{n^3}}\right)^2+\sqrt[3]{a+\frac{b}{n}+\frac{c}{n^2}+\frac{c}{n^3}}\left(3+\frac{1}{n}\right)+\left(3+\frac{1}{n}\right)^2\right)=$$

$$=\sqrt[3]{a^2}+3\sqrt[3]{a}+9,$$

and

$$\lim_{n\to\infty}\left(b-27+\frac{c-9}{n}+\frac{d-1}{n^2}\right)=b-27.$$

Hence, sequence (x_n) *is convergent only if* $a=27$.

Therefore $a=27$.

Problem 3. Evaluate the expression

$$\int_{-4}^{4}\frac{\sqrt{1+x^2+x^4}}{1+x+x^2+\sqrt{1+x^2+x^4}}dx.$$

Solution. *We have that*

$$\int_{-4}^{4}\frac{\sqrt{1+x^2+x^4}}{1+x+x^2+\sqrt{1+x^2+x^4}}dx=$$

$$= \int_{-4}^{0} \frac{\sqrt{1+x^2+x^4}}{1+x+x^2+\sqrt{1+x^2+x^4}}dx + \int_{0}^{4} \frac{\sqrt{1+x^2+x^4}}{1+x+x^2+\sqrt{1+x^2+x^4}}dx =$$

$$= \int_{4}^{0} \frac{\sqrt{1+(-x)^2+(-x)^4}}{1-x+(-x)^2+\sqrt{1+(-x)^2+(-x)^4}}d(-x) + \int_{0}^{4} \frac{\sqrt{1+x^2+x^4}}{1+x+x^2+\sqrt{1+x^2+x^4}}dx =$$

$$= \int_{0}^{4} \left(\frac{\sqrt{1+x^2+x^4}}{1-x+x^2+\sqrt{1+x^2+x^4}} + \frac{\sqrt{1+x^2+x^4}}{1+x+x^2+\sqrt{1+x^2+x^4}} \right) dx =$$

$$= \int_{0}^{4} \frac{2\sqrt{1+x^2+x^4}(1+x^2+\sqrt{1+x^2+x^4})}{(1+x^2+\sqrt{1+x^2+x^4})^2-x^2}dx = \int_{0}^{4} 1dx = 4.$$

Therefore

$$\int_{-4}^{4} \frac{\sqrt{1+x^2+x^4}}{1+x+x^2+\sqrt{1+x^2+x^4}}dx = 4.$$

Problem 4. Find the sum of all positive integers a less than 30, such that the domains of functions

$$y = \frac{\sin x}{9\cos^2 x - 1},$$

and

$$y = \frac{1}{9\cos^2 x - 1} + \frac{1}{9\cos 2x + a},$$

coincide.

Solution. *The domains of those functions coincide, if either the equation* $9\cos 2x + a = 0$ *does not have solution or the solutions of* $9\cos 2x + a = 0$ *are also solutions to* $9\cos^2 x - 1 = 0$.

In the first case, we obtain that $a = 10, 11, \ldots, 29$.

In the second case, from the following identity

$$9\cos 2x + a = 18\cos^2 x + a - 9,$$

we obtain that

$$\frac{9-a}{18} = \frac{1}{9}.$$

Thus, it follows that $a = 7$.

Hence, the sum of all such positive integers a (less than 30) is equal to 397.

Problem 5. Given that m, n, k are such real numbers, that

$$\int \sin x \ln(1+\sin x)dx = mx + n\cos x + k\cos x \ln(1+\sin x) + C.$$

Find $m^2 + n^2 + k^2$.

Solution. *We have that*

$$(mx+n\cos x+k\cos x\ln(1+\sin x)+C)' = m-n\sin x+k\cos x\cdot\frac{\cos x}{1+\sin x}-$$

$$-k\sin x\ln(1+\sin x) = m+k-(n+k)\sin x-k\sin x\ln(1+\sin x).$$

If $k=-1$, $n+k=m+k=0$, *then*

$$\int \sin x\ln(1+\sin x)dx = x+\cos x-\cos x\ln(1+\sin x)+C.$$

Now, let us prove that if for any x *belonging to the domain of the function* $\sin x\ln(1+\sin x)$ *it holds true*

$$mx+n\cos x+k\cos x\ln(1+\sin x) = x+\cos x-\cos x\ln(1+\sin x)+l, \qquad (7.110)$$

where m, n, k, l *are constants, then* $m=1$, $n=1$, $k=-1$, $l=0$.

If $\sin x=1$, *then from* (7.115), *we deduce that* $mx=x+l$. *Hence, the equation* $mx=x+l$ *has infinitely many solutions. Therefore* $m=1$, $l=0$.

If $x=0$, *then we obtain that* $n=1$.

If $x=\dfrac{\pi}{3}$, *then we obtain that* $k=-1$.

Therefore $m=1$, $n=1$, $k=-1$.

Thus, it follows that

$$m^2+n^2+k^2=3.$$

Problem 6. Evaluate the expression

$$\lim_{x\to 0}\frac{\tan(e^x-1)-\ln(1+\sin x)}{\sqrt[3]{1+x^2}-1}.$$

Solution. *Note that according to L'Hospital's rule, we have that*

$$\lim_{x\to 0}\frac{\ln(1+\sin x)-x}{x^2} = \lim_{x\to 0}\frac{\dfrac{\cos x}{1+\sin x}-1}{2x} = \lim_{x\to 0}\frac{\cos x-1-\sin x}{2x(1+\sin x)} =$$

$$= \lim_{x\to 0}\frac{-\sin^2\dfrac{x}{2}-\sin\dfrac{x}{2}\cos\dfrac{x}{2}}{x(1+\sin x)} = \lim_{x\to 0}\left(\frac{1}{2}\cdot\frac{\dfrac{-\sin\frac{x}{2}}{\frac{x}{2}}\cdot\sin\dfrac{x}{2}-\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\cos\dfrac{x}{2}}{1+\sin x}\right) = -\frac{1}{2}.$$

On the other hand, we have that

$$\lim_{x\to 0}\frac{\tan(e^x-1)-x}{x^2} = \lim_{x\to 0}\frac{\dfrac{e^x}{\cos^2(e^x-1)}-1}{2x} = \lim_{x\to 0}\frac{e^x-\cos^2(e^x-1)}{\cos^2(e^x-1)\cdot 2x}. =$$

$$= \lim_{x\to 0}\left(\frac{e^x-1}{x}\cdot\frac{1}{2\cos^2(e^x-1)}+\frac{\sin^2(e^x-1)}{2x\cos^2(e^x-1)}\right)=\frac{1}{2}.$$

Thus, it follows that

$$\lim_{x\to 0}\frac{\tan(e^x-1)-\ln(1+\sin x)}{\sqrt[3]{1+x^2}-1}=$$

$$=\lim_{x\to 0}\left(\frac{\tan(e^x-1)-x}{x^2}+\frac{x-\ln(1+\sin x)}{x^2}\right)\left(\sqrt[3]{(1+x^2)^2}+\sqrt[3]{1+x^2}+1\right)=$$

$$=\left(\frac{1}{2}+\frac{1}{2}\right)\cdot 3=3.$$

Problem 7. Let non-decreasing, continuous function $f(x)$ be defined on $\left[0,\frac{\pi}{2}\right]$. Given that

$$\int_0^{\frac{\pi}{2}} f(x)dx=10,$$

and

$$\int_0^{\frac{\pi}{2}} f(x)\sin^2 x dx=5.$$

Find

$$\int_0^{\frac{\pi}{4}} f(x)dx.$$

Solution. *Note that* $f\left(\frac{\pi}{2}-x\right)-f(x)$ *and* $\cos 2x$ *have the same sign. Therefore*

$$\left(f\left(\frac{\pi}{2}-x\right)-f(x)\right)\cos 2x\geq 0.$$

On the other hand

$$\int_0^{\frac{\pi}{2}}\left(f\left(\frac{\pi}{2}-x\right)-f(x)\right)\cos 2xdx=\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}-x\right)\cos 2xdx-\int_0^{\frac{\pi}{2}} f(x)\cos 2xdx=$$

$$=\int_{\frac{\pi}{2}}^{0} f(t)\cos 2\left(\frac{\pi}{2}-t\right)d\left(\frac{\pi}{2}-t\right)-\int_0^{\frac{\pi}{2}} f(x)\cos 2xdx=$$

$$-\int_0^{\frac{\pi}{2}} f(t)\cos 2t dt - \int_0^{\frac{\pi}{2}} f(x)\cos 2x dx = -2\int_0^{\frac{\pi}{2}} f(x)\cos 2x dx =$$

$$= -2\left(\int_0^{\frac{\pi}{2}} f(x)dx - 2\int_0^{\frac{\pi}{2}} f(x)\sin^2 x dx\right) = 0.$$

Thus, for $x \in \left[0, \frac{\pi}{2}\right]$*, it follows that*

$$\left(f\left(\frac{\pi}{2} - x\right) - f(x)\right)\cos 2x = 0.$$

Therefore, for $x \in \left[0, \frac{\pi}{2}\right]$*, it follows that*

$$f\left(\frac{\pi}{2} - x\right) - f(x) = 0.$$

Hence, we deduce that

$$10 = \int_0^{\frac{\pi}{2}} f(x)dx = \int_0^{\frac{\pi}{4}} f(x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)dx =$$

$$= \int_0^{\frac{\pi}{4}} f(x)dx + \int_{\frac{\pi}{4}}^{0} f\left(\frac{\pi}{2} - t\right) d\left(\frac{\pi}{2} - t\right) = \int_0^{\frac{\pi}{4}} f(x)dx + \int_0^{\frac{\pi}{4}} f(t)dt = 2\int_0^{\frac{\pi}{4}} f(x)dx.$$

We obtain that

$$\int_0^{\frac{\pi}{4}} f(x)dx = 5.$$

Problem 8. Let (x_n) be a sequence of positive real numbers, such that

$$\lim_{n\to\infty} \frac{\sqrt{1+x_{n+1}} - 31}{\sqrt{1+x_n} - 31} = 0.1.$$

Find the limit of the sequence (x_n).

Solution. *Let* n_0 *be a positive integer, such that for* $n = n_0, n_0 + 1, \ldots$ *it holds true*

$$0.05 < \frac{\sqrt{1+x_{n+1}} - 31}{\sqrt{1+x_n} - 31} < 0.15.$$

Therefore, if $n > n_0$, then

$$0.05 < \frac{\sqrt{1+x_{n_0+1}}-31}{\sqrt{1+x_{n_0}}-31} < 0.15.$$

$$0.05 < \frac{\sqrt{1+x_{n_0+2}}-31}{\sqrt{1+x_{n_0+1}}-31} < 0.15.$$

$$\dots$$

$$0.05 < \frac{\sqrt{1+x_{n+1}}-31}{\sqrt{1+x_n}-31} < 0.15.$$

Multiplying all these equations, we obtain that

$$|\sqrt{1+x_{n_0}}-31|\cdot 0.05^{n-n_0+1} < |\sqrt{1+x_{n+1}}-31| < |\sqrt{1+x_{n_0}}-31|\cdot 0.15^{n-n_0+1}.$$

Thus, it follows that

$$\lim_{n\to\infty}|\sqrt{1+x_{n+1}}-31| = 0.$$

We deduce that

$$\lim_{n\to\infty}\sqrt{1+x_{n+1}} = 31.$$

Therefore

$$\lim_{n\to\infty}(\sqrt{1+x_{n+1}})^2 = 31^2.$$

Hence

$$\lim_{n\to\infty} x_{n+1} = 960.$$

We obtain that

$$\lim_{n\to\infty} x_n = 960.$$

Problem 9. Solve the equation

$$\log_{25}(625x)\log_{25}(\log_{25}x) = \log_{25}x\log_{25}(\log_{25}(625x)).$$

Solution. *Note that the given equation is equivalent to the equation*

$$\frac{\ln(\log_{25}x)}{\log_{25}x} = \frac{\ln(\log_{25}x+2)}{\log_{25}x+2}. \tag{7.111}$$

Let us consider the function

$$f(x) = \frac{\ln x}{x},$$

on $(0, +\infty)$. *We have that*

$$f'(x) = \frac{1 - \ln x}{x^2}.$$

Therefore, function $f(x)$ *is increasing on* $(0, e]$ *and decreasing on* $[e, +\infty)$.

Now, let us prove the following lemma.

Lemma 7.14. *Let*

$$f(x) = \frac{\ln x}{x}.$$

Prove that $x = 2$ *is the unique solution of the equation* $f(x) = f(x+2)$.

Proof. *We have that*

$$f(2) = \frac{\ln 2}{2} = \frac{\ln 4}{4} = f(4).$$

If $0 < x < 1$, *then* $f(x) < 0 < f(x+2)$.

If $1 \leq x < 2$, *then* $f(x) < f(2) = f(4) < f(x+2)$. *Therefore* $f(x) < f(x+2)$.

If $2 < x < e$, *then* $f(x) > f(2) = f(4) > f(x+2)$.

If $x \geq e$, *then* $f(x) > f(x+2)$.

This ends the proof of the lemma.

According to the lemma, (7.111) is equivalent to $\log_{25} x = 2$. Thus, it follows that $x = 625$.

Problem 10. Evaluate the expression

$$\frac{2016}{\pi} \sum_{n=1}^{\infty} \arccos \frac{n^2 + n + 3}{\sqrt{n^2 + 3} \cdot \sqrt{n^2 + 2n + 4}}.$$

Solution. *Let us consider points* $A_n\left(n, \frac{n^2}{\sqrt{3}}\right)$ *of the graph of function* $y = \frac{1}{\sqrt{3}}x^2$, *where* $n = 1, 2, \ldots$

Let point O be the vertex of that parabola. We have that

$$\sum_{n=1}^{\infty} \angle A_n 0 A_{n+1} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}.$$

Note that from the equation

$$\overrightarrow{OA_n} \cdot \overrightarrow{OA_{n+1}} = |\overrightarrow{OA_n}||\overrightarrow{OA_{n+1}}| \cos(\angle A_n O A_{n+1}),$$

it follows that

$$\angle A_n O A_{n+1} = \arccos \frac{n^2 + n + 3}{\sqrt{n^2 + 3} \cdot \sqrt{n^2 + 2n + 4}}.$$

Therefore

$$\frac{2016}{\pi}\sum_{n=1}^{\infty}\arccos\frac{n^2+n+3}{\sqrt{n^2+3}\cdot\sqrt{n^2+2n+4}}=\frac{2016}{\pi}\cdot\sum_{n=1}^{\infty}\angle A_nOA_{n+1}=\frac{2016}{\pi}\cdot\frac{\pi}{3}=672.$$

Problem 11. Let $x_1, x_2, \ldots, x_{2016} \geq 0$ and $x_1+x_2+\cdots+x_{2016}=1$. Given that the greatest value of the expression

$$x_1^5(1-x_1)+x_2^5(1-x_2)+\cdots+x_{2016}^5(1-x_{2016}),$$

is equal to M. Find $(27M+14)^2$.

Solution. *At first, let us prove the following lemma.*

Lemma 7.15. *If $x, y \geq 0$ and $x+y \leq \frac{2}{3}$, then*

$$x^5(1-x)+y^5(1-y)\leq(x+y)^5(1-x-y). \tag{7.112}$$

Proof. If $x+y=0$, then (7.112) holds true.

If $x+y\neq 0$, then we have that

$$\frac{x^5(1-x)}{(x+y)^5}+\frac{y^5(1-y)}{(x+y)^5}\leq\left(\frac{x}{x+y}\right)^2(1-x)+\left(\frac{y}{x+y}\right)^2(1-y)=$$

$$=\frac{(x+y)^2(1-x-y)+xy(3(x+y)-2)}{(x+y)^2}\leq 1-x-y.$$

Thus, it follows that

$$x^5(1-x)+y^5(1-y)\leq(x+y)^5(1-x-y).$$

This ends the proof of the lemma.

Without loss of generality, one can assume that $x_1\leq x_2\leq\cdots\leq x_{2016}$. Therefore

$$2014x_1+2014x_2\leq x_3+x_4+\cdots+x_{2016}+x_3+x_4+\cdots+x_{2016}=2-2x_1-2x_2.$$

We deduce that

$$x_1+x_2\leq\frac{2}{2016}<\frac{2}{3}.$$

According to the lemma, we have that

$$x_1^5(1-x_1)+x_2^5(1-x_2)+\cdots+x_{2016}^5(1-x_{2016})\leq$$

$$\leq(x_1+x_2)^5(1-x_1-x_2)+x_3^5(1-x_3)+\cdots+x_{2016}^5(1-x_{2016}).$$

Note that numbers $x_1+x_2, x_3, \ldots, x_{2016}$ are non-negative and their sum is equal to 1.

Repeating these steps several times, we deduce that

$$x_1^5(1-x_1)+x_2^5(1-x_2)+\cdots+x_{2016}^5(1-x_{2016}) \le x^5(1-x)+y^5(1-y),$$

where x, y *are non-negative numbers and* $x+y=1$.

On the other hand, if $x_1=\cdots=x_{2014}=0$, $x_{2015}=x$, $x_{2016}=1-x$, *then we have that*

$$x_1^5(1-x_1)+x_2^5(1-x_2)+\cdots+x_{2016}^5(1-x_{2016}) = x^5(1-x)+y^5(1-y).$$

Therefore, M *is the greatest value of function* $x^5(1-x)+y^5(1-y)$ *on* $[0,1]$.

Let $f(x)=x^5(1-x)+y^5(1-y)$, *where* $x\in[0,1]$.

Note that

$$f(x)=x(1-x)-4(x(1-x))^2+2(x(1-x))^3.$$

Hence, M *is the greatest value of function* $g(t)=t-4t^2+2t^3$ *on* $\left[0,\frac{1}{4}\right]$.

Thus, it follows that

$$M=\frac{5\sqrt{10}-14}{27}.$$

We obtain that

$$(27M+14)^2=250.$$

Problem 12. Let continuous function $f(x)$ be defined on $[1,17]$. Given that the equation

$$\frac{f^2(x)-24f(x)+x^2-18x+125}{\sqrt{f^2(x)-36f(x)+x^2-2x+325}\cdot\sqrt{f^2(x)-12f(x)+x^2-34x+325}}=$$

$$=\frac{f^2(y)-24f(y)+y^2-18y+125}{\sqrt{f^2(y)-36f(y)+y^2-2y+325}\cdot\sqrt{f^2(y)-12f(y)+y^2-34y+325}}.$$

holds true for any numbers x, y belonging to $(1,17)$. Given also that $f(10)=15$. Find $2|f'(15)|$.

Solution. *Note that*

$$\frac{f^2(x)-24f(x)+x^2-18x+125}{\sqrt{f^2(x)-36f(x)+x^2-2x+325}\cdot\sqrt{f^2(x)-12f(x)+x^2-34x+325}}=$$

$$=\frac{(18-f(x))(6-f(x))+(1-x)(17-x)}{\sqrt{(18-f(x))^2+(1-x)^2}\cdot\sqrt{(16-f(x))^2+(17-x)^2}}.$$

Let us consider the points $A(1,18)$, $B(17,6)$, $X(x,f(x))$, $Y(y,f(y))$ *on the coordinate plane.*

The given condition can be rewritten as

$$\frac{\overrightarrow{XA}\cdot\overrightarrow{XB}}{|\overrightarrow{XA}|\cdot|\overrightarrow{XB}|}=\frac{\overrightarrow{YA}\cdot\overrightarrow{YB}}{|\overrightarrow{YA}|\cdot|\overrightarrow{YB}|},$$

or equivalently

$$\cos(\angle AXB)=\cos(\angle AYB).$$

Thus, it follows that

$$\angle AXB=\angle AYB. \tag{7.113}$$

Let us consider also the point $C(10,15)$.

From (7.113), *we obtain that for any* x *belonging to* $(1,17)$ *it holds true*

$$\angle AXB=\angle ACB. \tag{7.114}$$

Note that the points A, B, C are on the circle ω *defined by the equation* $x^2+y^2=325$.

From (7.114), *it follows that if* $x\in(1,17)$, *then point* X *is either on the minor arc ACB or on its symmetric arc with respect to line AB. Let us prove that the second case is not possible. We proceed by contradiction argument. Assume that point* X *is on the symmetric arc of the minor arc ACB with respect to line AB, then the graph of continuous function* $y=f(x)$ *intersects with line AB at the point with abscissa x, where* $1<x<17$. *This leads to a contradiction.*

Therefore, if $x\in(1,17)$, *then*

$$f(x)=\sqrt{325-x^2}.$$

Thus, it follows that

$$2|f'(15)|=2\left|\frac{-15}{\sqrt{325-15^2}}\right|=3.$$

7.4.15 Problem Set 15

Problem 1. Let $f(x)=x^{\sqrt[3]{2}}$. Evaluate the expression

$$f(f(f(1)))+f(f(f(2)))+\cdots+f(f(f(13))).$$

Solution. *Note that*

$$f(f(x))=\left(x^{\sqrt[3]{2}}\right)^{\sqrt[3]{2}}=x^{\sqrt[3]{4}},$$

and

$$f\left(f\left(f(x)\right)\right)=\left(f\left(f(x)\right)\right)^{\sqrt[3]{2}}=\left(x^{\sqrt[3]{4}}\right)^{\sqrt[3]{2}}=x^2.$$

Thus, it follows that

$$f\left(f\left(f(1)\right)\right)+f\left(f\left(f(2)\right)\right)+\cdots+f\left(f\left(f(13)\right)\right)=1^2+2^2+\cdots+13^2=819.$$

Problem 2. Evaluate the expression

$$\lim_{n\to\infty}\left(\frac{1^2-2\cdot1+3}{1^2+2\cdot1+3}\cdot\frac{2^2-2\cdot2+3}{2^2+2\cdot2+3}\cdot\dots\cdot\frac{n^2-2\cdot n+3}{n^2+2\cdot n+3}\cdot n^4\right).$$

Solution. *Note that*

$$k^2+2k+3=(k+2)^2-2(k+2)+3.$$

Thus, it follows that

$$\frac{1^2-2\cdot1+3}{1^2+2\cdot1+3}\cdot\frac{2^2-2\cdot2+3}{2^2+2\cdot2+3}\cdot\dots\cdot\frac{n^2-2\cdot n+3}{n^2+2\cdot n+3}=$$

$$=\frac{(1^2-2\cdot1+3)(2^2-2\cdot2+3)}{((n-1)^2+2(n-1)+3)(n^2+2n+3)}=\frac{6}{(n^2+2)(n^2+2n+3)}.$$

Hence, we obtain that

$$\lim_{n\to\infty}\left(\frac{1^2-2\cdot1+3}{1^2+2\cdot1+3}\cdot\frac{2^2-2\cdot2+3}{2^2+2\cdot2+3}\cdot\dots\cdot\frac{n^2-2\cdot n+3}{n^2+2\cdot n+3}\cdot n^4\right)=$$

$$=\lim_{n\to\infty}\frac{6}{\left(1+\frac{2}{n^2}\right)\left(1+\frac{2}{n}+\frac{3}{n^2}\right)}=6.$$

Problem 3. Let $f(x)=\dfrac{x}{\sqrt{1-x^2}}$. Find $\underbrace{f\left(f\left(\dots\left(f\right.\right.\right.}_{2016}\left(\frac{1}{\sqrt{2017}}\right)\left.\left.\left.\right)\dots\right)\right)$.

Solution. *Let us prove, by mathematical induction, that*

$$\underbrace{f\left(f\left(\dots\left(f\right.\right.\right.}_{n}(x)\left.\left.\left.\right)\dots\right)\right)=\frac{x}{\sqrt{1-nx^2}},$$

where $x^2<\dfrac{1}{2016}$ *and* $n=1,\dots,2016$.

Basis. *If* $n=1$, *then the statement holds true, as we have that* $f(x)=\dfrac{x}{\sqrt{1-1\cdot x^2}}$.

Inductive step. *Let*

$$\underbrace{f(f(\dots(f}_{k}(x))\dots)) = \frac{x}{\sqrt{1-kx^2}},$$

where $x^2 < \dfrac{1}{2016}$ *and* $k < 2016$. *Then*

$$\underbrace{f(f(\dots(f}_{k+1}(x))\dots)) = f(\underbrace{f(f(\dots(f}_{k}(x))\dots))) = f\left(\frac{x}{\sqrt{1-kx^2}}\right) =$$

$$= \frac{\dfrac{x}{\sqrt{1-kx^2}}}{\sqrt{1-\dfrac{x^2}{1-kx^2}}} = \frac{x}{\sqrt{1-(k+1)x^2}}.$$

Thus, it follows that

$$\underbrace{f(f(\dots(f}_{2016}\left(\frac{1}{\sqrt{2017}}\right))\dots)) = \frac{\dfrac{1}{\sqrt{2017}}}{\sqrt{1-\dfrac{2016}{2017}}} = 1.$$

Problem 4. Find the sum of all integer numbers belonging to the range of the function

$$f(x) = \sqrt{x-1} + \sqrt{11-x} + \cos \pi x.$$

Solution. *Note that*

$$f(x) = \sqrt{\left(\sqrt{x-1}+\sqrt{11-x}\right)^2} + \cos \pi x = \sqrt{10 + 2\sqrt{x-1}\sqrt{11-x}} + \cos \pi x \geq \sqrt{10} - 1.$$

On the other hand, $f(1) = \sqrt{10} - 1$. *Hence, the smallest value of function* $f(x)$ *is equal to* $\sqrt{10} - 1$.

We have that

$$f(x) = \sqrt{20 - \left(\sqrt{x-1} - \sqrt{11-x}\right)^2} + \cos \pi x \leq \sqrt{20} + 1 = 2\sqrt{5} + 1.$$

On the other hand, $f(6) = 2\sqrt{5} + 1$. *Hence, the greatest value of function* $f(x)$ *is equal to* $2\sqrt{5} + 1$.

Therefore, the range of continuous function $f(x)$ *is* $[\sqrt{10} - 1, 2\sqrt{5} + 1]$. *We obtain that all integer numbers belonging to the range of function* $f(x)$ *are 3, 4, 5 and their sum is equal to 12.*

Problem 5. Given that

$$f(x) = (x^2+x+1)^{20}(x^2-x+1)^{10}.$$

Evaluate the expression

$$\frac{6f^{(59)}(1)}{f^{(60)}(1)}.$$

Solution. *Note that*

$$(x^2+x+1)^{20} = (x^2)^{20} + 20(x^2)^{19}(x+1) + C_{20}^2(x^2)^{18}(x+1)^2 + \cdots + (x+1)^{20} =$$

$$= x^{40} + 20x^{39} + p(x),$$

where $p(x)$ is such polynomial that $deg(p(x)) = 38$. In a similar way, we obtain that

$$(x^2-x+1)^{10} = x^{20} - 10x^{19} + q(x),$$

where $q(x)$ is such polynomial that $deg(q(x)) = 18$.

Therefore, there exists a polynomial $r(x)$, such that $deg r(x) = 58$ and

$$(x^2+x+1)^{20}(x^2-x+1)^{10} = x^{60} + 10x^{59} + r(x).$$

Thus, it follows that

$$f^{(59)}(x) = 60!x + 10 \cdot 59!,$$

and

$$f^{(60)}(x) = 60!.$$

Hence, we obtain that

$$\frac{6f^{(59)}(1)}{f^{(60)}(1)} = \frac{6 \cdot 60! + 60 \cdot 59!}{60!} = 7.$$

Problem 6. Find the number of all couples (a,b), where $a,b \in \mathbb{R}$, such that

$$f(x) = \ln \frac{3 + a\sin x}{b + 5\sin x},$$

is an odd function.

Solution. *We have that the domain of function $f(x)$ is the set of solutions of the following inequality*

$$\frac{3 + a\sin x}{b + 5\sin x}.$$

Hence, if it is not an empty set, then it includes such an interval that for any x belonging to this interval it holds true

$$f(-x) = \ln \frac{3 - a\sin x}{b - 5\sin x} = -\ln \frac{3 + a\sin x}{b + 5\sin x} = -f(x).$$

Thus, it follows that

$$b^2 - 25\sin^2 x = 9 - a^2 \sin^2 x.$$

We deduce that $a^2 = 25$, $b^2 = 9$. *Thus, we obtain the couples (5, 3),* $(5, -3)$, $(-5, 3)$, $(-5, -3)$.

Now, let us verify how many couples among those couples satisfy the assumptions of the problem.

(5, 3) does not satisfy the assumptions of the problem, as function $f(x)$ *is defined at the point* $\arcsin \frac{3}{5}$ *and is undefined at the point* $-\arcsin \frac{3}{5}$.

(5, −3) does not satisfy the assumptions of the problem, as $D(f) = \emptyset$.

(−5, 3) satisfies the assumptions of the problem, as $D(f) = \left\{x : -\frac{3}{5} < \sin x < \frac{3}{5}\right\}$ *and* $f(-x) = -f(x)$.

(5, −3) also satisfies the assumptions of the problem.

Therefore, the number of all such couples is equal to 2.

Problem 7. Evaluate the expression

$$\int_{-1}^{5} \frac{1}{1 + 2^{\sqrt[3]{x-2}}}\,dx.$$

Solution. *Note that*

$$\int_{-1}^{5} \frac{1}{1 + 2^{\sqrt[3]{x-2}}}\,dx = \int_{-1}^{2} \frac{1}{1 + 2^{\sqrt[3]{x-2}}}\,dx + \int_{2}^{5} \frac{1}{1 + 2^{\sqrt[3]{x-2}}}\,dx =$$

$$= \int_{5}^{2} \frac{1}{1 + 2^{\sqrt[3]{4-t-2}}}\,d(4-t) + \int_{2}^{5} \frac{1}{1 + 2^{\sqrt[3]{x-2}}}\,dx = \int_{2}^{5} \frac{1}{1 + 2^{\sqrt[3]{2-t}}}\,dt + \int_{2}^{5} \frac{1}{1 + 2^{\sqrt[3]{x-2}}}\,dx =$$

$$= \int_{2}^{5} \Big(\frac{1}{1 + 2^{-\sqrt[3]{x-2}}} + \int_{2}^{5} \frac{1}{1 + 2^{\sqrt[3]{x-2}}}\Big)\,dx = \int_{2}^{5} 1\,dx = 3.$$

Thus, it follows that

$$\int_{-1}^{5} \frac{1}{1 + 2^{\sqrt[3]{x-2}}}\,dx = 3.$$

Problem 8. Evaluate the expression

$$\frac{100}{\pi}\lim_{n\to\infty}\left(\arccos\frac{1^4+2\cdot 1^3+3\cdot 1^2+2\cdot 1}{1^4+2\cdot 1^3+3\cdot 1^2+2\cdot 1+2}+\cdots+\arccos\frac{n^4+2n^3+3n^2+2n}{n^4+2n^3+3n^2+2n+2}\right).$$

Solution. *Let us prove that*

$$\arccos\frac{k^4+2k^3+3k^2+2k}{k^4+2k^3+3k^2+2k+2}=2\tan(k+1)-2\tan k, \tag{7.115}$$

where $k>0$. *We have that* $2\tan(k+1)-2\tan k>0$ *and* $2\tan(k+1)-2\tan k<2\tan(k+1)<\pi$. *Therefore*

$$\cos(2\tan(k+1)-2\tan k)=\cos(2(\tan(k+1)-\tan k))=-\frac{1-\tan^2(\tan(k+1)-\tan k)}{1+\tan^2(\tan(k+1)-\tan k)}=$$

$$=\frac{1-\left(\dfrac{1}{1+k(k+1)}\right)^2}{1+\left(\dfrac{1}{1+k(k+1)}\right)^2}=\frac{k^4+2k^3+3k^2+2k}{k^4+2k^3+3k^2+2k+2}.$$

This ends the proof of (7.115). *Hence, we obtain that*

$$\sum_{k=1}^{n}\arccos\frac{k^4+2k^3+3k^2+2k}{k^4+2k^3+3k^2+2k+2}=2\tan(n+1)-2\tan 1.$$

Thus, it follows that

$$\frac{100}{\pi}\lim_{n\to\infty}\left(\arccos\frac{1^4+2\cdot 1^3+3\cdot 1^2+2\cdot 1}{1^4+2\cdot 1^3+3\cdot 1^2+2\cdot 1+2}+\cdots+\arccos\frac{n^4+2n^3+3n^2+2n}{n^4+2n^3+3n^2+2n+2}\right)=$$

$$=\frac{100}{\pi}\lim_{n\to\infty}\left(2\tan(n+1)-\frac{\pi}{2}\right)=\frac{100}{\pi}\cdot\frac{\pi}{2}=50.$$

Problem 9. Find the smallest positive integer number a, such that there exists a function $f:\mathbb{R}\to\mathbb{R}$ satisfying the following assumptions

a) function $f(x)$ is twice differentiable at any point.

b) $f(-a)=f(a)=0$.

c) $f(x)>0$, if $x\in(-a,a)$.

d) $f(x)+7f''(x)\geq 0$, if $x\in(-a,a)$.

Solution. *Note that if* $a=5$, *then there exists such function. For example,* $f(x)=\cos\frac{\pi x}{10}$.

Now, let us prove that if there exists such function $f(x)$ for a positive number a, then $a > 4$.

Therefore, we deduce that the smallest positive integer a satisfying the assumptions of the problem is equal to 5.

We proceed the proof by contradiction argument. Assume that $a \leq 4$ and consider the following function

$$F(x) = \frac{\pi}{2a} f(x) \sin\left(\frac{\pi}{2a} x\right) + f'(x) \cos\left(\frac{\pi}{2a} x\right).$$

Note that $F(-a) = 0$, $F(a) = 0$, thus according to the Rolle's theorem there exists a number c, such that $-a < c < a \leq 4$ and $F'(c) = 0$.

On the other hand

$$F'(c) = \frac{\pi^2}{4a^2} f(c) \cos\left(\frac{\pi}{2a} c\right) + f''(c) \cos\left(\frac{\pi}{2a} c\right) = \cos\left(\frac{\pi}{2a} c\right)\left(\frac{\pi^2}{4a^2} f(c) + f''(c)\right) >$$

$$> \cos\left(\frac{\pi}{2a} c\right)\left(\frac{1}{7} f(c) + f''(c)\right) \geq 0.$$

Therefore $F'(c) > 0$. This leads to a contradiction.

Problem 10. Let $f : \mathbb{R} \to \mathbb{R}$ and for any real numbers x, y the following inequality holds true

$$f(y)(1 - \ln(1 + (x - y)^2)) \leq f(x).$$

Given that $f(3) = 14$. Find $f(14)$.

Solution. *From the given inequality, we obtain that*

$$p(y,x) \qquad f(x)(1 - \ln(1 + (y - x)^2)) \leq f(y).$$

Thus, if $0 < |x - y| < \sqrt{e - 1}$, then it follows that

$$-f(y) \ln(1 + (x - y)^2) \leq f(x) - f(y) \leq \frac{f(y) \ln(1 + (x - y)^2)}{1 - \ln(1 + (x - y)^2)}.$$

We deduce that if $0 < |x - y| < \sqrt{e - 1}$, then

$$-(x - y) f(y) \ln(1 + (x - y)^2)^{\frac{1}{(x-y)^2}} \leq \frac{f(x) - f(y)}{x - y} \leq \tag{7.116}$$

$$\leq \frac{f(y) \ln(1 + (x - y)^2)^{\frac{1}{(x-y)^2}}}{1 - \ln(1 + (x - y)^2)} (x - y),$$

and if $-\sqrt{e - 1} < x - y < 0$, then

$$\frac{f(y)\ln(1+(x-y)^2)^{\frac{1}{(x-y)^2}}}{1-\ln(1+(x-y)^2)}(x-y) \leq \frac{f(x)-f(y)}{x-y} \leq \tag{7.117}$$

$$\leq -\frac{f(y)\ln(1+(x-y)^2)^{\frac{1}{(x-y)^2}}}{1-\ln(1+(x-y)^2)}(x-y).$$

Note that

$$\lim_{x\to y}\Big(\frac{f(y)\ln(1+(x-y)^2)^{\frac{1}{(x-y)^2}}}{1-\ln(1+(x-y)^2)}(x-y)\Big) = \frac{f(y)\ln e}{1-0}\cdot 0 = 0,$$

and

$$\lim_{x\to y}(f(y)\ln(1+(x-y)^2)^{\frac{1}{(x-y)^2}}(x-y)) = f(y)\ln e\cdot 0 = 0.$$

Therefore, from (7.116) *and* (7.117)*, we obtain that*

$$0 \leq \lim_{x\to y}\frac{f(x)-f(y)}{x-y} \leq 0.$$

Thus, it follows that

$$\lim_{x\to y}\frac{f(x)-f(y)}{x-y} = 0.$$

Hence, for any real number y, we have that

$$f'(y) = \lim_{x\to y}\frac{f(x)-f(y)}{x-y} = 0.$$

We deduce that $f(x) = C$. On the other hand, from the condition $f(3) = 14$, it follows that $f(x) = 14$ for any real value of x.

Note that the following inequality

$$14(1-\ln(1+(x-y)^2)) \leq 14,$$

holds true for any real numbers x, y.

We obtain that $f(14) = 14$.

Problem 11. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable and $f'(x)$ be continuous in $\mathbb{R}$. Given that $f(4)-f(0) = 20$ and

$$\int_0^4 (f'(x))^2 dx = 100.$$

Find $f(3)-f(1)$.

Solution. *Note that*

$$\int_0^4 (f'(x)-5)^2 dx = \int_0^4 (f'(x))^2 dx - 10\int_0^4 f'(x)dx + \int_0^4 25dx =$$

$$= 100 - 10(f(4)-f(0)) + 100 = 0.$$

Thus, it follows that

$$f'(x) = 5.$$

Hence, we obtain that

$$f(x) = 5x + b,$$

where $0 \le x \le 4$. *Therefore*

$$f(3) - f(1) = 10.$$

Problem 12. Let $f : \mathbb{R} \to \mathbb{R}$ and for any real numbers x, y, z holds true the following inequality

$$|f(x)-f(y)||x+y| \le |f(x)-f(z)||x+z| + |f(z)-f(y)||z+y|.$$

Given that $f(0) = -5$ and $f(20) = 16$. Find $f(40)$.

Solution. *Let us consider the function* $g(x) = f(x)+5$. *Note that* $g : \mathbb{R} \to \mathbb{R}$ *and for any real numbers* x, y, z *the following inequality holds true*

$$|g(x)-g(y)||x+y| \le |g(x)-g(z)||x+z| + |g(z)-g(y)||z+y|. \tag{7.118}$$

On the other hand, $g(0) = 0$.

We denote by $p(a,b,c)$ *the inequality that we obtain from* (7.118)*, when* $x = a$, $y = b$, $z = c$.

Let us prove the following properties.

P1. *For any real numbers* x, y *the following inequality holds true*

$$|g(x)-g(y)||x+y| = |g(-x)-g(y)||-x+y|. \tag{7.119}$$

From $p(x,y,-x)$ *and* $p(-x,y,x)$*, we deduce that*

$$|g(x)-g(y)||x+y| \le |g(-x)-g(y)||-x+y|,$$

$$|g(-x)-g(y)||-x+y| \le |g(x)-g(y)||x+y|.$$

This ends the proof of (7.119).

P2. $g(x)$ *is an odd function, that is* $g(-x) = -g(x)$.

According to P1, for $y = 0$*, it follows that*

$$|g(x)| = |g(-x)|.$$

Let us consider the following two cases.
a) There exists a number $x_0 \neq 0$*, such that* $|g(-x_0)| = |g(x_0)|$*.*
According to P1, we have that

$$|g(x_0) - g(x)||x + x_0| = |g(-x_0) - g(x)|| - x_0 + x|.$$

Therefore

$$g(x) = \begin{cases} g(x_0), if\ \ x \neq 0 \\ 0, if\ \ x = 0. \end{cases}$$

From $p(0,2,1)$*, it follows that* $2|g(x_0)| \leq |g(x_0)|$*. Hence,* $g(x_0) = 0$*.*
b) If $g(-x) = -g(x)$ *for any* x*.*

P3. *If* $x \neq 0$ *is any real number, then either* $g(x) = \dfrac{g(1)}{x}$ *or* $g(x) = g(1)x$*.*
Indeed, according to P1 and P2, we obtain that

$$|g(x) - g(1)||x + 1| = |g(-x) - g(1)||x - 1| = |g(x) + g(1)||x - 1|.$$

Therefore, either $g(x) = g(1)x$ *or* $g(x) = \dfrac{g(1)}{x}$*.*

P4. *If* $k \neq 0$ *and* $g(x_1) = \dfrac{k}{x_1}$*,* $g(x_2) = \dfrac{k}{x_2}$*, then either* $|x_1| = |x_2|$ *or* $|x_1| = 1$ *or* $|x_2| = 1$*.*

We proceed the proof by contradiction argument. Assume that $|x_1| < 1 < |x_2|$*. From* $p(|x_1|, |x_2|, 1)$*, it follows that*

$$(|x_1| - 1)(|x_2| - 1) \geq 0.$$

This leads to a contradiction and ends the proof of P4.
Hence, either $g(x) = kx$ *or*

$$g(x) = \begin{cases} kx, if\ \ |x| \neq x_1, \\ \dfrac{k}{x}, if\ \ |x| = x_1, \end{cases}$$

where $x_1 > 0$ *and* $x_1 \neq 1$*.*

Note that from these functions only the function $g(x) = kx$ *satisfies to the condition* (7.118)*. Therefore, the following functions satisfy to condition* (7.118)

We obtain that $g(20) = f(20) + 5 = 21$*. Therefore,* $k = \dfrac{21}{20}$*,* $g(40) = 42$ *and* $f(40) = 37$*.*

7.5 Combinatorics

7.5.1 Problem Set 1

Problem 1. How many three-digit numbers (with distinct digits) are divisible by 11?

Solution. *The total number of all three-digit numbers divisible by 11 is 81. We have that* $\overline{abc} = 11(9a+b)+a-b+c$*, thus* $\overline{abc}$ *is divisible by 11 iff* $a-b+c=0$ *or* $a-b+c=11$*. The total number of three-digit numbers (having at least two same digits) and satisfying those conditions is respectively 13 and 4. Thus, the total number of all three-digit numbers with distinct digits and divisible by 11 is equal to* $81-17=64$.

Problem 2. There are three boys and seven girls. In how many ways is it possible to divide them into three different groups, such that each group consists of a boy, two groups consist of three people and the third group consists of four people?

Solution. *The total number of such division into three different groups is* $3! \cdot \frac{C_7^2 \cdot C_5^2}{2} = 630$.

Problem 3. In how many ways is it possible to put the book series of two different authors consisting of three and four books in the bookshelf, such that three books of the first author are in the correct consecutive order?

Solution. *We consider three books of the first author as one single book, then five books we can put in the bookshelf in* $5! = 120$ *ways, thus the solution of the problem is* $3! \cdot 120 = 720$.

Problem 4. Unknown size chessboard is divided into n parts, each part has the following form . Given that n is an odd number. Find the possible minimum value of n.

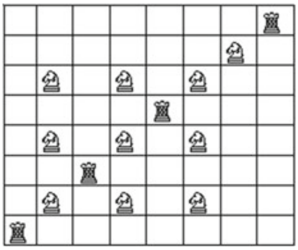
Solution. *Let* $m \times k$ *board is divided into n figures (having the following shape* *).*

Then $mk = 3n$*, thus m and k are odd numbers and either* $m \vdots 3$ *or* $k \vdots 3$*. Obviously, if* $3 \times k, (m \times 3)$ *board consists of n figures having the shape* *, then n should be an even number, which is impossible. Therefore,* $m \geq 5, k \geq 5$*, hence* $n = \frac{1}{3}mk \geq \frac{1}{3} \cdot 5 \cdot 9 = 15$*. Let us give an example for* $n = 15$.

1	4	4	7	7	13	13	15	15
1	1	4	7	8	8	13	14	15
2	2	5	5	8	10	10	14	14
2	3	5	6	9	10	11	12	12
3	3	6	6	9	9	11	11	12

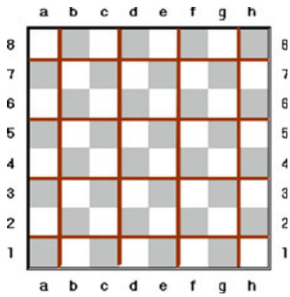
Problem 5. There are 10 knights and n rooks on a chessboard, such that none of them is under attack. Find the greatest possible value of n.

Solution. *Note that two rooks cannot be on the same horizontal or vertical (row or column), otherwise one of the pieces will be under attack. Therefore $n \leq 8$ and the number of chess squares which have no rook and are not under the attack is equal to $(8-n)^2$. Note that if $5 \leq n \leq 8$, then $(8-n)^2 \leq 9$, which means at least one of the knights is under attack. Thus $n \leq 4$. Let us give an example for $n = 4$.*



Problem 6. The sides of some squares of a chessboard are painted in red. How many sides at least one needs to paint, so that each square has at least two red sides?

Solution. *Note that there are no white squares with the same side, which means that for the white squares we need to paint in red at least two sides. Therefore, we need to paint in red at least 64 sides. Let us give an example of such painting.*



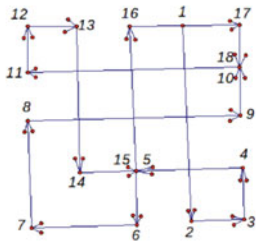
Problem 7. Given 999×1000 chessboard. It is necessary to paint some of squares (but not all of them), such that each unpainted square has exactly one painted *neighbouring* square (we call two squares neighbouring squares, if they have a common side). How many unpainted squares at least can have this board?

Solution. *Let us prove that the possible minimum number of unpainted squares is 999. If we paint all the columns except the last one, then that board satisfies our assumptions and the total number of unpainted squares will be 999. Let us now prove that the total number of unpainted squares cannot be less than 999. We proceed by a contradiction argument. Assume that the total number of unpainted*

squares is less than 999, then there exists at least one row which is fully painted. Therefore, there exist two consecutive rows, such that one is fully painted and the other one includes an unpainted square. Thus, that row should not include any painted square, we deduce that the total number of unpainted squares is not less than 1000, which is impossible.

Problem 8. Given 4×4 chessboard. Find the minimum number of sides (corresponding to the squares of the board), which need to be removed from the board, so that the rest of the figure could be drawn using a pencil and without lifting the pencil from the board (it is forbidden to pass over the same side twice).

Solution. *Suppose that after the removal of several sides the rest of the figure can be drawn without lifting the pencil from the board. The rest of the figure will be a graph. A vertice of a graph is called "odd" if there are odd number of sides starting from that vertice. Note that at the beginning the number of "odd" vertices is equal to 12, therefore it is necessary to remove at least five sides (if we remove less than five sides, then the number of "odd" vertices is greater or equal than 4, thus such graph cannot be drawn without lifting the pencil from the board). Note that when we remove five sides, the number of "odd" vertices cannot be less or equal than 2. Thus, we need to remove at least six sides. We give the following example (for six removed sides).*



Problem 9. Given seven points on the plane, the distance between them is expressed by numbers $a_1, a_2, \ldots, a_{21}$. What is the maximal number of times that we may have the same number among those 21 distances?

Solution. *Let pairwise distances a (for the points $A_1, A_2, \ldots, A_7$) are repeated at most k times. Denote by n_i the number of segments of length a starting from the point A_i. Without loss of generality, we can assume that $n_1 \geq n_2 \geq \cdots \geq n_7$. Note that $n_1 + n_2 \leq 9$, otherwise, on the circles with the centres A_1, A_2 and with the radius a there are at least $n_1 + (n_2 - 2) \geq 8$ points. This is impossible, as the total number of points is 7.*

Obviously, the distance between any two points A_1, A_2, A_3, A_4 is not equal to a. Let $A_iA_j \neq a$, $i \neq j$. In this case, on the circles with the centres A_i, A_j and with the radius a, there are at least $n_i + n_j - 2$ points. As A_i and A_j are not on that circles, then $n_i + n_j - 2 + 2 \leq 7$. Therefore $n_3 + n_4 \leq n_i + n_j \leq 7$, we deduce that $n_4 \leq 3$. Hence, $n_7 \leq n_6 \leq n_5 \leq 3$. We have that

$$k = \frac{1}{2}((n_1+n_2)+(n_3+n_4)+n_5+n_6+n_7) \leq \frac{1}{2}(9+7+3+3+3),$$

therefore $k \leq 12$*. Let us now give an example, where k can be equal to 12. If we consider the vertices of a regular hexagon (with the side length a) and its centre, then for those points the same distance is repeated 12 times. Thus* $k = 12$*.*

7.5.2 Problem Set 2

Problem 1. There are five teachers of mathematics, three teachers of physics and two teachers of chemistry. We choose some of them, such that there is a teacher chosen corresponding to each subject. In how many ways can we make such a choice?

Solution. *We can choose teachers of mathematics, physics and chemistry in* $2^5 - 1, 2^3 - 1$ *and* $2^2 - 1$ *ways, respectively. Therefore, the number of all the possible ways of such choices is equal to* $(2^5-1)(2^3-1)(2^2-1) = 651$*.*

Problem 2. Consider the points $A_i(i,1)$, $i = 1,2,\ldots,15$ and $A_i(i-15,4)$, $i = 16,17,\ldots,30$. Find the number of all isosceles triangles, with all vertices in a given set of points $A_1, A_2, \ldots, A_{30}$.

Solution. *The number of all isosceles triangles, such as for each of them the three vertices are from the points* $A_1, A_2, \ldots, A_{30}$ *and the base is on the line* $y = 1$ *or* $y = 4$*, is equal to* $2(C_7^2 + C_8^2) = 98$*.*

The number of all the isosceles right triangles, such as for each of them the three vertices are from the points $A_1, A_2, \ldots, A_{30}$ *and one of its legs is on the line* $y = 1$ *or* $y = 4$*, is equal to* $2 \cdot 24 = 48$*.*

The number of all the isosceles acute triangles, such as for each of them the three vertices are from the points $A_1, A_2, \ldots, A_{30}$ *and one of its legs is on the line* $y = 1$ *or* $y = 4$*, is equal to* $2 \cdot 20 = 40$*.*

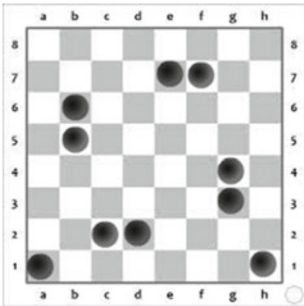
The number of all the isosceles obtuse triangles, such as for each of them the three vertices are from the points $A_1, A_2, \ldots, A_{30}$ *and one of its legs is on the line* $y = 1$ *or* $y = 4$*, is equal to* $2 \cdot 12 = 24$*.*

Thus, the number of all such isosceles triangles is equal to 210.

Problem 3. How many rooks at maximum can be placed on a chessboard so that each rook is not under attack of more than one rook?

Solution. *Let us show that if there are 11 rooks on the board, then at least one of them will be under attack of two rooks. We proceed by a contradiction argument. Assume there are 11 rooks on the board, such that each rook is not under attack of any other. In that case, at least three rows have two rooks. Note that each column including any of that two rooks (six rooks in total) does not have any other rook. Therefore, on the other two columns there are five rooks. Thus, one of the columns have three rooks, which leads to a contradiction.*

Let us give an example for 10 rooks, such that each rook is not under attack of more than one other rook.

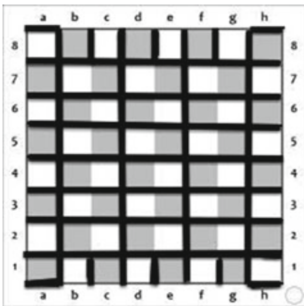


Problem 4. In how many ways can one divide the set of natural numbers into two disjoint subsets N_1 and N_2 satisfying the following condition: the difference of any two numbers from the same subset is not a prime number greater than 100?

Solution. *Consider an arbitrary positive integer n. Without loss of generality, we may assume that it belongs to the set N_1. Then, the number $n+103$ should be in the set N_2. Therefore, $n+2$ should be in the set N_1, as $n+103-(n+2)=101$ is a prime number. In a similar way, if n belongs to the set N_2, then $n+2$ should belong to N_2 also. Hence, we obtain that n and $n+2$ should be in the same set. We deduce that all the odd numbers belong to the same set as 1 and all the even numbers belong to the same set as 2. Thus, such division of subsets is unique, that is, all the odd numbers are in one set and all the even numbers are in the other set.*

Problem 5. Let some of the sides of the squares of a chess board be painted in red. What is the smallest number of sides, which need to be painted, so that each square has at least three red sides.

Solution. *Note that there are no two white squares which have a common side, therefore in each white square at least three sides must be painted in red. Moreover, for the border black squares at least two sides should also be painted in red. Thus, it follows that at least 98 sides must be painted in red. Let us give an example of such a painting.*



Problem 6. Consider 27 points on the plane, such that no three of them lie on a straight line. Some of these points are connected by segments, such that their number is more than 325. Given that for any four points A, B, C, D the following condition holds true: if A and B are connected, B and C are connected and C and D are connected, then A and D are connected as well. Find the number of all segments.

Solution. *Consider the points $A_1, A_2, \ldots, A_{27}$. Let A_1 and A_2 are not connected by a segment. Consider the following triples*

$$\{(A_1,A_3),(A_3,A_4),(A_4,A_2)\},\{(A_1,A_4),(A_4,A_5),(A_5,A_2)\},\ldots,$$

$$\{(A_1,A_{26}),(A_{26},A_{27}),(A_{27},A_2)\},\{(A_1,A_{27}),(A_{27},A_3),(A_3,A_2)\}.$$

Note that in one triple all the couples of the points are not connected with each other, thus the number of couples of non-connected points is not less than 26. Therefore, the number of all couples is not less than $\left(\frac{26\cdot 25}{2}+1\right)+26=\frac{27\cdot 26}{2}+1$. *This leads to a contradiction. Hence, we deduce that any two of these points are pairwise connected.*

Problem 7. Let the entries of 16×16 table be real numbers. Given that the sum of the numbers of each 4×4 square is not negative and the sum of the numbers of each 5×5 square is not positive. Denote the sum of all the entries of 16×16 table by D. Find the number of all the possible values of D.

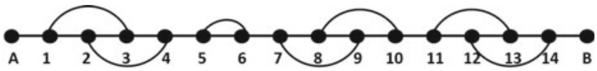
Solution. *Let us divide the* 16×16 *table into four* 8×8 *squares. If we show that for any of those four squares the sum of all the entries is equal to 0, then sum of all the entries of* 16×16 *table will be equal to 0 too, that is* $D=0$. *On the picture, each number written in the squares of* 8×8 *table indicates for how many* 4×4 *squares that square is a common square.*

1	*2*	*3*	*4*	*4*	*3*	*2*	*1*
2	*4*	*6*	*8*	*8*	*6*	*4*	*2*
3	*6*	*9*	*12*	*12*	*9*	*6*	*3*
4	*8*	*12*	*16*	*16*	*12*	*8*	*4*
4	*8*	*12*	*16*	*16*	*12*	*8*	*4*
3	*6*	*9*	*12*	*12*	*9*	*6*	*3*
2	*4*	*6*	*8*	*8*	*6*	*4*	*2*
1	*2*	*3*	*4*	*4*	*3*	*2*	*1*

Let us enumerate the columns and the rows of 8×8 *table by the numbers 1,2, …, 8 from the left to the right and from the top to the bottom, respectively. Let the number corresponding to the column i and the row j is denoted by $a_{i,j}$ (not the numbers on the picture, but the initial numbers). Denote by S the following sum* $(1\cdot a_{1,1}+2\cdot a_{1,2}+\cdots+1\cdot a_{1,8})+(2\cdot a_{2,1}+4\cdot a_{2,2}+\cdots+2\cdot a_{2,8})+\cdots+(1\cdot a_{8,1}+2\cdot a_{8,2}+\cdots+1\cdot a_{8,8})$. *We have that $S\geq 0$ and on the other hand $S\leq 0$, as a sum of*

all the entries of all 4×4 *squares. Therefore,* $S=0$ *and it holds true for the case, when the sum of all the entries of all* 4×4 *squares is equal to* 0. *Note that* 8×8 *table can be divided into four* 4×4 *squares, thus the sum of all the entries of* 8×8 *table is equal to 0.*

Problem 8. In how many ways can one go from the city A to the city B, if it is possible to pass through each of the places $1,2,\dots,14$ not more than once.



Solution. *Assume we go from the city A to the city B. Let us write on each of the curves (1,3), (2,4), (5,6), (7,9), (8,10), (11,13), (12,14) the corresponding number, which indicates how many times did we pass through that curve. Therefore, it will be a 7-tuple of zeros and ones. It holds true also the opposite statement, that is, every 7-tuple determines one way for going from the city A to the city B. Therefore, the total number of all the ways is equal to* $2^7=128$.

Problem 9. A ten-digit number $\overline{a_1a_2\dots a_{10}}$ is called "interesting", if its all digits are nonzero and each of $\overline{a_1a_2}$, $\overline{a_2a_3}$, $\overline{a_3a_4}$, $\dots$, $\overline{a_9a_{10}}$ is divisible by one of the numbers 13, 17, 23, 37. Denote by n the number of all "interesting" ten-digit numbers. Find $\frac{n}{6}$.

Solution. *Note that the numbers*

$$13,26,39,52,65,78,91,17,34,51,68,85,23,46,69,92,37,74$$

are divisible by $13,17,23,37$, *respectively. Let us also note that if we consider a digit a,* $(a\neq 0)$, *then two of the considered numbers have a as a last digit. Therefore, it follows that if we write a ten-digit number* $\overline{a_1a_2\dots a_{10}}$ *not from the left to the right, but from the right to the left, then its each digit (except* a_{10}*) can be chosen in two ways and* a_{10} *can be chosen in nine ways. Hence, we obtain that the number of "interesting" ten-digit numbers is equal to* $9\cdot 2^9=4608$.

7.5.3 Problem Set 3

Problem 1. Find the number of all six-digit positive integers with the distinct digits which are divisible by 11 and consist of the digits 1, 2, 3, 4, 5, 8.

Solution. *We have* $\overline{abcdef}=(10^5+1)a+(10^4-1)b+(10^3+1)c+(10^2-1)d+(10+1)e+(f+d+b-a-c-e)$, *thus* $\overline{abcdef}$ *is divisible by 11 iff, if* $11\mid f+d+b-a-c-e$.

Note that the sum of the digits 1, 2, 3, 4, 5, 8 is equal to 23, hence $a+c+e=6$ *or* $a+c+e=17$.

If $a+c+e=6$*, the number of such* 6*-digit numbers is* $3!\cdot 3!=36$. *If* $a+c+e=17$*, the number of such* 6*-digit numbers is again 36.*

Problem 2. Find the number of all three-digit positive integers with distinct digits, such as for any of them the biggest digit is the middle one.

Solution. *The number of* 3*-digit numbers which do not include 0 as one of the digits is equal to* $2\cdot C_9^3$. *On the other hand, the number of* 3*-digit numbers which include 0 as one of the digits is equal to* C_9^2. *Therefore,* $2\cdot C_9^3+C_9^2=204$.

Problem 3. Find the number of all four-digit positive integers $\overline{abcd}$ with the digits $1,2,3,4,5$, such as $\overline{ab}\neq 23$ and $\overline{bc}\neq 23$ and $\overline{cd}\neq 23$.

Solution. *The number of* 4*-digit numbers which can be written with digits 1, 2, 3, 4, 5 is equal to* 5^4. *On the other hand,* 4*-digit numbers which include 23 have the following form* $\overline{ab23}$, $\overline{a23d}$ *or* $\overline{23cd}$. *The number of* 4*-digit numbers which have the form* $\overline{ab23}$ *is equal to 25. The same for* $\overline{a23d}$ *and* $\overline{23cd}$*, their numbers are equal to 25 too. Note that the number 2323 is of the form* $\overline{ab23}$ *and* $\overline{23cd}$. *Therefore, we deduce* $5^4-74=551$.

Problem 4. Find the number of all positive divisors of 2015^8, such as any of them is not a square of a positive integer.

Solution. *We have that* $2015^8=5^8\cdot 13^8\cdot 31^8$*, thus the number of all (positive) divisors is equal to* 9^3 *and the number of divisors which are squares of positive integers is equal to* 5^3. *Therefore, we deduce* $9^3-5^3=604$.

Problem 5. Find the number of all seven-digit numbers, which start with the digit 1 and end with the digit 9, such as the difference of any two neighbour digits is 1 or 2.

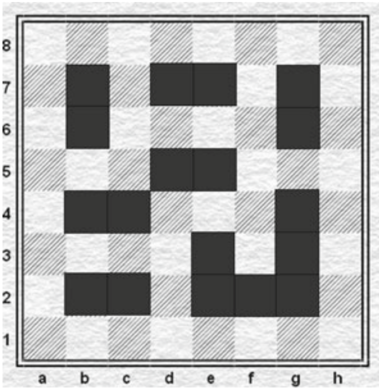
Solution. *Let the number* $\overline{a_1a_2a_3a_4a_5a_6a_7}$ *is such that* $a_1=1, a_7=9$ *and* $|a_{i+1}-a_i|=1$ *or* $|a_{i+1}-a_i|=2$*,* $i=1,2,\dots,6$. *Denote by* $|a_{i+1}-a_i|=\delta_i$*, for* $i=1,2,\dots,6$. *Note that* $\delta_1+\delta_2+\cdots+\delta_6=8$.

The number of 7*-digit numbers with* $\delta_i>0$*,* $i=1,2,\dots,6$ *is equal to* C_6^2.

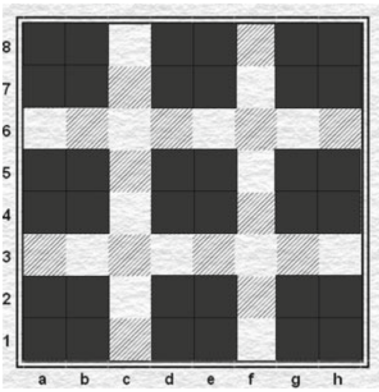
Let us divide into two groups all 7*-digit numbers with* $\delta_i<0$*,* $i=1,2,\dots,6$. *The first group includes all* 7*-digit numbers with* $\delta_i=-1$. *Note that the other* δ *numbers are equal to 1, 2, 2, 2, 2. The second group includes all* 7*-digit numbers with* $\delta_i=-2$. *Note that the other* δ *numbers are equal to 2, 2, 2, 2, 2. The number of the terms of the first group is equal to* $2\cdot C_5^2+5$. *The number of the terms of the second group is equal to 4. Hence, the answer is* 44.

Problem 6. Let n be a positive integer. Given that after placing (in a random way) n dominos on a chessboard there exists 2×2 square which does not have a square covered by any domino. Each domino covers exactly two squares of a chessboard. Find the possible greatest value of n.

Solution. *The following picture shows that the possible greatest value of n is smaller than 9.*



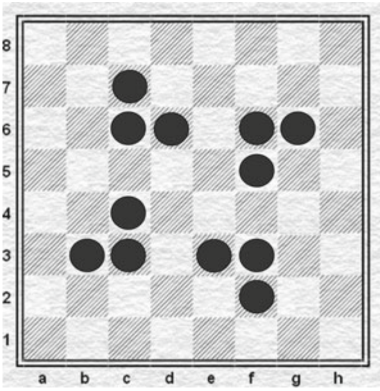
Note that after covering the chessboard by any eight dominos, there will exist a square among those nine squares (pictured in the image below) which does not intersect any domino (because every domino can intersect at most one of those nine squares).



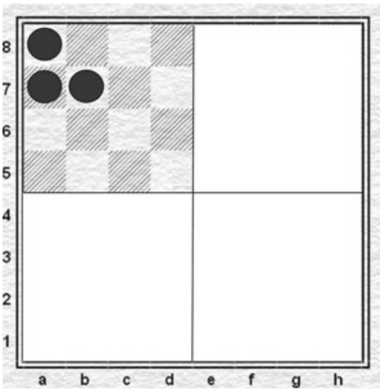
Therefore, the greatest possible value of n is 8.

Problem 7. Given that for any n black figures on a chessboard there exists a square, such that the white knight placed on that square does not attack any of these black figures. Find the possible greatest value of n.

Solution. *The following picture illustrates that $n < 12$.*



If the number of figures is equal to 11, then at least in one of 4×4 squares there will be at most two figures. Without loss of generality, we may assume that 4×4 square pictured in the image below has at most two figures.



Therefore, we can put the knight on one of the squares indicated in the image (as any square of this 4×4 square can be under attack of a knight located on at most one of those indicated squares). Hence, the possible greatest value of n is equal to 11.

Problem 8. Consider 4×9 rectangular grid. In how many ways can it be covered by L-shape trominos, such that each tromino covers exactly three squares and each square is covered by exactly one tromino?

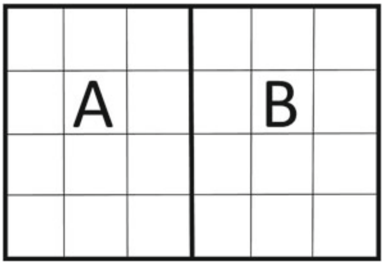
Solution. *Let us prove the following properties.*

P1. 4×3 *rectangle can be covered by L-shape trominos in four ways.*

The statement is obvious, as 2×3 *rectangle can be covered by L-shape trominos in two ways.*

P2. 4×6 *rectangle can be covered by L-shape trominos in 18 ways.*

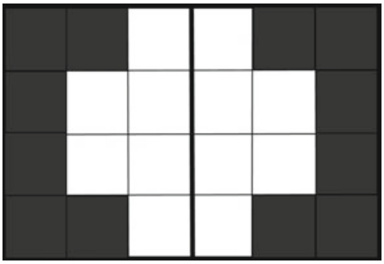
Let us consider the following two cases.



i) A and B (indicated on the image) are covered by L-shape trominos. The number of ways of such coverage is $4 \times 4 = 16$.

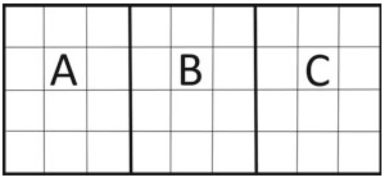
ii) If A and B are not covered by four L-shape trominos.

In that case, the corner squares will be covered in the following way.

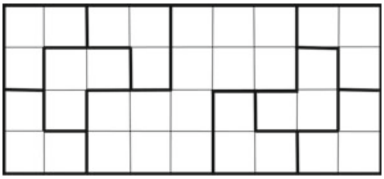


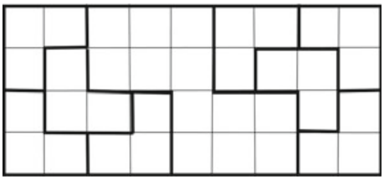
It is left to note that the rest can be covered in two ways.

Obviously, the number of ways of covering 4×9 *rectangle by L-shape trominos is equal to* $4 \cdot 4 \cdot 4 + 4 \cdot 2 + 2 \cdot 4 + n$, *where n is the number of all ways, such that for each of them A, B and C are not covered by* 4 *L-shape trominos.*



In order to find n, we need to repeat the steps of P2 and note that it holds true one of the following cases.

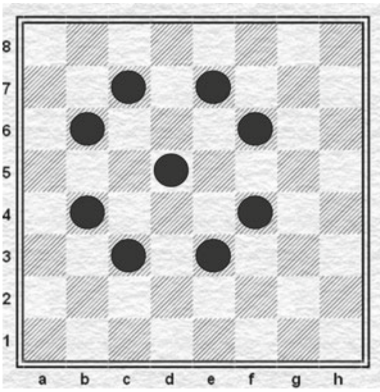




Hence, we deduce that $n = 8$.
Therefore, the total number of ways is equal to 88.

Problem 9. Given that from any n squares of a chessboard one can choose two squares, such that a knight needs at least three steps in order to go from the one square to the other one. Find the possible smallest value of n.

Solution. *The following picture illustrates that $n \geq 10$.*



Let us show that if we consider 10 squares on a chessboard, then one can choose two squares, such that a knight will need at least three steps in order to go from one square to another.

We proceed by a contradiction argument. Suppose there exist 10 squares, such that for any two of them a knight can go from one to another in one or two steps. Let us consider the smallest vertical (horizontal) section of the chessboard, which includes all that 10 squares. Obviously, the length of that section is not greater than 5. Therefore, all that squares are inside of 5×5 square. Let us enumerate its squares in the following way.

2	2	4	7	1
3	5	10	6	8
4	7	1	9	3
6	6	8	5	5
1	9	3	4	2

Note that all those 10 squares will have different numbers, such as a knight needs to make three steps in order to go from one square with the same number to another one (see the picture above). Hence, the square with the number 10 is one of those 10 squares. In a similar way, we may prove that any of the four coloured squares is one of those 10 squares. It is left to note that among the other squares there is none which is one of those 10 squares. We obtain that the number of squares is four, which is not possible. This leads to a contradiction. Therefore, the smallest value of n is equal to 10.

7.5.4 Problem Set 4

Problem 1. At least, with how many L-shape trominos one can cover a chessboard, such that each tromino covers exactly three squares and each square is covered by the same number of trominos?

Solution. *Let the chessboard is covered by n L-shape trominos. We have that* $3n = 64k$*, where k is the number of L-shape trominos covering the same square. Therefore* $n \geq 64$*. Let us give an example of such covering with* 64 *L-shape trominos. At first, we decide the chessboard into 16 squares of size* 2×2*. Afterwards, let us cover each* 2×2 *square by four L-shape trominos, such that each square is covered by three trominos. Hence, the smallest number of L-shape trominos for covering a chessboard is equal to 64.*

Problem 2. Find the number of all three-digit numbers, such that for each of them its digits are side lengths of some triangle.

Solution. *The number of all three-digit numbers, such that for each of them its digits are side lengths of an equilateral triangle, is equal to 9. The number of all three-digit numbers, such that for each of them its digits are side lengths of an isosceles but not equilateral triangle, is equal to* $3 \cdot 52 = 156$*. The number of all three-digit numbers, such that for each of them its digits are side lengths of a scalene triangle, is equal to* $6 \cdot 33 = 198$*. Therefore, the number of all three-digit numbers, such that for each of them its digits are side lengths of some triangle, is equal to 363.*

Problem 3. Consider 12 cards, such that on three of them is written the letter A, on three of them the letter B, on three of them the letter C and on the last ones the letter D. We choose 4 cards among them and put in some order. How many distinct combinations of letters can we have after such a choice?

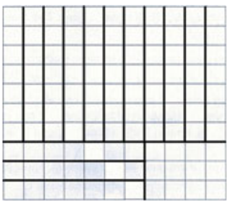
Solution. *The number of combinations with four distinct letters is equal to 24. The number of combinations with three same letters and one distinct letter is equal to* $4 \cdot 4 \cdot 3 = 48$*. The number of combinations with two the same and two the same but distinct from those two letters is equal to* $6 \cdot 6 = 36$*. The number of combinations with three distinct numbers is equal to* $12 \cdot 4 \cdot 3 = 144$*. Therefore, the number of all distinct combinations of letters after the required choice is equal to 252.*

Problem 4. A teacher invites guests to celebrate his birthday. Given that of any five guests there exist two guests who have met each other at the teacher's house. Given also that every guest visits teacher's birthday only once. At least with how many photos can he have the pictures of all guests?

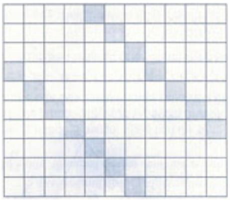
Solution. *Let us prove that with four photos the teacher can have the photos of all guests. In each photo, he can have all the guests, who are present at that particular moment. The first photo he can take at the moment, when some guests would like to leave. Then, the second photo he can take at the moment, when someone who leaves is not on the first photo. The third photo he can take at the moment, when someone who leaves is not on the previous photo and the same for the fourth photo. According to the conditions of the problem, every guest will be on some photo. If the first two guests (among five guests) come together and another guest comes after they leave, the next guest comes after the previous guest leaves and the last guest comes also after the previous guest, then obviously they cannot all be on less than four photos. Therefore, the smallest possible number of required photos is equal to 4.*

Problem 5. Consider 10×11 size rectangle drown on paper. At most, how many 1×7 size rectangles can one cut from 10×11 rectangle, such as the sides of any of them are parallel to the sides of 10×11 rectangle?

Solution. *We give the following example for cutting 14 rectangles (in the required way).*



Let us prove that we cannot cut 15 rectangles (in the required way). Consider the following figure.



Note that if the sides of some 1×7 rectangle are parallel to the sides of given rectangle, then the area of coloured part situated inside of it is equal to 1. Therefore, if our statement holds true, then the sum of areas of all covered squares should be at least 15, which leads to a contradiction.

Problem 6. At most, how many numbers can one choose among the numbers 1, 2, …, 2012, such that the difference of any two chosen numbers is not a prime number?

Solution. *Let us prove the following lemma.*

Lemma 7.16. *Let n be a positive integer. Given that $n \geq 4$. Then, from the numbers $1,2,3,\ldots,4n$ one can choose several (not more than n) numbers, such that the difference of any two numbers among those numbers is not a prime number.*

Proof. *Let A be the set of chosen numbers. Consider $x_1 \in A$, $x_2 \in A$, $x_3 \in A$, such that $x_1 < x_2 < x_3$. Consider the following two cases.*

Case 1. If $x_2 - x_1 = 1$ or $x_3 - x_2 = 1$, then $x_3 - x_1 \geq 9$.

Case 2. If $x_2 - x_1 \neq 1$ and $x_3 - x_2 \neq 1$, then $x_3 - x_1 \geq 8$.

Hence, for the both cases, we deduce that $x_3 - x_1 \geq 8$.

Consider $x_1, x_2, \ldots, x_6 \in A$, such that $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$. Let us prove that $x_6 - x_1 \geq 20$.

If $x_2 - x_1 \neq 1$ or $x_6 - x_5 \neq 1$, then $x_2 - x_1 \geq 4$ or $x_6 - x_5 \geq 4$. Let $x_2 - x_1 \geq 4$, then $x_4 - x_2 \geq 8$ and $x_6 - x_4 \geq 8$. Hence $x_6 - x_1 \geq 20$.

If $x_2 - x_1 = 1$ and $x_6 - x_5 = 1$, then $x_3 - x_2 \geq 8$, $x_4 - x_3 \geq 1$, $x_5 - x_4 \geq 8$. We deduce that $x_6 - x_1 \geq 19$, thus $x_6 - x_1 \geq 20$.

If $|A| \geq n+1$ and n is an even number, then

$$x_{n+1} - x_1 = (x_{n+1} - x_{n-1}) + (x_{n-1} - x_{n-3}) + \cdots + (x_3 - x_1) \geq 8 \cdot \frac{n}{2} = 4n,$$

where $x_1 < x_2 < \cdots < x_{n+1}$ and $x_1, x_2, \ldots, x_{n+1} \in A$. This leads to a contradiction.

If $|A| \geq n+1$ and n is an odd number, then $n+1 \geq 6$. Hence

$$x_{n+1} - x_1 = (x_{n+1} - x_{n-1}) + \cdots + (x_8 - x_6) + (x_6 - x_1) \geq 8 \cdot \frac{n-5}{2} + 20 = 4n.$$

This leads to a contradiction. This ends the proof of lemma.

From this lemma, we deduce that it is not possible to choose more than 503 numbers. We give the following example for 503 numbers: 1,5,9, ..., 2009.

Remark 7.2. *In the case, when $n = 3$ the lemma does not hold true.*

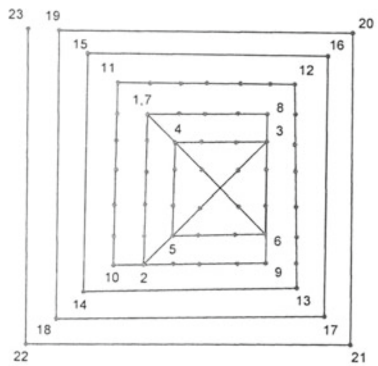
For example, the difference of any two numbers among $1, 2, 10, 11$ *numbers is not a prime number.*

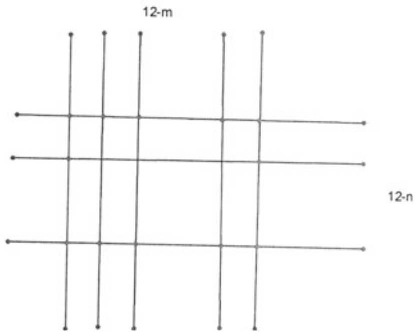
Problem 7. Consider a convex dodecagon (12 sided polygon) and its diagonals. Given that one cannot choose three diagonals, such that they have the same intersection point (located in the interior part of the dodecagon). Find the number of all triangles, such that for any of them all the sides are on the diagonals and any vertex is not one of the vertices of the dodecagon.

Solution. *Note that for any six vertices of a polygon there are exactly three diagonals that make such a triangle. Therefore, the number of such triangles is equal to* $C_{12}^{6} = 924$.

Problem 8. Without lifting a pencil from the paper how many line segments one should draw, such that they include all 144 vertices of some 11×11 square grid?

Solution. *Let us give an example of 22 line segments that include all 144 vertices of* 11×11 *square.*





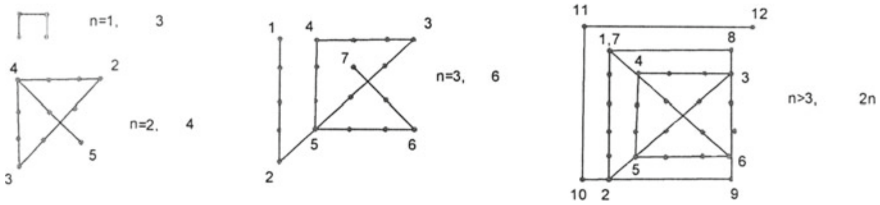
Let us now prove that if we draw k line segments without lifting the pencil from the paper that include all 144 vertices of 11×11, *then* $k \geq 22$. *Let n of those k segments are horizontal and m are vertical.*

If $n = 12$ (or $m = 12$), *then two horizontal line segments cannot be neighbours, thus* $k \geq 23$.

If $n = 11$ (or $m = 11$), *then for* $12 - m$ *points, situated on one horizontal line segment, we need at least* $12 - m$ *line segments.*

If $n < 11$ *and* $m < 11$, *then one needs at least* $22 - m - n$ *line segments for* $44 - 2m - 2n$ *points situated on the boundary of a convex hull of* $(12 - m)(12 - n)$ *intersection points of* $12 - n$ *horizontal and* $12 - m$ *vertical. Therefore* $k \geq m + n + (22 - m - n) = 22$.

Remark 7.3. *For $n \times n$, $(n \geq 2)$ square, the number of such line segments is equal to $2n$.*

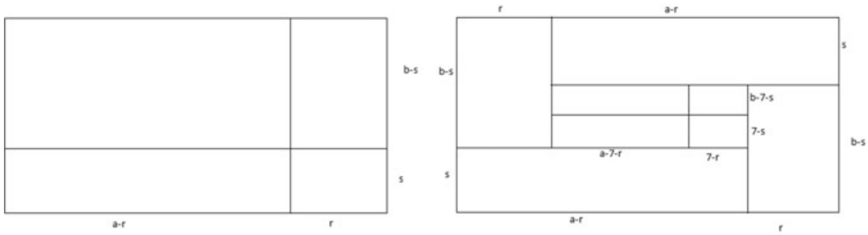


Problem 9. Find the minimum value of n, such that for any positive integers a, b, $(a > 6, b > 6)$ one can cut not more than n squares from $a \times b$ paper rectangle grid, in a way that the rest of the figure one can cut into 1×7 rectangles.

Solution. *Let us prove that such smallest number is 12.*

Lemma 7.17. *Let a and b be positive integers ($a > 6$, $b > 6$). Then, from $a \times b$ size paper rectangle, one can cut 12 squares in a way that the rest of the figure one can cut into 1×7 rectangles.*

Proof. *Let a and b be divisible by 7 with a remainder r and s, respectively. From the figure below, it follows that if we cut rs squares, then the rest of the figure one can cover by 1×7 rectangles.*



On the other hand, from the figure above it follows that if we cut $(7-r)(7-s)$ number of squares, then the rest of the figure one can cover by 1×7 rectangles. Note that $\min(rs, (7-r)(7-s)) \leq 12$.

By Problem 5, we have that from 10×11 rectangle one cannot cut five squares, such that the rest of the figure is possible to cover by 15 rectangles of 1×7 size. Therefore, one should cut at least 12 squares, such that the rest of the figure is possible to cover by 15 rectangles of 1×7 size. By this argument and the lemma, we end the proof of the problem.

7.5.5 Problem Set 5

Problem 1. Find the number of all three-digit numbers $\overline{abc}$, such that a quadratic expression $ax^2 + bx + c$ has a integer root.

Solution. *Note that the given quadratic polynomial has an integer root for all the three-digit numbers that end with 0. That root is 0. The total number of such three-digit numbers is equal to 90.*

If $c = 1$*, then the integer root of* $ax^2 + bx + c$ *can be only* -1*. Therefore,* $a - b + c = 0$*. The total number of such three-digit numbers is equal to 8.*

The number of such three-digit numbers that end with $2, 3, \ldots, 9$ *is equal to* 10, 7, 8, 4, 5, 2, 3, 2, *respectively.*

Hence, the total number of such three-digit numbers is equal to 139.

Problem 2. Find the number of all six-digit numbers, with distinct digits and belonging to the set $\{1, 2, 3, 4, 5, 6\}$, such that for each of those six-digit numbers digits 1 and 2, 5 and 6 are not consecutive digits.

Solution. *The total number of six-digit numbers with distinct digits that belong to* $\{1, 2, 3, 4, 5, 6\}$ *is equal to* $6! = 720$.

The number of such six-digit numbers, where 1 and 2, 5 and 6 are consecutive digits, is equal to $4! \cdot 2 \cdot 2 = 96$.

The number of such six-digit numbers, where 1 and 2 are consecutive digits, but 5 and 6 are not, is equal to $2 \cdot 5! - 96 = 144$.

In a similar way, the number of such six-digit numbers, where 5 and 6 are consecutive digits, but 1 and 2 are not, is equal to 144.

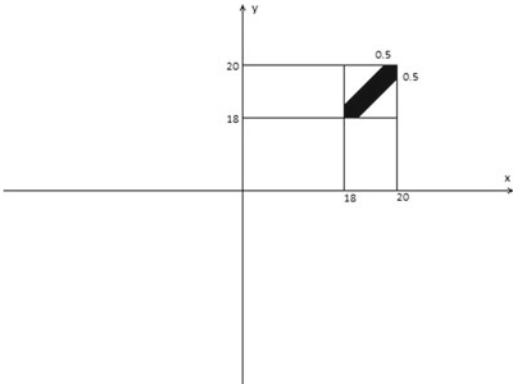
Therefore, the total number of required six-digit numbers is equal to $720 - 96 - 288 = 336$.

Problem 3. In how many ways can one put six identical books of mathematics and four different books of physics in a bookshelf, such that on one side of each book of physics the neighbouring book is a book of mathematics and on the other side the neighbouring book is a book of physics?

Solution. *Let us first put into the bookshelf the books of mathematics. Next, we choose any two neighbouring books among them and put in between two books of physics. Afterwards, we do the same one more time. Therefore, the total number of required ways of ordering the books will be equal to* $C_5^2 \cdot 4! = 240$.

Problem 4. Two friends agreed to go to a cafe and meet there at some point in time starting from 6 PM till 8 PM. If any of them comes not later than half past seven, then he needs to wait 30 minutes. If any of them comes later than half past seven, then he needs to wait till 8 PM. Let p be the probability of their meeting in the cafe. Find $16p$.

Solution. *Assume that the first person came at x o'clock and the second one at y o'clock. In Cartesian coordinate system, the point* (x, y) *is located in the square that is bounded by lines* $x = 18$*,* $x = 20$*,* $y = 18$*,* $y = 20$*. The condition that they will not meet is equivalent to* $|x - y| > 0.5$ *(see the picture below).*



Thus, $p = \dfrac{2^2 - 1.5^2}{2^2} = \dfrac{1.75}{4}$. *Hence, we obtain that* $16p = 7$.

Problem 5. Let the entries of 4×4 grid square be numbers 0 or 1, such that every column sum is equal to 2 and every row sum is equal to 2. How many such 4×4 grid squares are there?

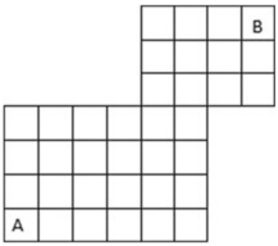
Solution. *Note that in each row and column there are two zeros and two ones. We can fill the first row in six different ways. After filling in the first row, let us consider two columns that include two zeros of the first row.*

If the second zeros of that columns are written in the same row, then the rest of the square can be filled in a unique way. The number of such options is equal to 3.

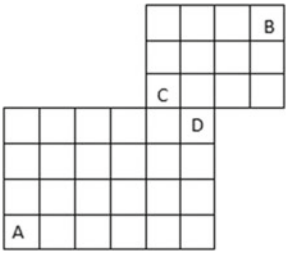
If the second zeros of that columns are written in different row, then the rest of the square can be filled in two ways. The number of such options is equal to 12.

Therefore, the total number of required squares is equal to $6 \times (12 + 3) = 90$.

Problem 6. Two squares are called neighbouring if they have a common side. In how many ways can one go from the square A to the square B? (see the figure below). From each square, one can move either to the up or to the right neighbouring square.



Solution. *Note that in order to go from A to B one needs to pass either through C or D (see the figure below).*



Moreover, every such path passes exactly through one of them (either C or D). The number of such paths passing through C is equal to $C_7^3 \cdot C_5^2$. *The number of such paths passing through D is equal to* $C_8^3 \cdot C_4^2$. *Therefore, the total number of such paths is equal to* $C_7^3 \cdot C_5^2 + C_8^3 \cdot C_4^2 = 686$.

Problem 7. A set is called a "good" set, if its elements belong to $\{1,2,\ldots,10\}$ and their sum is divisible by 4. Find the number of all "good" subsets of the set $\{1,2,\ldots,10\}$.

Solution. *We call a set an "even" set, consisting of the terms that belong to* $\{1,2,\ldots,10\}$, *if the sum of its members is an even number. Otherwise, it is called an "odd" set. Note that if A is an "even" set, then* $\{1,2,\ldots,10\}\ A$ *is an "odd" set. On the other hand, if A is an "odd" set (*$A \neq \{1,2,\ldots,10\}$*), then* $\{1,2,\ldots,10\}\ A$ *is an "even" set. Therefore, the total number of "even" sets is equal to 511.*

If 2 is a member of a "good" set, then we will eliminate it. If 2 is not a member of a "good" set, then we will add it. Hence, we deduce that the total number of "good" sets is equal to $\dfrac{511-1}{2} = 255$.

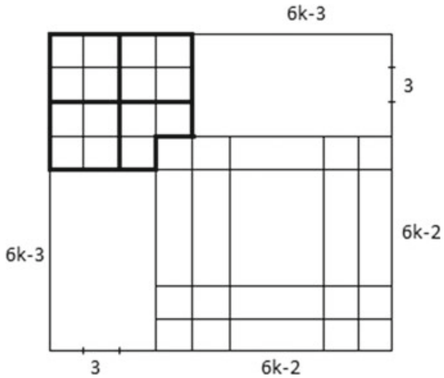
Problem 8. At least, how many 1×1 squares are required in order to cover any $n \times n$ square by 1×1, 2×2, 3×3 squares?

Solution. *Let us prove that with four* 1×1 *squares any square is possible to cover in the required way.*

If $2 \mid n$, *then we can cover it by* 2×2 *squares.*

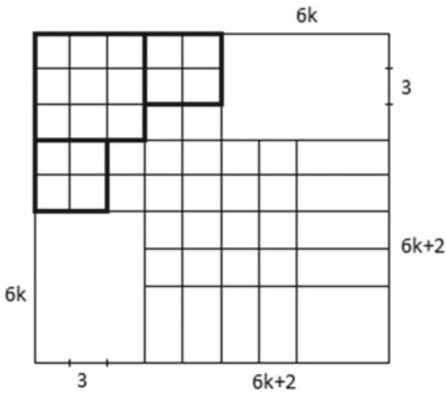
If $3 \mid n$, *then we can cover it by* 3×3 *squares.*

If $n = 6k+1$, *where* $k \in \mathbb{N}$, *then* $n \times n$ *size square can be covered in the following way:*



Here, we have used three 1×1 *squares.*

If $n = 6k + 5$*, see the figure below:*



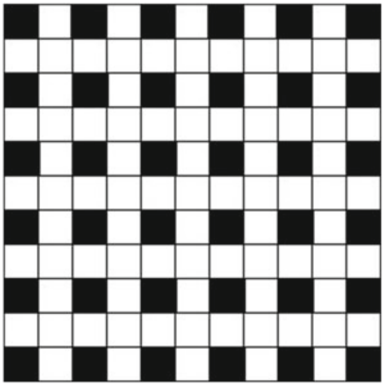
Note that 5×5 *square is not possible to cover only by* 2×2 *and* 3×3 *squares.*

If there is 3×3 *square, then there should be at least four* 1×1 *squares.*

If there is no 3×3 *square, then there should be at least five* 1×1 *squares. The proof of this statement is straightforward and can be done by painting the columns of* 5×5 *square in black and white (the painting of columns should be done in consecutive order).*

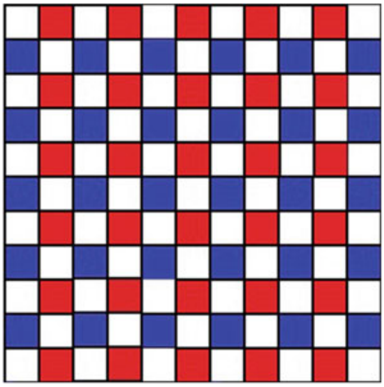
Problem 9. Consider 11×11 paper grid square. One cuts from it several figures of this form . Given that all the cuts are done along the sides of the squares. At least, how many squares could be left (after the cut) in the initial paper grid square?

Solution. *Note that, after the cut, every figure includes exactly one black square pictured below.*



At least, eight among those black squares are not covered by the given figures, otherwise the total number of all squares will not be less than $4(36-k)+k = 144-3k \geq 123$*, where k is the number of black squares that are not covered by given figures. The last inequality leads to a contradiction.*

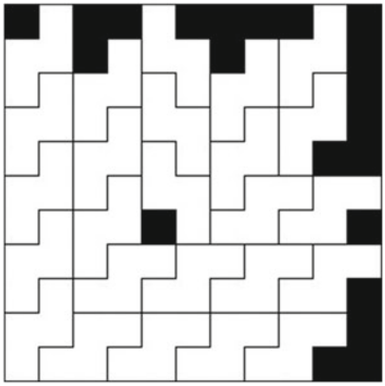
Note that in the picture below, at least three among the red squares are not covered by the given figures, otherwise the total number of all squares is not less than $28 \cdot 4 + 8 + 2 = 122$*. The last inequality leads to a contradiction.*



In a similar way, we can prove that at least four of the blue squares are not covered by the given figures. In that case, at least four of the red squares are not covered by the given figures. Note that at least ten among the black squares are not covered by the given figures, otherwise we obtain that $27 \cdot 4 + 9 + 8 > 121$*. The last inequality leads to a contradiction.*

Therefore, the number of the squares that are left (after the cut) is of $4n+1$ *form and is not less than 18. Hence, it is not less than 21.*

In the picture below, we give an example where the number of the squares that are left (after the cut) is equal to 21.

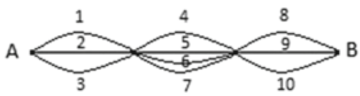


7.5.6 Problem Set 6

Problem 1. In how many different ways can one put into a bookshelf 7 numbered books of the same author, such that book number 2 is between books number 1 and 3?

Solution. *Let us consider the first three books (numbered from 1 to 3) together as one single book. Thus, we deduce that we need to put five books into a bookshelf. The total number of possible combinations is equal to* $5!$. *Note that for the first three books the possible combinations are the following:* I, II, III *or* III, II, I. *Hence, the number of required combinations is equal to* $2 \cdot 5! = 240$.

Problem 2. In how many different ways can one go from the city A to the city B and come back (see the figure below), if it is known that the ways for going and coming back do not include parts with the same numbers (e.g. the ways 1,4,8,10,5,2).



Solution. *The number of ways to go from the city A to the city B is equal to* $3 \cdot 4 \cdot 3$. *On the other hand, the number of ways to come back is equal to* $2 \cdot 3 \cdot 2$. *Therefore, the total number of ways to go from the city A to the city B and to come back is equal to* $36 \cdot 12 = 432$.

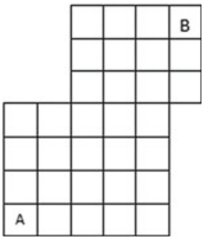
Problem 3. In how many different ways can one read the word "alternation"?
ALTERNATION
LTERNATION
TERNATION
ERNATION.
We start to read the first letter, then we read the letter on the right side or down of that letter.

Solution. *In order to read the word "alternation", one needs to start with the left letter of the row (on the top) and end with the right letter of the row (on the bottom). Moreover, one needs to move only to the right or down. Therefore, the number of the required ways is equal to* $C_{10}^0 + C_{10}^1 + C_{10}^2 + C_{10}^3 = 1 + 10 + 45 + 120 = 176$.

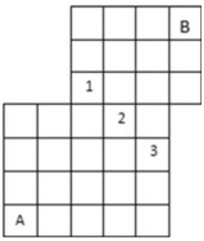
Problem 4. Let the entries of 3×3 grid square be numbers 0 or 1, such that the total sum is an even number. How many such squares are there?

Solution. *Let the entries of* 3×3 *square be the numbers 0 or 1. The number of such squares is equal to* 2^9. *Let us divide all such squares into couples. The squares A and B are called couples, if the sum of the entries of all their corresponding squares is equal to 1. Note that if the sum of all the entries of the square A is equal to m, then the sum of all the entries of the square B is equal to* $9 - m$. *Thus, only one of the sums (of all the entries of A and of all the entries of B) is an even number. Therefore, the number of required squares is equal to* $2^8 = 256$.

Problem 5. In how many ways can one go from the square A to the square B (see the figure below), if it is allowed to move either to the right or to the up "neighbouring" squares? Two squares are called "neighbouring" squares, if they have a common side.



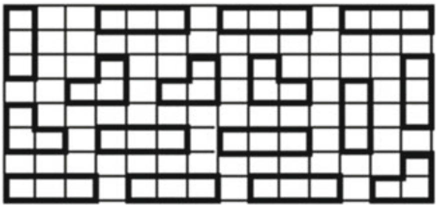
Solution. *Note that any way going from the square A to the square B is passing either through the squares* 1 *or* 2 *or* 3 *(see the figure below). Moreover, it passes through exactly one of those three (1,2,3) squares.*



The number of ways passing through the square 1 is equal to $C_6^2 \cdot C_5^2$. *The number of ways passing through the square 2 is equal to* $C_7^3 \cdot C_4^2$. *The number of ways passing through the square 3 is equal to* $C_7^3 \cdot C_3^1$. *Therefore, the total number of the required ways is equal to* $C_6^2 \cdot C_5^2 + C_7^3 \cdot (C_4^2 + C_3^1) + C_7^3 \cdot C_3^1 = 570$.

Problem 6. At most, how many trominos can one put on 8×14 grid rectangle, such that each tromino covers exactly three squares of the grid rectangle and any two trominos do not have a common point.

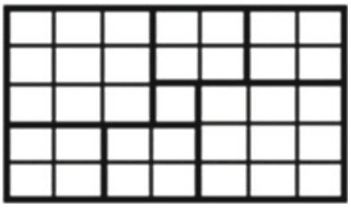
Solution. *Let us consider all the vertices of the squares of* 8×14 *grid rectangle. We have that the total number of vertices is equal to* $9 \cdot 15 = 135$. *Note that every tromino includes exactly eight such vertices. It is given that any two trominos do not have a common point, thus the total number of trominos is not more than 16. Below, we provide an example for 16 trominos.*



Problem 7. At least, how many squares one needs to remove from 2011×2015 grid rectangle, such that the rest of the figure is possible to cut into 2×2, 3×3 squares?

Solution. *At first, note that* 2011×2015 *grid rectangle is not possible to cover only with* 2×2 *and* 3×3 *squares. Let us enumerate the columns of the grid rectangle with numbers* $1, 2, \ldots, 2015$ *(the enumeration is done from the left to the right). Now, let us colour the columns* $1, 2, \ldots, 2015$ *of the grid rectangle in white and the columns* $2, 4, \ldots, 2014$ *in black. Note that, the difference of total numbers of white and black squares in* 2×2 *or* 3×3 *squares is a multiple of 3. On the other hand, 2011 is not a multiple of 3. This ends the proof of the statement.*

Let us now show that one can cover 2011×2015 *grid rectangle with one* 1×1 *square and* 2×2, 3×3 *squares. In the figure below is given one way of coverage for* 5×7 *grid rectangle.*



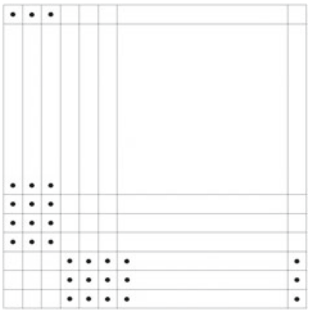
In order to end the proof, it is left to note that if for the coverage of $m \times n$ *grid rectangle one* 1×1 *is sufficient, then it is sufficient also for the coverage of* $m \times (n+6)$ *and* $(m+6) \times n$ *grid rectangles. We have that* $2011 = 6k + 7$ *and* $2015 = 6m + 5$, *where* $k, m \in \mathbb{N}$. *Therefore, the smallest number of necessary* 1×1 *squares is equal to 1.*

Problem 8. On some squares of 150×150 grid square, one puts some playing cards (one playing card per square). Given that, for any playing card, the row or the column including that playing card has no more than three playing cards. At most, how many playing cards can one put on 150×150 grid square?

Solution. *Assume that on the columns are the following playing cards: $a_1, a_2, \dots, a_{150}$. Assume also that on the rows are the following playing cards: $b_1, b_2, \dots, b_{150}$, such that $a_1 \geq a_2 \geq \cdots \geq a_{150}$ and $b_1 \geq b_2 \geq \cdots \geq b_{150}$. Let m and k be positive integers, such that $a_m \geq 4$, $a_{m+1} \leq 3$ and $b_k \geq 4$, $b_{k+1} \leq 3$. According to the condition of the problem, we have that $a_1 + a_2 + \cdots + a_m \leq b_{k+1} + b_2 + \cdots + b_{150} \leq 3(150 - k)$. Therefore, the total number of all playing cards is not more than $3(150-k) + 3(150-m) = 3(300-k-m)$.*

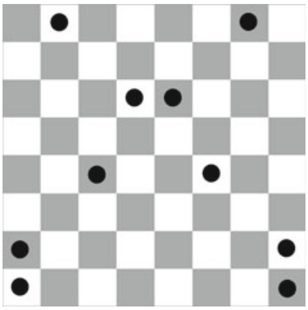
If $k + m \geq 6$, then we obtain that the total number of all playing cards is not more than 882.

If $k + m \leq 5$, then without loss of generality we can assume that $m \leq 2$. Hence, the total number of all playing cards is not more than $2 \cdot 150 + 148 \cdot 3 < 882$. In the figure below is provided an example for the case of 882 playing cards.

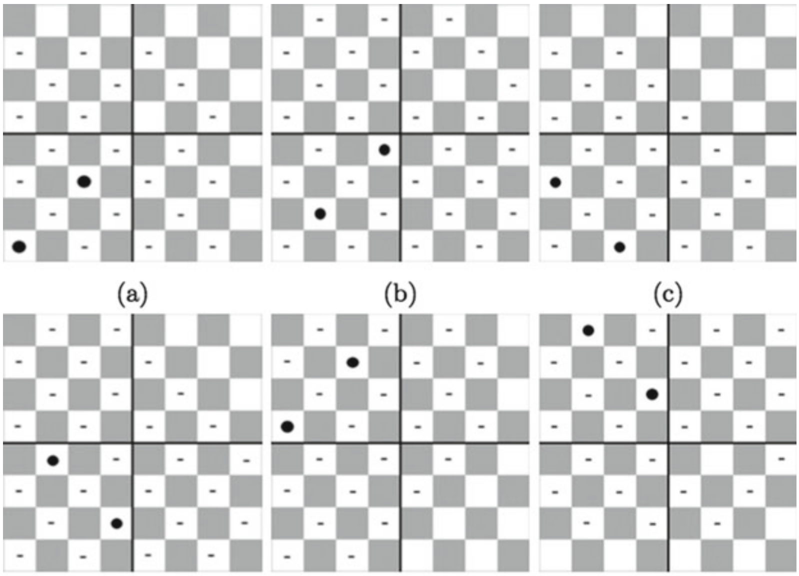


Problem 9. A figure on the chessboard is called a "fast" knight, if it makes simultaneously two steps of a regular knight. At most, how many fast knights can one put on the chessboard, such that none of them is under attack?

Solution. *Let us provide an example of 10 "fast" knights, such that none of them is under attack (see the figure below).*



Let us now prove that from 11 "fast" knights one can choose two "fast" knights that are under attack of each other. It is enough to prove that two of six "fast" knights located on the white squares are under attack of each other. We proceed the proof by contradiction argument. Assume that there are six "fast" knights that are located on the white squares and are not (pairwise) under attack. Let us divide the chessboard into four 4×4 *squares. Therefore, in one of them there are at least two "fast" knights. The following cases are possible:*



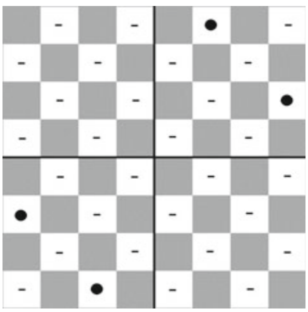
(a) (b) (c) (d) (e) (f)

In each figure, we denote by "-" all the white squares there are no "fast" knights.

In the figures (b) and (d), the number of "fast" knights is not more than 4.

On the other hand, in the figures (e) and (f) the number of "fast" knights is not more than 5. This leads to a contradiction.

It is left to note that the cases (a) and (c) are equivalent. On the other hand, the case of the figure (c) may be deduced to the following case:



In this case also, the number of "fast" knights is equal to 4. This leads to a contradiction.

7.5.7 Problem Set 7

Problem 1. In how many different ways can one put into a bookshelf 6 numbered books of the same author, such that the books number 1 and 3 are not (simultaneously) the neighbours of the book number 2?

Solution. *At first, let us find in how many different ways one can put the book number 2 in between the books number 1 and 3. The answer is* $2 \cdot 4! = 48$. *Thus, we obtain that the solution of the problem is* $6! - 48 = 720 - 48 = 672$.

Problem 2. At most, how many numbers can one choose from the numbers 1,2, ..., 14, such that pairwise subtractions of all the chosen numbers are distinct positive numbers?

Solution. *Assume that we have chosen n numbers from the numbers* $1,2,\ldots,14$, *such that they have distinct positive pairwise subtractions. Hence,* $\frac{n(n-1)}{2} \leq 13$. *Thus,* $n \leq 5$.

All positive subtactions of the numbers $1,8,9,12,14$ *are distinct. Moreover, they are equal to* $1,2,3,4,5,6,7,8,11,13$. *Therefore, the greatest possible value of n is equal to 5.*

Problem 3. On each edge of a triangular pyramid are given four points. Let us consider these 24 points and four vertices of the pyramid. Find the number of all lines passing through any two points from those 28 points.

Solution. *Let us consider three different cases:*

a) Line a passes through two points of the same edge, the number of such lines is equal to 6.

b) Line a passes through two points of the same lateral face, but it does not include any edge. The number of such lines is equal to $4 \cdot (3 \cdot 4 + 3 \cdot 16) = 240$.

c) Line a passes through two points located on the opposite edges, but does not belong to the plane including the lateral face. The number of such lines is equal to $16 \cdot 3 = 48$.

Hence, the total number of such lines is equal to 294.

Problem 4. During the hockey championship, any two teams of 10 participant teams have played with each other exactly once. Given that all the teams have obtained different final scores. At most, how many games could win the team on the last place? Note that in hockey the victory is 2 scores, draw is 1 score, defeat is 0 score.

Solution. *Let the team on the last position has n victories. Then, the sum of the scores of all teams is equal to* $45 \cdot 2 = 90$. *On the other hand, it is not less than* $2n + 2n + 1 + \cdots + 2n + 9 = 20n + 45$. *Therefore,* $90 \geq 20n + 45$. *Hence,* $n \leq 2$. *Let us give an example, where the team on the last position has only two victories.*

	1	2	3	4	5	6	7	8	9	10	Sum
1		2	2	0	0	2	2	2	2	2	14
2	0		2	1	0	2	2	2	2	2	13
3	0	0		1	1	2	2	2	2	2	12
4	2	1	1		2	0	1	0	2	2	11
5	2	2	1	0		0	1	0	2	2	10
6	0	0	0	2	2		1	0	1	2	8
7	0	0	0	1	1	1		2	0	2	7
8	0	0	0	2	2	2	0		0	0	6
9	0	0	0	0	0	1	2	2		0	5
10	0	0	0	0	0	0	0	2	2		4

Problem 5. The entries of 8×8 grid square are positive integers. Given that in any two squares having a common side are written numbers, such that the ratio of the greater one to the smaller one is equal to 2. At most, how many pairwise distinct numbers can be written in the given 8×8 grid square?

Solution. *Let a be the smallest number written in the square and b be any number written in the square. Let* $x_1 = a$, x_2, $\ldots$, $x_n = b$, *where the numbers* x_i *and* x_{i+1}, $(i = 1, \ldots, n-1)$ *are written in the squares having a common side and* $n \leq 15$. *We have that* $x_{i+1} \leq 2x_i$ *and* $b = 2^k a$, *where* $k \in \mathbb{N}$. *Hence,* $b \leq 2^{14} a$. *Therefore, the number of pairwise distinct numbers is not more than 15. Below we provide an example that it is equal to 15.*

2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}
2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}
2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}
2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9
2	2^2	2^3	2^4	2^5	2^6	2^7	2^8
1	2	2^2	2^3	2^4	2^5	2^6	2^7

Problem 6. Consider n sets, such that the following properties hold true:
a) any set has exactly three elements,
b) there exists a set that does not intersect at most with two sets,
c) any element can belong to at most three sets.
Find the greatest possible value of n.

Solution. *According to property b), there exists a set A that intersects with at least* $n-3$ *sets. On the other hand, according to properties a) and c) we have that* $n-3 \leq 6$. *Hence,* $n \leq 9$. *Now, we provide an example of nine sets (satisfying the assumptions of the problem)*

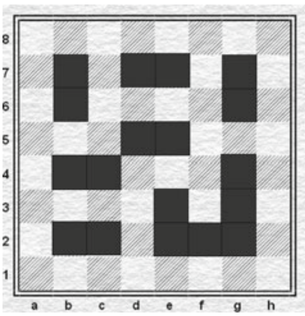
$$\{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,5,9\},\{1,4,8\},\{2,4,7\},\{2,5,8\},\{3,6,9\},\{3,6,7\}.$$

Problem 7. Find the number of all 8-digit numbers with digits 1, 2 or 3, such that any of them either does not have the digit 2 or for any digit 2 one of the neighbours is digit 1 and the other neighbour is digit 3.

Solution. *Denote by* x_n *the number of all n-digit numbers with the digits 1,2,3, such that for any of them the digit 2 is in between the digits 1 and 3. We have that* $x_1 = 2$ *and* $x_2 = 4$. *Note that* $x_{n+2} = x_n + 2x_{n+1}$. *One can prove this formula by considering the following two cases: the second to the last digit of the number with* $n+2$ *digits is equal to 2 or is not equal to 2. Therefore,* $x_3 = 10$, $x_4 = 24$, $x_5 = 58$, $x_6 = 140$, $x_7 = 338$, $x_8 = 816$.

Problem 8. Let n be a positive integer. Given that after placing (in a random way) n dominos on 9×10 rectangular grid, there exists 2×2 square which does not have a square covered by any domino. Each domino covers exactly two squares of a chessboard. Find the possible greatest value of n.

Solution. *The following picture shows that the possible greatest value of n is smaller that 12.*



The greatest value of n is equal to 11, as the number of 2×2 *distinct squares is equal to* $8 \cdot 9 = 72$. *Note that from 11 dominos, each of them can have a common square with, at most, six* 2×2 *squares (all together 66 squares). Hence, there is* 2×2 *square that does not have a common square with any of those 11 dominos.*

Problem 9. The entries of 4×4 grid square are the numbers 0,1 or 2, such that any row and column sum of the grid square is equal to 2. Find the number of all such grid squares (different from each other).

Solution. *Note that any grid square satisfying to the assumptions of the problem can be uniquely represented as the sum of two other grid squares (with the entries 0 and 1). Moreover, in any row and column of each square number 1 is written only once. Let us call such square a "good square":*

$$\begin{array}{|c|c|c|c|}\hline 2&0&0&0\\\hline 0&1&0&1\\\hline 0&0&1&1\\\hline 0&1&1&0\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&0&0&0\\\hline 0&1&0&0\\\hline 0&0&0&1\\\hline 0&0&1&0\\\hline\end{array} + \begin{array}{|c|c|c|c|}\hline 1&0&0&0\\\hline 0&0&0&1\\\hline 0&0&1&0\\\hline 0&1&0&0\\\hline\end{array}$$

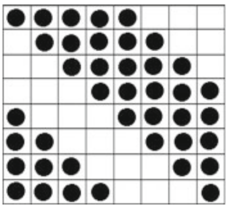
Note that the number of good squares is equal to 24. Let us enumerate them as $A_1, A_2, \ldots, A_{24}$. On the other hand, any square satisfying the assumptions of the problem is of the form $A_i + A_j$, where $i \leq j$. Hence, their number is equal to $1 + 2 + \cdots + 24 = 300$.

7.5.8 Problem Set 8

Problem 1. On some squares of 8×8 grid square are placed candies. Given that the number of candies on each row and on each column is not more than five and it is odd number. At most, how many candies can be placed on the squares of the given grid square?

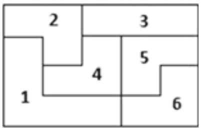
Solution. *We have that on each row the number of candies is not more than 5. Thus, the total number of candies placed on* 8×8 *grid square is not more than 40.*

We provide the following example, where the total number of candies placed on 8×8 *grid square is equal to 40.*



Problem 2. Assume that a grid rectangle is divided into n parts of the forms A, B and C, (such that there are parts of all these forms). Find the smallest possible value of n.

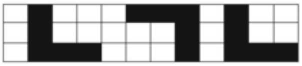
Solution. *In the figure below, we split* 4×5 *grid rectangle into six parts of the given forms.*



A gird rectangle is called "beautiful", if it is possible to divide into the figures of A, B, C forms, such that there are at least one figure A, one figure B and one figure C.

Lemma 7.18. *Let n*$\mathbb{N}$*, then* $3\times n$ *grid rectangle cannot be "beautiful".*

Proof. Proof of the lemma by contradiction argument. Assume that $3\times n$ *grid rectangle is "beautiful" and the number of A figures is equal to m, the number of B or C figures is equal to k. We have that* $5m+3k=3n$*. Thus, it follows that* $3\mid m$*.*



Therefore, one can choose two A figures, such that between them there is no other figure of form A and the number of squares in between them is not divisible by 3. Hence, that part is not possible to divide into B, C figures. This leads to a contradiction.

According to the lemma, if $a\times b$ *grid rectangle is "beautiful", then* $a\geq 4$ and $b\geq 4$.

If $n=3$*, then* $S=11$*, where S is the area of* $a\times b$ *grid rectangle. But,* $S=11$ *leads to a contradiction.*

If $n=4$*, then either* $S=14$ *or* $S=16$*. Both cases lead to a contradiction.*

If $n=5$*, then either* $S=17$ *or* $S=19$ *or* $S=21$*. These cases lead to a contradiction.*

Therefore, the smallest possible value of n is equal to 6.

Problem 3. A man puts n identical coins into eight begs, such that the number of coins in difference begs are different. The begs are for his wife and seven children. Given that no matter which beg chooses the wife, she can distribute her all coins to children, such that all children have equal number of coins. Find the smallest possible value of n.

Solution. *Let the wife took the beg with smallest number of coins. According to the assumptions of the problem, the number of coins in that beg is equal to the sum of seven non-negative integers. Thus, there are at least* $0+1+2+3+4+5+6=21$ *coins in that beg. Hence, we obtain that*

$$n\geq 21+22+23+24+25+26+27+28=196.$$

Let us given an example for 196. For example, if the man puts 21, 22, 23, 24, 25, 26, 27, 28 coins in eight begs, then no matter which beg the wife chooses, she can make the number of coins in the begs of all children to be equal to 28.

Problem 4. Find the number of all finite sequences, such that for any of them hold true the following properties:

a) the first term is equal to 0.

b) every term is greater either by 1 or by 10 than the previous term.

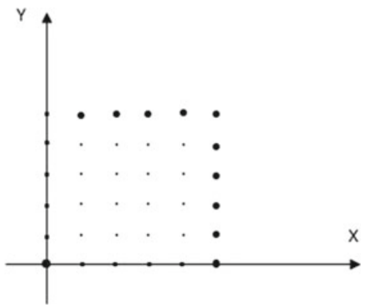
c) any term does not have a digit greater than 5 and at least one of the digits of the last term is equal to 5.

Solution. *Let us choose on the coordinate plane the points* (m,n)*, where* $m,n \in \mathbb{Z}$*,* $0 \leq m \leq 5$*,* $0 \leq n \leq 5$*.*

Let us substitute any sequence by one path, in the following way: we start at point $(0,0)$*. If its any term is equal to the previous term plus 1, then we move one unit to the right side. On the other hand, if its any term is equal to the previous term plus 10, then we move one unit above.*

For example, we substitute sequence 0,1,2,3,4,5 by the path

$$(0,0) \to (0,1) \to (0,2) \to (0,3) \to (0,4) \to (0,5).$$



The total number of sequences is equal to the number of paths that one can go from the point $(0,0)$ *to one of the points* $(0,5)$*,* $(1,5)$*,* $(2,5)$*,* $(3,5)$*,* $(4,5)$*,* $(5,5)$*,* $(5,0)$*,* $(5,1)$*,* $(5,2)$*,* $(5,3)$*,* $(5,4)$*. That is* $2(C_5^0 + C_6^1 + C_7^2 + C_8^3 + C_9^4) + C_{10}^5 = 672$*.*

Problem 5. Find the number of all fourteen-digit numbers with digits 1,2,3, such that any of them either does not have a digit 2 or digit 2 is in between digits 1 and 3. Moreover, any two neighbouring digits are not equal.

Solution. *Denote by* x_n *the number of all n-digit numbers, such that any two of its neighbouring digits are distinct and if one of the digit is equal to 2, then the neighbouring digits of digit 2 are digits 1 and 3.*

We have that $x_1 = 2$*,* $x_2 = 2$*.*

Let $n \geq 3$, consider one of those n-digit number and the following cases:

a) If the digit in penultimate position is equal to 2.

Then, deleting the last two digits of this n-digit number, we obtain that this $(n-2)$-digit number satisfies the assumptions of the problem.

b) If the digit in penultimate position is not equal to 2.

Then, deleting the last digit of this n-digit number, we obtain that this $(n-1)$-digit number satisfies the assumptions of the problem.

Note that obviously $x_{n+2} = x_{n+1} + x_n$, for $n = 1,2,3$.

Therefore, it follows that

$$x_1 = 2, x_2 = 2, x_3 = 4, x_4 = 6, x_5 = 10, x_6 = 16, x_7 = 26, x_8 = 42,$$

$$x_9 = 68, x_{10} = 110, x_{11} = 178, x_{12} = 288, x_{13} = 466, x_{14} = 754.$$

Problem 6. In how many ways can one cover 3×10 grid rectangle by dominos, such that any domino covers exactly two squares and that every square is covered exactly by one domino?

Solution. *If n is an even, then we denote by x_n the total number of different ways of covering 3×10 grid rectangle by dominos.*

If n is odd, then we denote by x_n the total number of different ways of covering 3×10 grid rectangle by dominos after deleting the bottom left corner square.

We have that $x_1 = 1$, $x_2 = 3$.

Note that if $k \geq 2$, then $x_{2k} = 2x_{2k-1} + x_{2k-2}$ and $x_{2k-1} = x_{2k-2} + x_{2k-3}$.

Therefore, we obtain that

$$x_3 = 4, x_4 = 11, x_5 = 15, x_6 = 41, x_7 = 56, x_8 = 56, x_9 = 209, x_{10} = 571.$$

Problem 7. We have chosen arbitrarily five numbers from the numbers $1,2,\ldots,n$. Given that from those five numbers, one can choose some numbers, such that the chosen numbers can be divided into two groups with equal sums (a group can consist of only one term). Find the greatest possible value of n.

Solution. *Let us prove that the greatest possible value of n is equal to 12.*

Assume that one has chosen the numbers x_1, x_2, x_3, x_4, x_5 from the numbers $1,2,\ldots,12$ and $x_1 < x_2 < x_3 < x_4 < x_5$. Let us consider the following cases:

a) If $x_1 + x_2 < x_3$. Consider the interval $[x_3, x_3 + x_4 + x_5]$. From 31 possible sums of the numbers x_1, x_2, x_3, x_4, x_5 the following sums $x_1, x_2, x_1 + x_2, x_1 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5$, $x_1 + x_2 + x_3 + x_4 + x_5$ do not belong to the considered interval. The other 25 sums belong to this interval. On the other hand, the total number of positive integers belonging to this interval is $x_4 + x_5 + 1 \leq 24$. According to Dirichlet's principle, two of those 25 sums are equal.

b) If $x_3 < x_1 + x_2 < x_4$, then from 31 possible sums of the numbers x_1, x_2, x_3, x_4, x_5 the following sums $x_1, x_2, x_1 + x_2, x_1 + x_3 + x_4 + x_5, x_2 + x_3 + x_4 + x_5$, $x_1 + x_2 + x_3 + x_4 + x_5$ do not belong to interval $[x_3, x_3 + x_4 + x_5]$. The other 25 sums belong to

this interval. On the other hand, the total number of positive integers belonging to this interval is $x_4+x_5+1 \le 24$. *According to Dirichlet's principle, two of those 25 sums are equal.*

c) If $x_4 < x_1+x_2 < x_5$, *then from 31 possible sums of the numbers* x_1, x_2, x_3, x_4, x_5 *only 24 belong to interval* $[x_3, x_3+x_4+x_5]$. *On the other hand, the total number of positive integers belonging to this interval is* $x_4+x_5+1 \le 23$. *According to Dirichlet's principle, two of those 24 sums are equal.*

d) If $x_1+x_2 > x_5$, *then from 31 possible sums of the numbers* x_1, x_2, x_3, x_4, x_5 *only 21 belong to interval* $[x_3, x_3+x_4+x_5]$. *On the other hand, the total number of positive integers belonging to this interval is* $x_3+x_4+1 \le 20$. *According to Dirichlet's principle, two of those 21 sums are equal.*

Hence, $x_3+x_4 \ge 20$. *Thus, it follows that* $x_4 = 11$, $x_5 = 12$ *and* $x_3 \ge 9$.

The following cases are possible :

$$*,8,9,11,12(8+12=9+11), \;\; 6,7,*,11,12(6+12=7+11),$$

$$7,8,9,11,12(8+12=9+11), \;\; 4,9,10,11,12(4+9+10=11+12),$$

$$5,8,10,11,12(5+8+10=11+12), \;\; *,9,10,11,12(9+12=10+11),$$

$$6,8,10,11,12(6+12=8+10), \;\; 7,8,10,11,12(7+12=8+11).$$

e) If $x_1+x_2 \in \{x_3, x_4, x_5\}$, *then the statement is proved.*

In order to finish the solution, note that if from the numbers $1,2,\ldots,12,13$ *we choose the numbers 3, 6, 11, 12, 13, then the assumptions of the problem do not hold true.*

Problem 8. In how many ways can one cover 3×11 grid rectangle by dominos after deleting the bottom left corner square, such that any domino covers exactly two squares and that every square is covered exactly by one domino?

Solution. *See the solution of Problem 6. We need to find* x_{11}. *We have that*

$$x_{11} = x_{10} + x_9 = 571 + 209 = 780.$$

Problem 9. Arrangement of 12 dominos on 9×10 grid rectangle is called "convenient", if the following two properties hold true:

a) any domino covers exactly two squares of 9×10 grid rectangle

b) any 2×2 square that covers exactly four squares of 9×10 grid rectangle has a common square at least with one domino.

Find the number of all "convenient" arrangements.

Solution. *Let us at first prove the following lemma:*

Lemma 7.19. *If on* 9×10 *grid rectangle one has a "convenient" arrangement of 12 dominos, then none of the dominos has a common point with the border of the rectangle and any two dominos do not have a common point.*

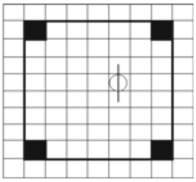
Proof. Proof of the lemma by contradiction argument. Assume that for example domino A has a common point with the border of given grid rectangle or dominos B and C have a common point.

Let us consider all 2×2 *grid squares consisting of the squares of* 9×10 *grid rectangle. Note that their total number is equal to* $8 \cdot 9 = 72$.

Note that any domino can have a common square at most with six dominos of those 72 dominos. Moreover, domino A can have a common square at most with four squares, and dominos B and C together at most with 11.

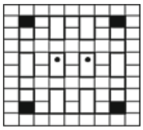
In any case, there is 2×2 *grid square that does not have a common square with any of those dominos. Hence, it follows that the arrangement of dominos is not "convenient".*
This leads to a contradiction.
Hence, this ends the proof of the lemma.

Let us proceed the solution.
According to the lemma, 12 dominos of "convenient" arrangement are placed inside of 7×8 *grid rectangle denoted by* Φ.



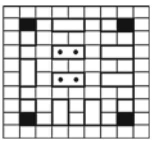
Obviously, the border squares of rectangle Φ *are covered by dominos. Considering 16 possible cases, using the lemma, we deduce that the number of "convenient" arrangements is equal to 9.*

Fig. 7.1 1, possible



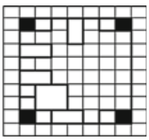
.4

Fig. 7.2 4, possible



.4

Fig. 7.3 4, impossible



.4

Fig. 7.4 1, impossible

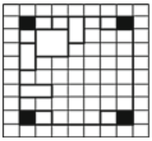


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Fig. 7.5 2, possible

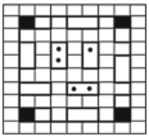

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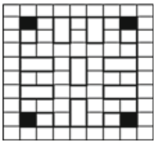
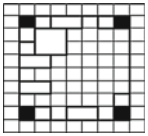
Fig. 7.6 2, impossible



.3

Fig. 7.7 2, impossible



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7.5.9 Problem Set 9

Problem 9. Given that n mathematicians take part in a mathematical conference. Let the following properties hold true:

a) Any two participants have a common acquaintance participant.

b) Any participant is acquainted with not more than three participants.

Find the possible greatest value of n.

Solution. *Assume that from those n mathematicians A and B are acquaintances. According to the assumption of the problem, they have a common acquaintance C. If any of A, B, C is not acquainted with the others, then property b) does not hold true.*

Let A is acquainted with D. Then, the common acquaintance of A and D is either B or C. Without loss of generality, one can assume that it is B. We have that a) and b) hold true for A, B, C, D.

If, except those, there is some participant E, then the common acquaintance of A and E is C. Hence, E and C do not have a common acquaintance. This leads to a contradiction.

Therefore, the greatest possible value of n is equal to 4.

Problem 2. In how many ways can one cover 3×18 grid rectangle with L-shape trominos, such that each L-shape tromino covers exactly three squares and each square is covered only by one tromino?

Solution. *Note that the trominos, converting the top and bottom left squares, create* 3×2 *grid rectangle (in two ways). Therefore, the answer is* $2^9 = 512$.

Problem 3. Find the number of all sequences $x_1, x_2, \ldots, x_8$, such that for any of them it holds true:

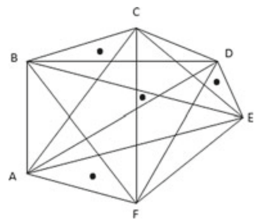
a) $x_i \neq x_j$, if $i \neq j$.
b) $x_i \in \{1, 2, \ldots, 8\}$, where $i = 1, 2, \ldots, 8$.
c) $3 \mid x_i + x_{i+1} + x_{i+2}$, where $i = 1, 2, \ldots, 6$.

Solution. *We have that for* $1 \leq i \leq 5$ *it holds true* $3 \mid x_i + x_{i+1} + x_{i+2}$ *and* $3 \mid x_{i+1} + x_{i+2} + x_{i+3}$. *Thus, it follows that* $3 \mid x_{i+3} - x_i$. *Hence,* x_1, x_4, x_7 *are divisible by* 3 *with the same remainder. In a similar way, we deduce that* x_2, x_5, x_8 *are divisible by* 3 *with the same remainder. On the other hand,* x_3, x_6 *are divisible by* 3 *with the same remainder. Note that the opposite statement holds true also.*

Therefore, the number of required sequences is equal to $2 \cdot 3! \cdot 3! \cdot 2 = 144$.

Problem 4. Let $ABCDEF$ be a convex hexagon. Given that its diagonals AD, BE and CF do not intersect at one point. At least, how many points does one need to choose inside of $ABCDEF$, such that inside of twenty triangles created by vertexes A, B, C, D, E, F there is at least one point from the chosen points?

Solution. *Note that inside of each of the triangles ABC, ACD, ADE, AEF should be a point. Hence, the number of required points is not less than 4. Let us provide the following example for four points:*



Therefore, one needs to choose at least four points.

Problem 5. In a king's wine cellar, there are 120 barrels of wine that are numbered by $1, 2, \ldots, 120$ numbers. The king knows that one of his servants has poisoned one of the barrels, but he does not know the number of the poisoned barrel. Given that after drinking any portion of the poisoned wine the person dies the next day. Given also that the king can make his 10 servants to taste the wine from any barrel he chooses. At least, how many servants does the king need to send to the wine cellar to taste the wines in order to know (the next day) the number of the poisoned barrel?

Solution. *Let the king has sent to the wine cellar the servants $A_1, A_2, \ldots, A_n$. Note that the set of servants tasting the wine of any barrel is a subset of $A_1, A_2, \ldots, A_n$. Therefore, in order the king to know (the next day) the number of the poisoned barrel, it is necessary that the subsets corresponding to those barrels are pairwise distinct.*

Thus, it follows that $2^n \geq 120$. We deduce that $n \geq 7$.

If $n = 7$, then the number of subsets is equal to 128. Hence, from 120 barrels one can find the poisoned one.

We obtain that the answer is 7.

Problem 6. Find the number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_7 = 15,$$

such that it holds true $1 \leq x_i \leq 3$, $i = 1, 2, \ldots, 7$.

Solution. *Let $x_i = 2 + y_i$, $i = 1, \ldots, 7$. Hence, $y_i \in \{-1, 0, 1\}$ and $y_1 + y_2 + \cdots + y_7 = 1$. Note that the number of solutions of the last equation is equal to*

$$C_7^1 + C_7^2 \cdot C_5^1 + C_7^3 \cdot C_4^2 + C_7^4 = 357.$$

Problem 7. Given that if any n squares of 19×61 grid rectangle are painted in black, then there is a L-shape tromino consisting of three black squares. Find the possible smallest value of n.

Solution. *In the figure below 610 squares are painted in black and there is no tromino with the black squares.*

Let us prove that if any 611 trominos are painted in black, then there is a L-shape tromino with three black squares.

Let us at first prove the following lemmas:

Lemma 7.20. *If $n \in \mathbb{N}$ and any $2n + 2$ squares of $2 \times (2n + 1)$ grid rectangle are painted in black, then either there is a L-shape tromino with three black squares or in considered grid rectangle there are $n + 1$ rows that follow unpainted rows.*

The proof of this lemma can be easily done using mathematical induction.

Lemma 7.21. *If $n \in \mathbb{N}$ and any $4n + 3$ rows of $3 \times (2n + 1)$ grid rectangle are painted in black, then there is a L-shape tromino with three black squares.*

Let us separate from $3 \times (2n+1)$ grid rectangle $2 \times (2n+1)$ grid rectangle. According to the assumption of the lemma at least $2n+2$ squares of the separated grid rectangle are painted black. If at least $2n+3$ squares of the separated grid rectangle are painted black, then according to the first lemma there is a L-shape tromino with three black squares.

If exactly $2n+2$ squares of the separated grid rectangle are painted in black, then the column that is not included in the separated grid square is fully painted. Therefore, according to the first lemma, there is a L-shape tromino with three black squares.

Lemma 7.22. *If $n \geq k$, $n, k \in \mathbb{N}$ and any $(k+1)(2n+1)+1$ squares of $(2n+1) \times (2k+1)$ grid rectangle, then there is a L-shape tromino with three black squares.*

The proof of this lemma by mathematical induction.

Basis. *If $k = 1$, then the statement holds true, as it is equivalent to the second lemma.*

Inductive step. *Let us show that if the statement holds true for $k = m$, where $m \in \mathbb{N}$, $m+1 \leq n$, then it also holds true for $k = m+1$.*

Let us separate from $(2n+1) \times (2m+3)$ grid rectangle $2 \times (2n+1)$ right grid rectangle. If at least $2n+3$ squares of separated grid rectangle are painted in black, then according to the first lemma there is a L-shape tromino with three black squares.

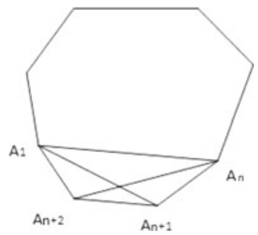
If the number of black squares in the separated grid rectangle is not more than $2n+1$, then in the other $(2n+1) \times (2m+1)$ grid rectangle the number of black squares is not less than $(m+2)(2n+1)+1-(2n+1) = (m+1)(2n+1)+1$. This ends the proof of the statement.

If the number of the black squares in the separated grid rectangle is equal to $2n+2$, then it is sufficient to consider the case, when in the separated grid rectangle painted $n+1$ rows follow the unpainted rows. Let us consider the third column (from the right), if at least $n+2$ squares of this column are painted in black, then there are two among them that have a common side. Therefore, there is a L-shape tromino with three black squares. Otherwise, the number of black squares in the initial grid rectangle is not more than $2n+2+(n+1)+m(2n+2)$. This leads to a contradiction, as $2n+2+(n+1)+m(2n+2) < (m+2)(2n+1)+1$.

The last lemma ends the solution.

Problem 8. At most, how many diagonals can one consider in a convex 101-gon, such that any of them has a common (inner) point with not more than one diagonal (from the considered diagonals)?

Solution. *Let us denote by $f(n)$ the maximum number of diagonals in a convex n-gon, ($n \geq 4$), such that any of those diagonals has a common (inner) point with no more than one diagonal. Note that $f(n+2) \geq f(n)+3$, (see the figure below).*



By mathematical induction, one can easily deduce that

$$f(n) \geq \left\lceil \frac{3n-8}{2} \right\rceil.$$

Let $n \geq 5$. Consider $f(n)$ diagonals of a convex n-gon such that any of those diagonals has a common (inner) point with no more than one diagonal, then at least one of those diagonals does not have any common (inner) point with the other diagonals. In order to prove this statement, let us consider two intersecting diagonals and note that one of the sides of the quadrilateral (such that its vertexes are the endpoints of those diagonals) is the required diagonal.

If $n \geq 5$ and we consider $f(n)$ diagonals of a convex n-gon such that any of those diagonals has a common (inner) point with no more than one diagonal, then at least one of those diagonals does not have any common (inner) point with the other diagonals, then we choose that diagonal from those $f(n)$ diagonals, such that it does not have any inner point with the other diagonals. Assume that the convex n-gon is divided into a convex k-gon and $n-k+2$-gon. Therefore, $f(n) \leq f(k)+f(n-k+2)+1$.

Thus, by mathematical induction, one can easily deduce that

$$f(n) \leq \left\lceil \frac{3n-8}{2} \right\rceil.$$

Hence, if $n \geq 5$, then $f(n) = \left\lceil \frac{3n-8}{2} \right\rceil$. Note that this formula holds true for $n = 4$ too. We have that $f(101) = 147$.

Problem 9. A 10×10 table consists of 100 unit cells. A *block* is a 2×2 square consisting of 4 unit cells of the table. A set C of n blocks covers the table (i.e. each cell of the table is covered by some block of C), but no $n-1$ blocks of C cover the table. Find the greatest possible value of n.

Solution. *Consider an infinite table divided into unit cells. Any 2×2 square consisting of 4 unit cells of the table we also call a block.*

Fix arbitrary set M of blocks lying on the table. Now, we will consider arbitrary finite sets of unit cells of the table covered by M. For any such Φ denote by $|\Phi|$ the least possible number of blocks of M that cover all cells from Φ.

We have the following properties:

P1) *If* $\Phi_1 \subseteq \Phi_2$*, then* $|\Phi_1| \leq |\Phi_2|$.

P2) $|\Phi_1 \cup \Phi_2| \leq |\Phi_1| + |\Phi_2|$.

P3) *For the set A shown in Fig. 1, we have* $|A| = 2$*; for the set B shown in the Fig. 2, we have* $|B| = 3$.

Fig.1 (right) and Fig. 2 (left)

Fig. 7.8*

P4) *Let C be any rectangle* 3×6 *of the table. Then* $|C| \leq 10$.

This estimate is proved by consideration of different ways in which the cells X and Y (see Fig. 3–8) can be covered by the blocks of M (Fig. 9,10 and 11).

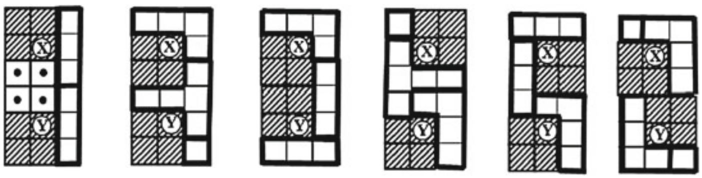


Fig. 3 - Fig. 8 (from right to left)

Fig. 7.9*

For these figures we have, respectively, the following estimates:

Fig. 3: Case 1) $|C| \leq 2+2+3+3+1+1$ *or Case 2)* $|C| \leq 1+1+1+1+1+1+1+1$.

Fig. 4: $|C| \leq 3+3+1+1+1$.

Fig. 5: $|C| \leq 2+2+3+1+1$.

Fig. 6: $|C| \leq 3+3+1+1+1$.

Fig. 7: $|C| \leq 2+2+3+1+1$.

Fig. 8: $|C| \leq 3+3+1+1+1+1$.

Remark 7.4. In the Fig. 3, the first case means that the four marked cells are covered by at most three blocks: the second case means that the marked cells are covered by four different blocks.

Remark 7.5. The Fig. 8 presents the only case where $|C|$ *can attain the value 10; in all other figures, we have that* $|C| \leq 9$.

P5) *Let D be any* 6×6 *square of the table. From previous properties, it follows that* $|D| \leq 20$*. We claim that in fact* $|D| \leq 19$*. This easily follows from the Fig. 9 and remark 7.5 (using two different ways of dividing D into two rectangles* 3×6*).*

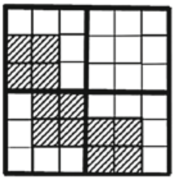


Fig. 9

Fig. 7.10*

Now, we can finish the solution of the problem. Let E be given 10×10 table, D be its central 6×6 square. We have $|D| \leq 19$. One can easily verify that $|E \setminus D| \leq 20$ (applying the properties P1)-P4)). So, $|E| \leq |D| + |E \setminus D| \leq 19 + 20 = 39$. On the other hand, Fig. 10 shows that $n = 39$ can be attained.

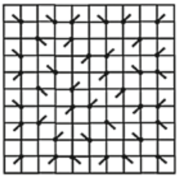


Fig. 10

Fig. 7.11*

In this figure, the marked points are the centres of blocks participating in the covering. For convenience, we marked by half-diagonals those unit cells which are covered by the only block.

7.5.10 Problem Set 10

Problem 1. In how many ways can one put six ordered books in the bookshelf, such that the neighbouring books of the book number 4 are not the books number 1, 2 and 3?

Solution. *The number of such arrangements is equal to*

$$2 \cdot 2 \cdot 4! + 2 \cdot 4! = 144.$$

Problem 2. Find the number of all solutions (x, y, z) of equation $x + y + z = 64$, such that x, y, z are positive integers and y is an even number.

Solution. *If* $y = 2k$, *where* $k = 1, 2, \ldots, 31$, *then* $x + z = 64 - 2k$. *Hence, the number of* $(x, 2k, z)$ *triples is equal to* $63 - 2k$.

Therefore, the number of (x, y, z) *triples satisfying the assumptions of the problem is equal to* $61 + 59 + \cdots + 1 = 31^2 = 961$.

Problem 3. Find the number of all positive integers smaller than 10^4, such that any of them is divisible by 11 and the sum of its digits is equal to 19.

Solution. *Note that such positive integer is either a three-digit number or four-digit number.*

Let $11 \mid \overline{abc}$ *and* $a+b+c=19$, *then* $11 \mid a-b+c$. *Hence,* $11 \mid 19-2b$. *Thus, it follows that* $b=4$ *and* $a+c=15$.

Therefore, such three-digit numbers are 649, 748, 847 and 946.

Let $11 \mid \overline{abcd}$ *and* $a+b+c+d=19$, *then* $11 \mid a-b+c-d$. *Hence, either* $a+c=15$, $b+d=4$ *or* $a+c=4$, $b+d=15$.

If $a+c=15$, $b+d=4$, *the number of four-digit numbers* $\overline{abcd}$ *is equal to* $4 \cdot 5 = 20$.

If $a+c=4$, $b+d=15$, *then the number of four-digit numbers* $\overline{abcd}$ *is equal to* $4 \cdot 4 = 16$.

Thus, the number of positive integers satisfying the assumptions of the problem is equal to 40.

Problem 4. Find the number of three-digit numbers, such that any of them has exactly eight divisors.

Solution. *A three-digit number with exactly eight divisors has one of the following forms:* p^7 *or* p^3q *or* pqr, *where* p, q, r *are pairwise distinct prime numbers.*

A three-digit number of the form p^7 *is only* $2^7 = 128$.

	$p=2$	$p=3$	$p=5$	$p=7$	*total*
number of three-digit numbers of form p^3q	*25*	*10*	*4*	*1*	*40*

$p<q<r$	$p=2, q=3$	$p=2, q=5$	$p=2, q=7$
number of three-digit numbers of form pqr	*32*	*21*	*16*

$p<q<r$	$p=2, q=11$	$p=2, q=13$	$p=2, q=17$
number of three-digit numbers of form pqr	*9*	*6*	*3*

$p<q<r$	$p=2, q=19$	$p=3, q=5$	$p=3, q=7$
number of three-digit numbers of form pqr	*1*	*15*	*11*

$p<q<r$	$p=3, q=11$	$p=3, q=13$	$p=3, q=17$
number of three-digit numbers of form pqr	*5*	*3*	*1*

$p<q<r$	$p=5, q=7$	$p=5, q=11$	*total*
number of three-digit numbers of form pqr	*5*	*2*	*130*

Therefore, the number of three-digit numbers that have eight divisors is equal to 171.

Problem 5. Given that 64 mathematicians take part in a mathematical conference, such that among any three, at least two participants are acquaintances. Find the smallest possible number of all couples of acquaintances.

Solution. *Let the number of participants is equal to n, $n \geq 3$. Given that among any three, at least two participants are acquaintances. Denote by $f(n)$ the smallest possible number of all couples of acquaintances.*

Let us prove the following properties.

P1. *If $n \in \mathbb{N}$ and $n \geq 5$, then $f(n) \geq n-2+f(n-2)$.*

In order to prove this property, from n participants, let us choose participants A and B that are not acquaintances. Hence, for the rest $n-2$ participants the assumptions of the problem hold true and any of those participants either an acquaintance of A or B. Therefore, $f(n) \geq n-2+f(n-2)$.

P2. *If $k \in \mathbb{N}$ and $k \geq 2$, then $f(2k) \geq k(k-1)$.*

Note that if $k \in \mathbb{N}$, then $f(2k+1) \geq k^2$.

We have that $f(3) = 1$, $f(4) = 2$, the proof follows from P1 (using mathematical induction).

P3. *If $k \in \mathbb{N}$, then $f(2k) = k(k-1)$ and if $k \in \mathbb{N}$, then $f(2k+1) = k^2$.*

Let $k \geq 3$, let us prove that $f(2k) \leq k(k-1)$.

Indeed, if we divide those $2k$ people into two groups, such that in each group there are k participants and in any group any two participants are acquaintances. Therefore, the assumptions of the problem hold true and the number of all couples of acquaintances is equal to $k(k-1)$.

In a similar way, let us divide $2k+1$ people $(k \geq 2)$ into two groups, such that in one group there are k and in the other one $k+1$ people. In each group, any two participants are acquaintances. Hence,

$$f(2k+1) \leq \frac{k(k-1)}{2} + \frac{(k+1)k}{2} = k^2.$$

Using P2 ends the proof of P3

Therefore, we obtain that $f(64) = 992$.

Problem 6. A grasshopper is in the centre of the rightmost square of 1×14 grid rectangle. It can jump one or two squares either to the right or to the left. A 'successful journey' is a journey consisting of 13 jumps, such that the grasshopper has managed to be on all squares of 1×14 grid rectangle. Find the number of all successful journeys.

Solution. *Let us enumerate (from left to right) the squares of 1×14 grid rectangle by numbers $1, 2, \ldots, 14$. Assume that after 13th jump the grasshopper is on x_1 square, after 12th jump on x_2 square and so on, after the first jump on x_{13} square and at the beginning it was on x_{14} square.*

According to the assumptions of the problem, we have that:

a) $x_i \neq x_j$, if $i \neq j$, where $i, j \in \{1, 2, \ldots, 14\}$.

b) $x_i \in \{1, 2, \ldots, 14\}$, where $i = 1, 2, \ldots, 14$.

c) $|x_i - x_{i+1}| \in \{1,2\}$*, where* $i = 1,2,\dots,13$.
d) $x_{14} = 14$.
We need to find the number of x_1*,* $x_2,\dots,$ x_{14} *sequences.*
Denote by a_n*,* $(n \geq 2)$ *the number of* x_1*,* $x_2,\dots,$ x_n *sequences, such that it holds true:*
a) $x_i \neq x_j$*, if* $i \neq j$*, where* $i,j \in \{1,2,\dots,n\}$.
b) $x_i \in \{1,2,\dots,n\}$*, where* $i = 1,2,\dots,n$.
c) $|x_i - x_{i+1}| \in \{1,2\}$*, where* $i = 1,2,\dots,n-1$.
d) $x_n = n$.
Note that

$$a_n = a_{n-1} + a_{n-3} + 1,\ n = 4,5,6,\dots. \tag{7.120}$$

Therefore, $a_2 = 1$*,* $a_3 = 2$*,* $a_4 = 4$*, from* (7.120) *it follows that* $a_5 = 6$*,* $a_6 = 9$*,* $a_7 = 14$*,* $a_8 = 21$*,* $a_9 = 31$*,* $a_{10} = 46$*,* $a_{11} = 68$*,* $a_{12} = 100$*,* $a_{13} = 147$*,* $a_{14} = 216$.
Hence, the number of all successful journeys is equal to 216.

Problem 7. Consider any 301 rectangles with positive integer sides, such that all sides are less than or equal to 100. Among those rectangles are chosen the maximum number of rectangles A_1, A_2, $\dots$, A_k, such that any of them, except the last one, is possible to cover by the next one. Find the smallest possible value of k.

Solution. *Note that the perimeters of rectangles* 1×100*,* 2×99*,* $3\times 98,\dots,$ 50×51 *are equal. Hence, none of the rectangles covers the other one. Therefore, if we take seven rectangles of the first type and six rectangles of the second type then* $k \leq 7$.

Let us now prove that if we have rectangles $a_i \times b_i$*, where* $a_i, b_i \in \mathbb{N}$*,* $a_i \leq 100$*,* $b_i \leq 100$ *and* $a_i \leq b_i$*,* $i = 1,2,\dots,301$*, then we can choose seven among them, such that any of them is possible to cover with the next one. Hence,* $k \geq 7$.

Instead of each rectangle $a_i \times b_i$ *consider point* (a_i, b_i) *on the coordinate plane. Now, consider the following sets*

$$M_1 = \{(1,1),(1,2),(1,3),\dots,(1,100),(2,100),(3,100),\dots,(100,100)\},$$

$$M_2 = \{(2,2),(2,3),\dots,(2,99),(3,99),\dots,(99,99)\},$$

$$\dots$$

$$M_{50} = \{(50,50),(50,51),(51,51)\}.$$

Note that any of points (a_i, b_i) *belongs to one of the sets* M_1*,* M_2*,* $\dots$*,* M_{50}.
Therefore, according to Dirichlet's principle seven of these points belong to the same set. Hence, for the seven rectangles corresponding to those seven points the required condition holds true. Thus, it follows that $k = 7$.

Problem 8. Consider all 10×10 grid squares, such that the entries of any of them are numbers from 1 to 100 (written in a random way and in any square of any 10×10 grid square is written only one number). For any 10×10 grid square

consider all positive differences of numbers written in any two squares that have at least one common vertex. Let M be the greatest among those differences. Find the possible smallest value of M.

Solution. *Let us consider the following table.*

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Note that $M = 11$. *Thus, the required number is not more than 11.*

Let us now prove that if the entries of 10×10 *are the numbers* $1, 2, \dots, 100$ *(written in a random way), such as in any square is written one number, then there are two squares with at least one common vertex, such that the difference of the numbers written in that squares is not less than 11.*

Hence, we will deduce that the required number is not less than 11. Therefore, it is equal to 11.

Indeed, let us consider the squares, where are written the numbers 1 and 100.

Note that one can choose squares $A_1, A_2, \dots, A_{10}$*, such that squares* A_i *and* A_{i+1}*,* $i = 1, \dots, 9$ *have a common vertex.*

On the other hand, number 1 is written in the square A_1 *and number 100 is written in the square* A_{10}.

Let number a_i *is written in the square* A_i*, where* $i = 1, \dots, 10$. *Hence, we have that*

$$99 = a_{10} - a_1 = (a_{10} - a_9) + (a_9 - a_8) + \dots + (a_2 - a_1).$$

Therefore, there exists a number i, such that

$$a_{i+1} - a_i \geq 11.$$

This ends the proof of the statement.

Problem 9. Find the number of all sequences $x_1, x_2, \dots, x_{13}$, such that for any of them it holds true

a) $x_i \neq x_j$, if $i \neq j$, where $i, j \in \{1, 2, \dots, 13\}$.
b) $x_i \in \{1, 2, \dots, 13\}$, where $i = 1, 2, \dots, 13$.
c) $|x_i - x_{i+1}| \in \{1, 2\}$, where $i = 1, 2, \dots, 12$.

Solution. *Denote by a_n, $n > 1$ the number of sequences $x_1, x_2, \dots, x_n$, such that it holds true*

a) $x_i \neq x_j$, if $i \neq j$, where $i, j \in \{1, 2, \dots, n\}$.

b) $x_i \in \{1, 2, \dots, n\}$, where $i = 1, 2, \dots, n$.

c) $|x_i - x_{i+1}| \in \{1, 2\}$, where $i = 1, 2, \dots, n-1$.

d) $x_n = n$.

Note that $a_2 = 1$, $a_3 = 2$, $a_4 = 4$ and $a_k = a_{k-1} + a_{k-3} + 1$, $k = 4, 5, 6, \dots$

Thus, it follows that $a_2 = 1$, $a_3 = 2$, $a_4 = 4$, $a_5 = 6$, $a_6 = 9$, $a_7 = 14$, $a_8 = 21$, $a_9 = 31$, $a_{10} = 46$, $a_{11} = 68$, $a_{12} = 100$, $a_{13} = 147$.

Note that the number of sequences satisfying the assumptions of the problem and the condition $x_{12} = 13$ is equal to $a_{11} + a_{10} = 114$.

In a similar way, the number of sequences satisfying the assumptions of the problem and the condition $x_{11} = 13$ is equal to $a_9 + a_8 = 52$.

Therefore, the answer is $2 + 2(3 + 10 + 23 + 52 + 114 + 147) = 700$.

7.5.11 Problem Set 11

Problem 1. Find the number of all three-digit numbers, such that for any of them the sum of all digits is a square number.

Solution. *We need to find the number of all three-digit numbers, such that for any of them the sum of all digits is equal to either 1 or 4 or 9 or 16 or 25.*

The number of such three-digit numbers is equal to 1, 10, 45, 66, 6, respectively. Therefore, the number of all three-digit numbers, such that for any of them the sum of all digits is a square number, is equal to 128.

Problem 2. Find the number of all solutions (x, y, z) of the equation $x + y + z = 100$, such that x, y, z are positive integers and $13 \mid x + z$.

Solution. *Note that $y \in \{9, 22, 35, 48, 61, 74, 87\}$. Therefore, the number of all solutions (x, y, z) satisfying the assumptions of the problem is equal to*

$$90 + 77 + 64 + 51 + 38 + 25 + 12 = 357.$$

Problem 3. The squares of 3×3 grid square are painted in either red or blue or yellow. Given that any two squares with a common side are painted in different colours. In how many such ways can one paint 3×3 grid square?

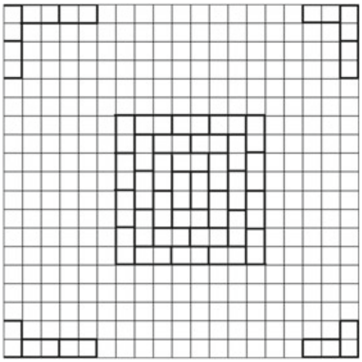
Solution. *If the central square is red, then consider the four squares having a common side with it. The number of all possible ways is equal to $8 + 4 \cdot 4 + 4 \cdot 4 + 2 + 4 \cdot 4 + 8$. Therefore, the answer is $3 \cdot 66 = 198$.*

Problem 4. One covers 2016×2016 grid square by dominos, such that every domino covers exactly two squares and every square is covered by only one domino. Let k be the number of 2×2 grid squares, such that any of them is covered by exactly two dominos. Find the smallest possible value of k.

Solution. *At first, let us prove that if* 2016×2016 *grid square is covered by dominos, then there exists* 2×2 *grid square that is covered by exactly two dominos. The proof by contradiction argument. Assume that there does not exist* 2×2 *grid square that is covered by exactly two dominos.*

Note that if we try to cover the squares of the diagonal of 2016×2016 *grid square, starting from the left bottom square, then the right top square is not possible to cover.*

Now, let us provide an example, where only one 2×2 *grid square (of* 2016×2016 *grid square) is covered by exactly two dominos.*



Therefore, the smallest possible value of k is equal to 1.

Problem 5. Find the number of all sequences $x_1, x_2, \ldots, x_{100}$, such that for any of them it holds true

a) $x_i \neq x_j$, if $i \neq j$, where $i, j \in \{1, 2, \ldots, 100\}$.

b) $x_i \in \{1, 2, \ldots, 100\}$, where $i = 1, 2, \ldots, 100$.

c) $|x_i - x_{i+1}| = 1$, where $i = 1, 2, \ldots, 94$.

Solution. *Note that* $x_i \neq 1$, *where* $i = 2, 3, \ldots, 94$.

If $x_1 = 1$, *then* $x_2 = 2$, $x_3 = 3$, $\ldots$, $x_{95} = 95$. *Hence, the number of such sequences is equal to 5!.*

If $x_{95} = 1$, *then the number of such sequences is equal to 5!.*

If $x_{96} = 1$, *then the number of such sequences is equal to 4!, and so on.*

Therefore, the number of all such sequences is equal to

$$2 \cdot 5! + 4! + 3! + 2! + 1 = 273.$$

Problem 6. Fourteen participants took part in a chess championship, such that any two participants played together only once. Given that the sum of the final points of the participants in the first three places is six times more than the sum of the final points of the participants in the last four places. Find the sum of the final points of the other seven participants, if a win is worth one point to the victor and none to the loser, a draw is worth a half point to each player.

Solution. *Let us denote by x_i the sum of the final points of the participant in the i-th place, where $i = 1, 2, \ldots, 14$.*

Note that the participants in the last four places have played together six games, thus it follows that

$$x_{11} + x_{12} + x_{13} + x_{14} \geq 6.$$

On the other hand,

$$x_1 + x_2 + x_3 \leq 3 + 3 \cdot 11.$$

According to the assumptions of the problem, we have that

$$x_1 + x_2 + x_3 = 6(x_{11} + x_{12} + x_{13} + x_{14}).$$

Thus, it follows that

$$x_1 + x_2 + x_3 = 36,$$

and

$$x_{11} + x_{12} + x_{13} + x_{14} = 6.$$

Therefore, the sum of the final points of the other seven participants is equal to $91 - 42 = 49$.

Problem 7. Find the smallest number n, such that among any n numbers chosen from the numbers $1, 2, \ldots, 1000$ there are two numbers, such that their ratio is equal to 3.

Solution. *We call a "chain" of length k the sequence $b_1, b_2, \ldots, b_k$, where $b_i \in \{1, 2, \ldots, 1000\}$, $i = 1, \ldots, k$, $3 \nmid b_1$, $b_{i+1} = 3b_i$, $i = 1, \ldots, k-1$ and $3b_k > 1000$.*

Let us consider the following chains (1, 3, 9, 27, 81, 243, 729), (2, 6, 18, 54, 162, 486), $\ldots$, where any next chain is constructed in the following way: we consider the smallest positive integer a that is not included in none of the previous chains (obviously $3 \nmid a$), then we consider the chain $a, 3a, 9a, \ldots$

Note that if the ratio of two numbers is equal to 3, then these numbers are consequent terms of the same chain. On the other hand, in the "chain" of length k one can choose, at most $[\frac{k+1}{2}]$ terms such that any two among them are not consequent terms.

Therefore, from the numbers $1, 2, \ldots, 1000$ *one can choose*

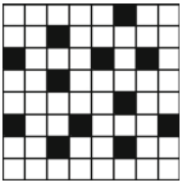
$$4+2\cdot 3+5\cdot 3+17\cdot 2+49\cdot 2+148\cdot +445=750,$$

numbers, such that there are no two numbers with the ration equal to 3.

On the other hand, if we choose 751 numbers, then there is a "chain" of length k one can choose, such that we have chosen from it more than $[\frac{k+1}{2}]$ *terms. Therefore, at least two of the chosen terms are consequent terms. Hence, their ratio is equal to 3.*

Problem 8. At least, how many squares of 8×8 grid square does one need to paint in black, such that at least one of the squares of any 2×3 grid rectangle (consisting of the six squares of considered 8×8 grid square) is black?

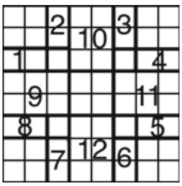
Solution. *In the figure below, we provide an example of 12 black squares, such that the assumptions of the problem hold true.*



Let us prove that one cannot choose 11 black squares on 8×8 *grid square, such that any* 2×3 *grid rectangle (consisting of the six squares of considered* 8×8 *grid square) includes a black square. Proof by contradiction argument. Assume that any* 2×3 *grid rectangle includes at least one black square from some 11 black squares.*

Now, let us prove the following properties:

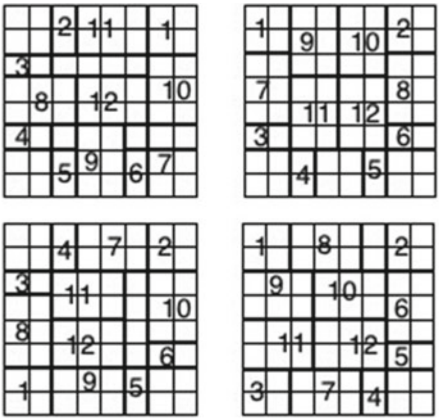
P1. *At least one of the four corner* 2×2 *grid squares of* 8×8 *grid square includes a black square. Proof by contradiction argument.*



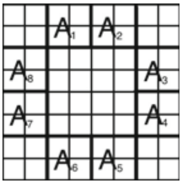
In this case, in the figure above, the rectangles $1, 2, \ldots, 12$ *include a black square. This leads to a contradiction.*

P2. *All (four) corner* 2×2 *grid squares of* 8×8 *grid square include a black square.*

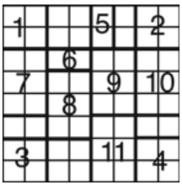
The proof in a similar way, as for P1. See the figure below.



P3. *At least one of the squares A_1, $A_2, \ldots,$ A_8 (see the figure below) does not include a black square.*

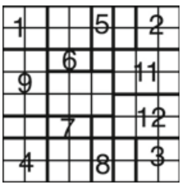


Otherwise, the number of black square is not less than 12.



Assume that A_1 does not include a black square, then A_6 does not include a black square (see the figure above).

On the other hand, the figure below shows that there does not exist such 11 squares.

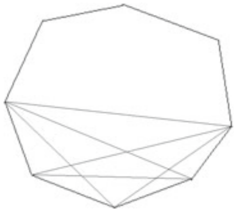


This leads to a contradiction.

Problem 9. At most, how many diagonals can one choose in a convex 500-gon, such that any of them has a common inner point with no more than two chosen diagonals?

Solution. *Denote by $f(n)$ the maximum number of diagonals of a convex n-gon, such that any of them has a common inner point with no more than two diagonals.*

Note that $f(n+3) \geq f(n)+6$, (see the figure below) and $f(3)=0$, $f(4)=2$, $f(5)=5$.



Hence, by mathematical induction, we obtain that

$$f(n) \geq 2n-6+\left[\frac{n+1}{3}\right]-\left[\frac{n}{3}\right].$$

Let us prove that if $n \geq 4$, $n \in \mathbb{N}$ and we consider $f(n)$ diagonals, such that any of them has a common inner point with no more than two diagonals, then

$$f(n) \leq 2n-6+\left[\frac{n+1}{3}\right]-\left[\frac{n}{3}\right].$$

Consider the following cases:

If there is a diagonal among those $f(n)$ diagonals that do not have a common inner point with any of those diagonals, then consider this diagonal. Let us assume that the considered convex n-gon is divided by this diagonal into a convex k-gon and $(n-k+2)$-gon. Therefore,

$$f(n) \leq f(k)+f(n-k+2)+1.$$

Thus, by mathematical induction, it follows that

$$f(n) \leq 2n-6+\left[\frac{n+1}{3}\right]-\left[\frac{n}{3}\right].$$

If any diagonal among those $f(n)$ diagonals has a common inner point with (at least) one of those diagonals, let us consider any of those diagonals as a line segment connecting the chain of line segments, such that any line segment is a side of the considered n-gon. Now, among those diagonals consider a diagonal, such that the chain of line segments corresponding to it has the minimum number of line segments. Let us denote this minimal number of line segments by p.

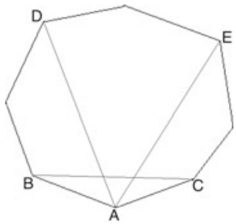
If $p \geq 3$, then according to the assumption of the problem, there are two neighbouring vertexes of the considered n-gon corresponding to not more than two diagonals. Thus, it follows that

$$f(n) \leq f(n-2)+3. \tag{7.121}$$

If $p = 2$, then according to the assumption of the problem, there is a vertex of the considered n-gon corresponding to either one or two diagonals. Therefore,

$$f(n) \leq f(n-1)+2, \tag{7.122}$$

otherwise according to the figure below, it follows that either diagonals BE, DC are not considered or they are both considered simultaneously.



In the first case, if diagonal BE is not considered, then let us consider this diagonal and do not consider diagonal AD. Hence, we obtain $(n-1)$-gon, such that the assumption of the problem holds true. Thus,

$$f(n) \leq f(n-1)+2.$$

In the second case, we obtain that there is no diagonal that has a common inner point with segment BD. Therefore, it is one of the sides of the considered n-gon, as it cannot be one of the considered $f(n)$ diagonals. Otherwise, note that by adding this diagonal the assumption of the problem holds true, this leads to a contradiction with the definition of $f(n)$.

In a similar way, one can prove that line segments DE and EC are also sides of the considered n-gon. Therefore, $n = 5$. We deduce that

$$f(n) \leq 2n-6+\left[\frac{n+1}{3}\right]-\left[\frac{n}{3}\right].$$

Taking into consideration (7.121) and (7.122), by mathematical induction, we obtain that

$$f(n) \leq 2n-6+\left[\frac{n+1}{3}\right]-\left[\frac{n}{3}\right].$$

Hence, it follows that

$$f(n) = 2n - 6 + \left[\frac{n+1}{3}\right] - \left[\frac{n}{3}\right].$$

We deduce that

$$f(500) = 995.$$

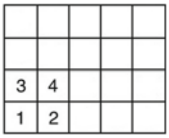
7.5.12 Problem Set 12

Problem 1. Find the number of all two-digit numbers, such that each of them is divisible by all its digits.

Solution. *Let two-digit number $\overline{ab}$ be such that $a \mid \overline{ab}$ and $b \mid \overline{ab}$. Therefore $a \mid b$ and $b \mid 10a$. Hence, such two-digit numbers are 11, 12, 15, 22, 24, 33, 36, 44, 48, 55, 66, 77, 88, 99. Thus, it follows that the number of all such two-digit numbers is equal to 14.*

Problem 2. Let every square of 4×5 grid rectangle be painted in one of the following colours: red, blue, yellow or green. Given that any two squares with a common vertex are painted in different colours. In how many distinct ways can the given grid rectangle be painted in such a way?

Solution. *Note that square 1 is possible to paint in four ways, square 2 in three ways, square 3 in two ways and square 4 in one way (see the figure below).*



On the other hand, after painting squares $1,2,3,4$, *the first two columns can be painted in four ways. Afterwards, let us paint all the other columns.*

The total number of such ways of painting will be equal to 11.

Therefore, the answer is $4! \cdot 11 = 264$.

Problem 3. In how many distinct ways can one choose three numbers among the numbers $1, 2, \ldots, 20$, such that their sum is divisible by 3?

Solution. *The sum of three integers is divisible by 3, if either those numbers are divisible by 3 with the same remainder or with the pairwise distinct remainders. Therefore, the answer is*

$$C_7^3 + C_7^3 + C_6^3 + 7 \cdot 7 \cdot 6 = 384.$$

Problem 4. In how many distinct ways can one arrange the numbers $1, 2, \ldots, 9$ on a circle, such that the sum of any three consequently written numbers is divisible by 3?

Solution. *Note that the condition of the problem holds true, if and only if any three consequently written numbers are divisible by 3 with distinct remainders. Therefore, the answer is*

$$2 \cdot 2 \cdot 3! \cdot 3! = 144.$$

Problem 5. In how many distinct ways can one divide the set of natural numbers into two subsets, such that in any subset the ratio (quotient) of any two elements is not a prime number?

Solution. *Let the set of natural numbers be divided into subsets A and B, such that in any subset the quotient of any two elements is not a prime number*

Let $1 \in A$. Note that all prime numbers belong to B. We have that any prime number greater than 1 is possible to represent as $n = p_1^{\alpha_1} \cdot \dots \cdot p_k^{\alpha_k}$, where $p_1, \dots, p_k$ are distinct prime numbers and $\alpha_1, \dots, \alpha_k$ are positive integers. Let us denote by $h(n)$ the sum $\alpha_1 + \dots + \alpha_k$.

By mathematical induction, with respect to $h(n)$, we deduce that if $h(n)$ is an even number, then $n \in A$. Otherwise, if $h(n)$ is an odd number, then $n \in B$.

Basis. *If $h(n) = 1$, then n is a prime number. Thus $n \in B$.*

Inductive step. *Assume that the statement holds true for $h(n) = m$, $m \in \mathbb{N}$, let us prove that it holds true for $h(n) = m + 1$.*

Let $n = p_1^{\alpha_1} \cdot \dots \cdot p_k^{\alpha_k}$ and $\alpha_1 + \dots + \alpha_k = m + 1$.

If m is an even number and $\alpha_1 > 1$, then

$$h(p_1^{\alpha_1 - 1} \cdot \dots \cdot p_k^{\alpha_k}) = m.$$

Thus, it follows that

$$p_1^{\alpha_1 - 1} \cdot \dots \cdot p_k^{\alpha_k} \in A.$$

We deduce that

$$n = p_1 \cdot p_1^{\alpha_1 - 1} \cdot \dots \cdot p_k^{\alpha_k} \in B.$$

If $\alpha_1 = 1$, then

$$h(p_2^{\alpha_2} \cdot \dots \cdot p_k^{\alpha_k}) = m,$$

and

$$p_2^{\alpha_2} \cdot \dots \cdot p_k^{\alpha_k} \in A, \quad n \in B.$$

If m is an odd number, the proof is in a similar way as in the previous case.

This ends the proof of the statement.

We have obtained that subsets A and B are defined uniquely.

On the other hand, obviously subsets A and B obtained in such a way satisfy the assumptions of the problem.

Hence, the number of required divisions is equal to 1.

Problem 6. Find the number of all sequences $x_1, x_2, \dots, x_{101}$, such that for any of them it holds true:

a) $x_i \in \{1,2,\dots,101\}$, where $i = 1,2,\dots,101$.
b) $|x_i - x_j| \geq |i-j|$, where $i,j \in \{1,2,\dots,101\}$.

Solution. *According to b), we have that*

$$|x_{101} - x_1| \geq 100.$$

On the other hand, according to a), we have that

$$|x_{101} - x_1| \leq 100.$$

Thus, it follows that

$$|x_{101} - x_1| = 100.$$

Consider two cases.

Case I. $x_1 = 1$, $x_{101} = 101$.

Let us prove that $x_i = i$, $i = 2,3,\dots,100$.

We have that

$$100 = |x_i - x_1| + |x_{101} - x_i| \geq |i-1| + |101-i| = 100.$$

Therefore, $x_i - x_1 = i-1$. *Hence,* $x_i = i$.

Case II. $x_1 = 101$, $x_{101} = 1$.

Let us consider the following sequence $y_i = 102 - x_i$. *Note that for* (y_i) *hold true* a) *and* b) *conditions. Therefore, according to Case I, we obtain that* $y_i{=}i$. *Hence* $x_i = 102 - i$, $i{=}1,2,\dots,101$. *Hence, the number of all such sequences* (x_n) *is equal to 2.*

Problem 7. One writes the numbers 1, 2, ..., 12 in the squares of 3×4 grid rectangle, such that in each square is written only one number. Afterwards, one considers the product of three numbers written in each column. Find the smallest possible value of the greatest product among those four products.

Solution. *Note that, in the figure below, the greatest product is equal to 162.*

7	9	10	12
5	6	8	11
4	3	2	1
140	162	160	132

Let us prove that the greatest product cannot be less than 162. Proof by contradiction argument. Assume that there is a grid rectangle A, such that the greatest product is less than 162. Now, consider the column of grid rectangle A that does not include the numbers 1, 2, 3.

Without loss of generality, one can assume that this column is the first one. Then, it is possible the following cases:

6			
5			
4			

a)

7			
5			
4			

b)

8			
5			
4			

c)

In the cases a) and c), let us consider (from the other three columns) the column that does not include the numbers 1, 2. The product of the numbers of that column is not less than $3 \cdot 7 \cdot 8 = 168$. *This leads to a contradiction.*

In the case b), we have that in one of the columns are written the numbers 3, 6, 8 (see the figure below).

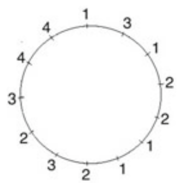
7	8		
5	6		
4	3		

In the figure above, one of the empty columns does not include the number 1. Therefore, the produce of the numbers written in that column is not less than $2 \cdot 9 \cdot 10 = 180$. *This leads to a contradiction.*

Hence, the smallest possible value of the greatest product (among those four products) is equal to 162.

Problem 8. Let the circle be divided into 13 equal arcs. At most, in how many colours does one need to paint those 13 separation points, such that for any colouring there are three points of the same colour that are the vertices of an isosceles triangle?

Solution. *In the figure below is given an example of colouring in four colours, where there are no three points of the same colour that are the vertices of an isosceles triangle.*



Let us prove that if we colour in three colours, then there are three points of the same colour that are the vertices of an isosceles triangle.

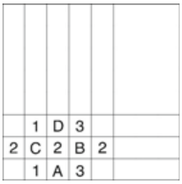
According to the Dirichlet's principle, there are at least five points of the same colour. Assume that points A_1, A_2, A_3, A_4, A_5 *are painted in the first colour. Assume that there are no three points among those five points that are the vertices of an isosceles triangle. For any points* A_i *and* A_j*, the midpoint of one of two arcs* A_iA_j *is one of those 13 points; therefore, two of the quadrilaterals with the vertices* A_1, A_2, A_3, A_4, A_5 *are trapezoids. Thus, it follows that there are three points among those five points that are the vertices of an isosceles triangle. This leads to a contradiction.*

Problem 9. Let each square of $n \times n$ grid square be painted in one of the given three colours, such that any square has at least two neighbour (having a common side) squares painted in different colours. Moreover, for any square the number of neighbour squares painted in different colours are equal. Find the possible greatest value of n.

Solution. *Let us give an example of* 4×4 *grid square satisfying the assumptions of the problem (see the figure below).*

1	2	3	1
3	2	3	2
3	2	3	2
1	2	3	1

Now, let us prove that if $n \geq 5$, *then there does not exist such grid square. Consider the figure below.*



Note that the neighbour squares of the square A are painted in 1,2,3 colours. The neighbour squares of the square B are painted in colours 2,3 colours (see the figure). The colours of the neighbour squares of the square C can be uniquely identified. On the other hand, for the square D the assumptions of the problem does not hold true. Therefore, the greatest possible value of n is equal to 4.

Problem 10. A 6-digit number is called “interesting”, if it has distinct digits and is divisible by 999. Find the number of all “interesting” numbers.

Solution. *Note that an “interesting” is of the form* $999 \cdot \overline{abc}$.

On the other hand, if $c = 0$, *then we have that*

$$999 \cdot \overline{abc} = \overline{abc000} - \overline{abc} = \overline{a(b-1)9(9-a)(10-b)0}.$$

The number of “interesting” numbers of this form is equal to $8 \cdot 6 = 48$.

If $c \neq 0$, *then we have that*

$$999 \cdot \overline{abc} = \overline{abc000} - \overline{abc} = \overline{ab(c-1)(9-a)(9-b)(10-c)}.$$

The number of “interesting” numbers of this form is equal to $8 \cdot 7 \cdot 6 + 8 \cdot 6 = 384$.

Therefore, the number of all “interesting” numbers is equal to 432.

Problem 11. There are 2016 excellent pupils in the city and they all take part in the annual meeting of the excellent pupils. Given that any participant has the same number of acquaintances (excellent pupils). Moreover, for any boy the number of female acquaintances is bigger than the number of male acquaintances, and for any girl, the number of female acquaintances is not less than the number of male acquaintances. Given also that the number of female excellent pupils is not more than 1120. Find the smallest possible value of the number of the acquaintances of each pupil.

Solution. *Denote by m the number of male excellent pupils and by n the number of female excellent pupils. We have that* $n \leq 1120$. *Therefore,* $m \geq 896$. *Let us denote by k the number of the acquaintances of each pupil (taking part in the annual meeting). Consider the number of couples of female–male acquaintances. Let either* $k = 2l$ *or* $k = 2l + 1$, $l \in \mathbb{Z}$. *This number is not less than* $m(l+1)$. *On the other hand, this number is not more than* nl. *Thus, it follows that* $nl \geq m(l+1)$. *Hence,*

$$l \geq \frac{m}{n}(l+1) \geq \frac{4}{5}(l+1).$$

Therefore, $l \geq 4$ *and* $k \geq 8$.

Let us consider 224 groups, such that in each group there are nine pupils (five female and four male) and any two pupils in the considered group are acquaintances. Moreover, any two pupils from the different groups are not acquaintances. Therefore, the assumptions of the problem hold true and $k = 8$. *Hence, the answer is 8.*

Problem 12. Given that 20 teams took part in the volleyball championship, such that any two teams have played together only once. Team A is called "stronger" than team B, if either A won B or there is a team C, such that A won C and C won B. A team is called a "champion", if it is "stronger" than any other team. Given that there is no team that won all the others. Find the smallest possible value of the number of "champions".

Solution. *Let us prove the following properties.*

P1. *In the end of the championship, the team that has most scores is a "champion".*

Proof by contradiction argument. Assume that, in the end, team A has most scores, but it is not a "champion". In this case, there is a team B, such that A is not "stronger" than B. Therefore, A has lost to B. On the other hand, any team that has lost to A has lost to B too. Thus, B has more scores than A. This leads to a contradiction.

P2. *If A is a "champion" and it did not win all the other teams, then one of the teams that have won A is a "champion".*

Denote by M *the set of teams that have won A. According to P1, there is a team B belonging to set* M, *such that it is "stronger" than all the other teams belonging to set* M.

B is a "champion", as it has won A. Therefore, it is "stronger" than all the other teams that have lost A.

P3. *The number of "champions" is not less than 3.*

According to P1, there is a team A that is a "champion". On the other hand, according to P2 there is a team C that has won B and is a "champion".

This ends the proof of the statement.

P4. *The number of "champions" can be equal to 3.*

Let us enumerate the teams with the numbers 1, 2, ..., 20.

If team a has won team b, then let us denote it by a → b.

Now, let us provide an example with three "champions".

$$1 \to 2 \to 3 \to 1, \ \ 4 \to 5 \to 6 \to \cdots \to 20 \to 4, \ \ i \to j,$$

if $i \in \{1,2,3\}$, $j \in \{4,5,\ldots,20\}$ *and the other games of teams* $4,5,\ldots,20$ *have random results.*

In this case, "champions" are only teams 1, 2, 3.

Therefore, the smallest possible number of "champions" is 3.

7.5.13 Problem Set 13

Problem 1. In how many ways can one split the set $\{1,2,3,4,5,6,7,8\}$ into two disjoint subsets, such that in each subset the sum of all elements is a prime number?

Solution. *Note that these sums can be equal to 5, 31 or 7, 29 or 13, 23 or 17, 19.*

Let the sum of some elements of the set $\{1,2,3,4,5,6,7,8\}$ *is equal to n. Denote by* a_n *the number of all such equations. Hence, we need to find* $a_5 + a_7 + a_{13} + a_{17}$.

Note that $a_5 = 3$, $a_7 = 5$, $a_{13} = 11$, $a_{17} = 13$. *Therefore* $a_5 + a_7 + a_{13} + a_{17} = 32$.

Problem 2. In how many ways can one put six different fruits in three plates, such that at least in two plates the number of fruits is equal?

Solution. *At first, let us find in how many ways one can put six different fruits in three plates. Let us enumerate the plates by numbers 1, 2, 3 and enumerate each fruit by the number of the plate, where that fruit will be placed. Hence, we need to find the number of six-digit numbers consisting of digits 1, 2, 3. We have that it is equal to* $3^6 = 729$.

Now, let us find in how many ways can one put six different fruits in three plates, such that there are no two plates with equal number of fruits.

If the number of fruits in the plates is equal to 1, 2, 3, then the number of all such placements is equal to $6 \cdot C_6^1 \cdot C_5^2$.

If the number of fruits in the plates is equal to 0, 1, 5 or 0, 2, 4, then the number of all such placements is equal to $6 \cdot C_6^1 + 6 \cdot C_6^2$.

Thus, it follows that the number of all placements is equal to

$$729-6\cdot C_6^1\cdot C_5^2-6\cdot C_6^1-6\cdot C_6^2=243.$$

Problem 3. Let 14 teams take part in a volleyball championship. Given that any two teams have played with each other only once. A group consisting of several teams is called the strongest, if any team (except the teams belonging to that group) has lost against at least one of the teams belonging to that group. Given that in any championship there exists a strongest group consisting of n teams. Find the smallest possible value of n.

Solution. *At first, let us prove that in any championship there exists a strongest group consisting of three teams. Note that the number of victories is equal to 91, therefore there is a team that has at least seven victories. Let team A has won teams* A_1*,* A_2*,* A_3*,* A_4*,* A_5*,* A_6*,* A_7*. In a similar way, among the other six teams there is a team B that has won at least three teams among those teams. Assume that team B has won teams* B_1*,* B_2*,* B_3*. Let one of the other two teams has won the other team. Assume that team C has won team* C_1*. Thus, it follows that A, B, C is the strongest group.*

Now, let us provide an example there does not exist a strongest group consisting of two teams.

If team a has won team b, then denote it by $a\to b$*. Let us enumerate the teams by numbers* $1,2,\ldots,14$*. In the following example, the results of the teams that are not written are not important:*

$$1\to2\to3\to1,\ \ 4\to5\to6\to4,\ \ 7\to8\to9\to\cdots\to14\to7,\ \ i\to j,$$

if $i\in\{1,2,3\}$*,* $j\in\{4,5,6\}$ *or* $i\in\{4,5,6\}$*,* $j\in\{7,\ldots,14\}$ *or* $i\in\{7,\ldots,14\}$*,* $j\in\{1,2,3\}$*.*

Hence, the smallest possible value of n is equal to 3.

Problem 4. Find the smallest possible value of the number of diagonals of a convex hexagon, such that none of those diagonals is parallel to any side of the hexagon.

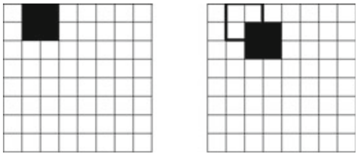
Solution. *Note that a convex hexagon has nine diagonals. In a convex hexagon, any side can be parallel at most to one diagonal. Thus, it follows that at least three diagonals are not parallel to any side.*

Now, let us provide an example of a convex hexagon, such that six diagonals are parallel to the sides.

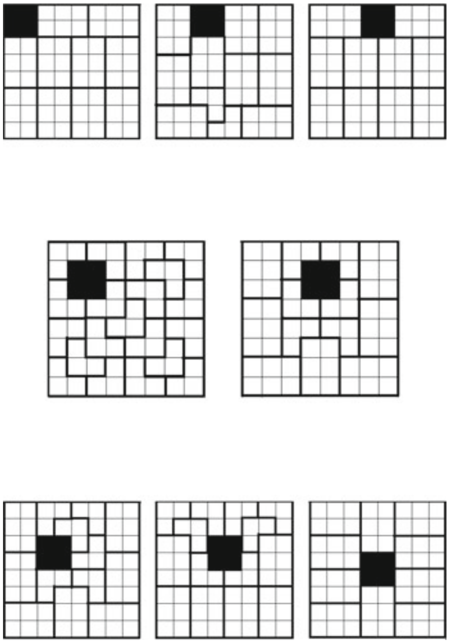
Let the vertices of a convex hexagon ABCDEF be $A(0,0)$*,* $B(0,2)$*,* $C(6,3)$*,* $D(12,2)$*,* $E(14,1)$*,* $F(12,0)$*. Note that* $AB\parallel DF$*,* $BC\parallel AD$*,* $CD\parallel BF$*,* $DE\parallel CF$*,* $EF\parallel AC$*,* $AF\parallel BD$*. Therefore, the answer is 3.*

Problem 5. In how many ways can one take 2×2 grid square away from 8×8 grid square, such that the obtained figure is possible to divide into twenty L-shape trominos?

Solution. *From the figure below, one deduces that there are 16 ways, such that it is not possible to divide into trominos.*



From the figure below, it follows that for the other 33 cases, it is possible to divide into trominos.



We obtain that the answer is 33.

Problem 6. Given six boxes, such that for any two boxes one of them is possible to put inside of the other one. In how many ways can one arrange (put inside of each other) these boxes? (e.g. one of the possible arrangements is $\{(6,4,1),(5,3),2\}$, in this example the boxes are enumerated by numbers $1,2,\dots,6$).

Solution. *The boxes can be arranged in* $1,2,\dots,6$ *places. Therefore, the total number of all possible arrangements is equal to*

$$1+(C_6^1+C_6^2+\frac{1}{2}C_6^3)+(C_6^4+C_6^3\cdot C_3^2+\frac{1}{6}C_6^3\cdot C_4^2)+(C_6^3+\frac{1}{2}C_6^2\cdot C_4^2)+C_6^2+1=203.$$

Problem 7. Find the number of all non-empty subsets of set $\{1,2,\dots,11\}$, such that for any of them the sum of all elements is divisible by 3.

Solution. *At first, let us find the number of all subsets of set* $\{1,2,4,5,7,8,10,11\}$, *such that for any of them the sum of all elements is divisible by 3.*

Note that the sum of all elements is equal to 96. Therefore, it is sufficient to find the number of subsets that have at most four elements.

The number of such subsets that have two elements is equal to $C_4^1 \cdot C_4^1 = 16$.

The number of such subsets that have three elements is equal to $C_4^2 \cdot C_4^2 = 36$.

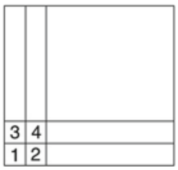
Hence, we obtain that the number of all such subsets is $1+2(16+8)+36 = 85$.

Note that we need to find the number of subsets of the form $A \cup B$ *and* C, *where* A *is a subset of set* $\{1,2,4,5,7,8,10,11\}$, B *is a subset of set* $\{3,6,9\}$ *and* C *is a non-empty subset of set* $\{3,6,9\}$.

Therefore, the final answer is $85 \cdot 8 + 7 = 687$.

Problem 8. Let each square of $n \times n$ $(n > 1)$ grid square be painted in one of the following colours: red, blue, yellow or green. Given that any two squares with at least one common vertex are painted in different colours. Denote by A_n the number of all possible $n \times n$ grid squares painted in such a way. Find $\dfrac{A_{19}}{A_{10}}$.

Solution. *Note that square 1 is possible to paint in four ways, square 2 in three ways, square 3 in two ways and square 4 in one way (see the figure below).*



Note that after enumeration of squares 1, 2, 3, 4, the squares of the first and the second columns can be enumerated in 2^{n-2}.

Note that if in the second column there are squares of three different colours, then the third and all the other columns are possible to enumerate in a single way.

If in the second column there are squares of two different colours, then the third and all the other columns are possible to enumerate in two ways.

Thus, it follows that

$$A_n = 4! \cdot (2^{n-2} - 1 + 2^{n-2}) = 4! \cdot (2^{n-1} - 1).$$

Therefore

$$\frac{A_{19}}{A_{10}} = \frac{4! \cdot (2^{18} - 1)}{4! \cdot (2^9 - 1)} = 2^9 + 1 = 513.$$

Problem 9. At most, how many vertices of a regular 26-gon can one choose, such that any three vertices among the chosen ones are the vertices of a scalene triangle?

Solution. *Consider a regular 26-gon $A_1A_2A_3 \ldots A_{26}$. One can easily verify that if we choose vertices A_1, A_2, A_4, A_5, A_{14}, A_{15}, A_{17}, A_{18}, then any three vertices among the chosen ones are the vertices of a scalene triangle.*

Now, let us prove that if we choose any nice vertices of a regular 26-gon, then there are three vertices among the chosen ones that are the vertices of an isosceles triangle.

Note that regular 13-gons

$$A_1A_3A_5A_7A_9A_{11}A_{13}A_{15}A_{17}A_{19}A_{21}A_{23}A_{25},$$

and

$$A_2A_4A_6A_8A_{10}A_{12}A_{14}A_{16}A_{18}A_{20}A_{22}A_{24}A_{26}.$$

According to Dirichlet's theorem. at least five vertices of one of those 13-gons are chosen.

Let us prove that if five vertices of regular 13-gon $C_1C_2 \ldots C_{13}$ are chosen, then there are three vertices among them that are the vertices of an isosceles triangle.

We proceed the proof by contradiction argument. For any chosen vertices C_i and C_j, the midpoint of one of the arcs C_iC_j is a vertex of that regular 13-gon and is not chosen. Therefore, we obtain that two quadrilaterals among the quadrilaterals with four vertices from the chosen five vertices are trapezoids. Thus, it follows that there are three vertices among the chosen ones that are the vertices of an isosceles triangle. This leads to a contradiction. Hence, the answer is 8.

Problem 10. Let in some squares of 11×11 grid square are written asterisks. Two asterisks are called neighbours, if the squares corresponding to them have at least one common vertex. Given that any asterisk has at most one neighbour asterisk. Find the possible greatest number of all asterisks.

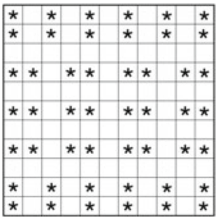
Solution. *Denote by n the number of all asterisks. Let us write on any vertex of each square of 11×11 grid square the number of asterisks, such that any of those asterisks is in the square with that vertex.*

The sum of all such numbers is equal to $4n$. On the other hand, the number of vertices with number 2 is not more than n. Therefore

$$4n \leq 12^2 + n.$$

Thus, it follows that $n \leq 48$.

We provide an example for 48 asterisk, such that the assumptions of the problem hold true (see the figure below).



Problem 11. Consider 64 balls having eight different colours, such that there are eight balls of each colour. At least, how many balls does one need to take from those balls in order to be able to put them in the line, such that for any two different colours there is a ball with two neighbour balls of these colours.

Solution. *At first, let us prove that if we put less than or equal to 31 balls in a line, then the assumptions of the problem do not hold true.*

We proceed the proof by contradiction argument. Assume that there are less than or equal to 31 balls satisfying the assumptions of the problem. According to Dirichlet's principle, the number of balls of some colour is not more than 3. On the other hand, these balls need to be the neighbours of at least seven other balls. This leads to a contradiction.

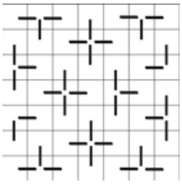
Now, let us provide an example of such 32 balls. In this example, we enumerate the colours by numbers 1, 2, 3, 4, 5, 6, 7, 8.

$$6,7,8,8,4,5,8,2,3,8,7,1,2,3,4,5,6,4,5,1,2,6,3,2,4,1,7,5,6,7,3,1.$$

Therefore, the answer is 32.

Problem 12. Find the smallest positive integer n, such that the following statement holds true: if 7×7 grid square is covered (in a random way) by n rectangles, such that any rectangle covers exactly two squares of 7×7 grid square, then one can take away one of those n rectangles, in such a way that the rest of the rectangles again cover the grid square.

Solution. *Let us provide an example of coverage (of 7×7 grid square) with 37 rectangles, such that after taking away any of those 37 rectangles the other 36 rectangles cannot cover the grid square. Thus, it follows that $n \geq 38$.*



Now, let us prove that if 7×7 *grid square is covered by 38 such rectangles, then one can take away one of those rectangles, such that the rest of the rectangles (37 rectangles) cover the grid square.*

We proceed the proof by contradiction argument. Assume that after taking away any of those 38 rectangles the other 37 rectangles cannot cover the grid square.

Therefore, for any rectangle there is a square, such that it is covered only by that rectangle. Let us write the symbol $*$ *on all such squares. Note that the number of all symbols* $*$ *is equal to* $38+m$, *where* m *is the number of rectangles covering two symbols* $*$. *On the other hand, the number of blank (without* $*$*) squares is equal to* $49-(38+m) = 11-m$ *and the rectangles covering any of those* $11-m$ *squares are creating figures consisting of* 3, 4, 5 *squares. Denote by* k, l, p *the number of these figures, respectively. Thus, it follows that*

$$k+l+p = 11-m,$$

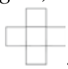
and

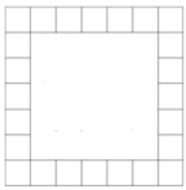
$$3k+4l+5p = 49-2m.$$

We obtain that

$$k-p = 4(k+l+p)-(3k+4l+5p) = -5-2m.$$

Hence, we deduce that $p \geq 5+2m$.

On the other hand, as 7×7 *grid square is covered by* m *rectangles,* k *figures consisting of three squares,* l *figures of type*  *and* p *figures of type* . *Hence, 24 squares in the figure below are covered by those figures.*



Therefore

$$24 \leq 2m+3(k+l)+p = 2m+3(11-m-p)+p,$$

or

$$2p \leq 9-m.$$

This leads to a contradiction, as $p \geq 5+2m$.

Thus, it follows that $n = 38$.

7.5.14 Problem Set 14

Problem 1. In how many ways is it possible to put five different candies in three different plates, if in the first plate is allowed to put not more than one, in the second plate not more than two and in the third plate not more than three candies?

Solution. *Let us consider the following cases.*

If we do not put any candy in the first plate, then the number of such placements is equal to C_5^2.

If we put only one fruit in the second plate, then the number of such placements is equal to $C_5^1 \cdot C_4^1$.

If we put two candies in the third plate, then the number of such placements is equal to $C_5^1 \cdot C_4^2$.

Thus, it follows that the number of all such placements is equal to

$$C_5^2 + C_5^1 \cdot C_4^1 + C_5^1 \cdot C_4^2 = 60.$$

Problem 2. Let each square of 3×3 grid square be painted in one of the following colours: red, blue or yellow. Given that any two squares having a common side are painted in different colours. Given also that in each colour are painted exactly three squares. In how many ways is it possible to paint (as it was described) this grid square?

Solution. *Let us consider all possible ways of painting three squares that are on the diagonal.*

If these three squares are painted in three different colours, then all the other squares can be painted in only one way. The number of all such grid squares is equal to 6.

If these three squares are painted in the same colour, then we obtain that the number of all such grid squares is equal to 6.

If these three squares are painted in two different colours, the we deduce that the number of all such grid squares is equal to $2 \cdot 6 \cdot 2 = 24$.

Therefore, the grid square is possible to paint (as it was described) in 36 different ways.

Problem 3. In how many ways is it possible to put ten different fruits in two different plates, such that in any plate there are at least three fruits?

Solution. *At first, let us find out in how many ways can one put ten different fruits in two different plates. Any fruit is possible to put either in the first plate or in the second plate, thus the number of ways (of such placements) is equal to the number of ten-digit numbers with digits 1 and 2. Note that the number of such ten-digit numbers is equal to* 2^{10}. *Therefore, the total number of ways (of placements satisfying the assumptions of the problem) is equal to*

$$2^{10} - C_{10}^0 - C_{10}^1 - C_{10}^2 - C_{10}^8 - C_{10}^9 - C_{10}^{10} = 1024 - 2(1 + 10 + 45) = 912.$$

Problem 4. Find the smallest positive integer n, such as any square of 2016×2016 grid square is possible to paint in one of the given n colours, such that any  figure (consisting of five squares) does not have squares painted in the same colour.

Solution. *Note that $n \geq 5$, as the squares of considered figures are painted in different colours. Let us paint the squares of 2016×2016 grid square in one of the colours 1, 2, 3, 4, 5, such that the assumptions of the problem hold true (see the figure below).*

1	2	3	4	5	1	2	3		1
4	5	1	2	3	4	5	1		4
2	3	4	5	1	2	3	4		2
5	1	2	3	4	5	1	2		5
3	4	5	1	2	3	4	5		3
1	2	3	4	5	1	2	3		1
1	2	3	4	5	1	2	3		1

Therefore, the smallest possible value of n is equal to 5.

Problem 5. Seven boys and some girls take part in an event. Given that any two boys have danced with distinct number of girls and any two girls have danced with equal number of boys. At least, how many people took part in that event?

Solution. *Denote by m the number of girls. According to the assumptions of the problem, we have that $m + 1 \geq 7$. Therefore $m \geq 6$.*

If $m = 6$, then boys have danced with 0, $1, \ldots, 6$ girls, respectively. Thus, the sum is equal to the number of couples (boy–girl) danced together. On the other hand, this number should be divisible by 6. Note that $6 \nmid 21 = 0 + 1 + \cdots + 6$. We obtain that $m \geq 7$.

Let us provide an example for $m = 7$.

boys \ girls	1	2	3	4	5	6	7
1	+						
2	+	+					
3	+	+	+				
4				+	+	+	+
5			+	+	+	+	+
6		+	+	+	+	+	+
7	+	+	+	+	+	+	+

Here, $+$ means, for example, in the square $(4,7)$, that boy number 4 has danced with girl number 7.

Thus, it follows that at least 14 people took part in that event.

Problem 6. Find the number of words consisting of two letters A, three letters O and seven letters B, such that any of them does not include consequent (neighbour) vowel letters. For example, *BABABBOBOBOB*.

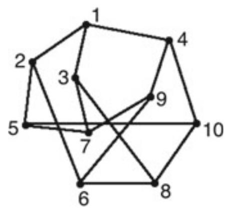
Solution. *The number of words consisting of two letters A and three letters O is equal to* $C_5^2 = 10$. *Now, let us put in between each two neighbour letters one letter B. Note that the other three letters B are possible to put in* $C_6^3 + 2C_6^2 + C_6^1 = 56$ *ways.*

Therefore, the total number of such words is equal to $10 \cdot 56 = 560$.

Problem 7. Let n participants take part in a mathematical conference. Given that any participant is acquainted with at most three other participants. Given also that any two participants are either acquaintances or have a mutual acquaintance. Find the greatest possible value of n.

Solution. *Let A be one of the participants. We have that A can be acquainted with at most three other participants. Moreover, those participants can be acquainted with at most two other participants (beside A). Therefore* $n \leq 1 + 3 + 3 \cdot 2 = 10$.

Now, let us provide an example of ten participants, such that the assumptions of the problem hold true. In this example, instead of participants we consider ten points and the acquaintances are connected by a line segment.



Problem 8. Consider a set consisting of six positive integers. For any non-empty subset (of this set) consider the arithmetic mean of all its elements. At most, how many times can the same number be a term of the sequence consisting of those arithmetic means?

Solution. *Let* a_1, a_2, a_3, a_4, a_5, a_6 *be the considered six positive integers. Consider the equal terms of this sequence with the greatest total number of terms, let the value of each of these terms be equal to a.*

If we denote by $b_i = a_i - a$, $i = 1, \ldots, 6$, *then in the sequence consisting of the arithmetic means of all terms of any subset of the set* $\{b_1, b_2, b_3, b_4, b_5, b_6\}$ *the number of terms equal to 0 is the greatest one. Let k numbers among the numbers* b_1, b_2, b_3, b_4, b_5, b_6 *be non-positive.*

If $k \leq 3$, *then in the considered sequence the number of terms equal to 0 is not more than 9.*

Indeed, when $k = 3$ *and* $b_1 < b_2 < b_3 \leq 0 < b_4 < b_5 < b_6$, *then the arithmetic mean is equal to 0. If we add to numbers* $b_1, b_2, b_3, b_1 + b_2, b_1 + b_3, b_2 + b_3, b_1 + b_2 + b_3$, *perhaps except to one of them, numbers* b_4, b_5, b_6, *if* $b_6 = b_5 + b_4$, *then we*

can obtain at most nine terms equal to 0. For example, for numbers -3, -2, -1, *1, 2, 3, we have the sums of corresponding subsets* -3, -2, -1, -5, -4, -3, -6 *and we obtain 0, if we consider* $-3+3$, $-3+1+2$, $-2+2$, $-1+1$, $-3-2+3+2$, $-3-1+3+1$, $-2-1+1+2$, $-2-1+3$, $-3-2-1+1+2+3$. *Here, as initial numbers, one can consider numbers 1, 2, 3, 5, 6, 7.*

If $k=2$, *then the number of 0 terms is not more than 9, as the same positive sum cannot be obtained more than three times.*

If $k=1$, *let* $0<b_2<b_3<b_4<b_5<b_6$, *then the sequence* b_2+b_3, b_2+b_4, b_2+b_5, b_2+b_6, b_3+b_4, b_3+b_5, b_3+b_6, b_4+b_5, b_4+b_6, b_5+b_6 *cannot have more than three equal terms, therefore 0 can be there not more than eight times.*

If $k\geq 4$, *then by considering numbers* $-b_1$, $-b_2$, $-b_3$, $-b_4$, $-b_5$, $-b_6$ *we deduce this case to the previous case.*

Therefore, the same number can be a term of the sequence consisting of those arithmetic means, at most, nine times.

Problem 9. Consider 64 balls of eight different colours. Given that there are eight balls of each colour. At least, how many balls is possible to put in a row, such that for any two colours (not obliged to be different) there is a ball having neighbour balls of these colours?

Solution. *At first, let us prove that if one places not more than 39 balls in a row, then the assumptions of the problem do not hold true.*

We proceed the proof by contradiction argument. Assume that there are not more than 39 balls in a row, such that the assumptions of the problem hold true. According to Dirichlet's principle, the number of balls of one of the colours is not more than 4. This leads to a contradiction. Let us now provide an example for 40 such balls. Here, we enumerate the colours by numbers 1, 2, ..., 8.

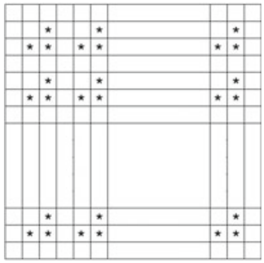
Example: 3, 4, 5, 6, 7, 5, 1, 2, 4, 3, 5, 6, 8, 7, 1, 2, 5, 6, 5, 6, 2, 1, 1, 7, 1, 7, 3, 8, 7, 4, 4, 3, 4, 3, 2, 8, 2, 8, 8, 6.

Therefore, the answer is 40.

Problem 10. Let in some squares of 40×40 grid square be written asterisks (only one asterisk in each square). Given that the total number of asterisks is equal to n. Find the greatest possible value of n, such that for any placement of asterisks there exists 3×3 grid square having less than three asterisks.

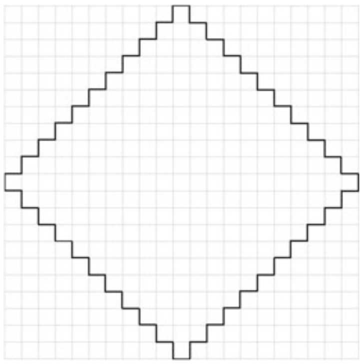
Solution. *Note that one can choose 169 different* 3×3 *grid squares from* 40×40 *grid square, such that none of them has a common square. Therefore, if the number of asterisks is equal to* $169\cdot 3-1=506$, *then according to Dirichlet's principle one of the chosen 169 grid squares has not more than two asterisks.*

Let us provide an example of 507 asterisks, such that any 3×3 *grid square has exactly three asterisks.*

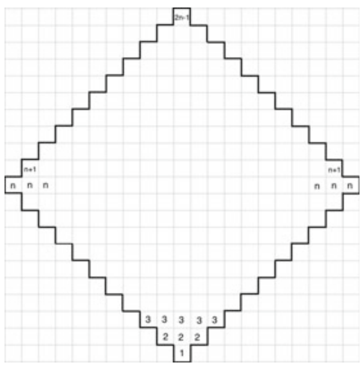


Hence, the greatest possible value of n is equal to 506.

Problem 11. Find the greatest possible number of dominos that is possible to place inside the following figure, such that each domino covers two squares and each square is covered by only one domino.



Solution. *Let us write consequent positive integers, in the squares of the considered figure, in the following way (see the figure below).*



Denote by x_n the greatest number of dominos that can be placed inside of the given figure. Hence, we need to find x_{11}.

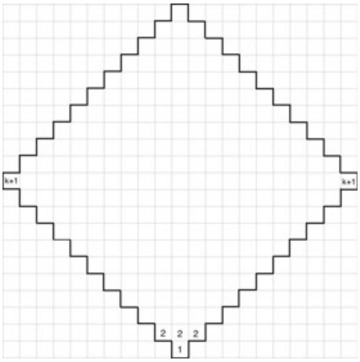
By mathematical induction method, let us prove that

$$x_n = (n-1)^2, n = 1, 2, \dots$$

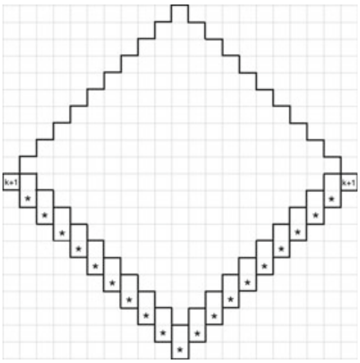
Basis. *We have that* $x_1 = 0 = (1-1)^2$.

Inductive step. *Let the statement holds true for* $n \leq k$, $x_n = (n-1)^2$, *where* $k \in \mathbb{N}$. *Prove that* $x_{k+1} = k^2$.

Let in the figure below x_{k+1} *dominos are placed, such that the sum of the numbers written in the squares covered by all dominos is the smallest possible (denote that sum by* S*).*



Note that in the figure below the squares with $*$ *are covered by dominos as it is illustrated in the figure.*



If the square with the number 1 is not covered by any domino, then according to the definition of x_{k+1} *the square having a common side with this square is covered by some domino. Let us take away that domino and instead of it consider a domino covering the square with the number 1. Hence, we obtain a smaller sum* (S). *This leads to a contradiction.*

We obtain also that the squares with $*$ (where are written numbers 2,3,..., k) can be covered by dominos, as it is illustrated in the figure above.

Thus, it follows that

$$x_{k+1} = x_k + 2k - 1 = (k-1)^2 + 2k - 1 = k^2.$$

This ends the proof of the statement.

Therefore, we deduce that $x_{11} = 100$.

Problem 12. Given that some cities among n cities located on the island are pairwise connected by airways, such that each city is connected at most with three other cities, and if there is no connection between any two cities, then it is possible to go from one city to the another one at least in two different ways, every time passing through only one transit city. Find the greatest possible value of n.

Solution. *Instead of these n cities, let us consider n points on the plane.*

If there is an air connection between two cities, then we connect corresponding points by a line segment. Denote by m the number of figures consisting of two line segments with the same vertex. According to the assumptions of the problem, we have that any point is a vertex of at most three line segments.

Thus, it follows that

$$m \leq 3n. \tag{7.123}$$

On the other hand, any two points not connected by a line segment are the vertices of at least two of those m figures.

Therefore

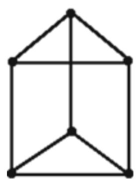
$$m \geq n(n-4). \tag{7.124}$$

From (7.123) *and* (7.124), *we deduce that* $n \geq 7$.

If $n = 7$, *then we obtain that* $m \leq 21$ *and* $m \geq 21$. *Thus, it follows that* $m = 21$. *Hence, any point is a vertex of three line segments. Therefore, the total number of all line segments is equal to* $\frac{7 \cdot 3}{2}$. *This leads to a contradiction.*

We obtain that $n \leq 6$.

Let us provide an example of six points, such that the assumptions of the problem hold true (see the figure below).



Hence, the greatest possible value of n is equal to 6.

7.5.15 Problem Set 15

Problem 1. Find the number of all permutations $x_1, x_2, \ldots, x_8$ of numbers $1,2,\ldots,8$, such that $x_1-x_2+x_3-x_4+x_5-x_6+x_7-x_8=16$.

Solution. *Note that*

$$x_1+x_3+x_5+x_7 \leq 5+6+7+8=26,$$

and

$$x_2+x_4+x_6+x_8 \geq 1+2+3+4=10.$$

Thus, it follows that

$$x_1-x_2+x_3-x_4+x_5-x_6+x_7-x_8 \leq 16.$$

Hence, we obtain that

$$\{x_1,x_3,x_5,x_7\}=\{5,6,7,8\},$$

and

$$\{x_2,x_4,x_6,x_8\}=\{1,2,3,4\}.$$

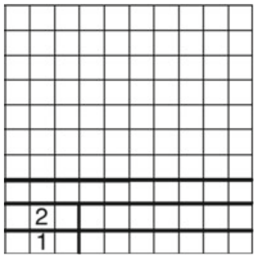
Therefore, the number of all such permutations is equal to $4!\cdot 4!=576$.

Problem 2. In how many ways can one transport three girls and seven boys by three boats, such that in every boat there are one girl and at least two boys?

Solution. *A group of seven boys can be divided into three groups (consisting of at least two persons) in* $C_7^2\cdot C_5^2 : 2! = 105$ *ways. Thus, it follows that the answer of the problem is equal to* $3!\cdot 105=630$.

Problem 3. Let every cell of 10×10 square grid be painted in one of the following colours: red, blue or yellow. Given that the cells of any rectangle that consists of three cells of the given square grid are painted in different colours. In how many ways is possible to paint, as described above, the given square grid?

Solution. *Note that the cells of figure 1 (see the figure below) are possible to paint in six ways.*



Note that after any painting of figure 1, the first row is painted in a unique way and the figure 2 in two ways. On the other hand, after any painting of figure 2, the second row is painted in a unique way. Continuing in this way, we obtain that all rows are painted in a unique way.

Therefore, the given square grid is possible to paint (as described in the problem) in $6 \cdot 2 = 12$ *ways.*

Problem 4. Find the number of words consisting of two letters A, three letters O and five letters B, such that for any of them the number of couples, consisting of two consequent vowels, is equal to 1. For example, *BAABOBBBOBO*.

Solution. *Note that there are four possibilities for writing those two consequent vowels: AA, AO, OA, OO. Let us call those couples "vowels". Therefore, we have* $2+3-1=4$ *vowel letters and five consonant letters.*

Hence, three of those five consonant letters need to be written in between the vowel letters and the number of ways of writing the other two consonant letters is equal to the number of non-negative solutions of the following equation

$$x_1+x_2+x_3+x_4+x_5=2.$$

On the other hand, the number of non-negative solutions of the given equation is equal to C_6^4.

Thus, it follows that the number of such words is equal to

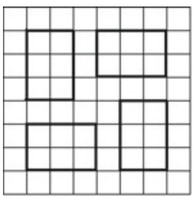
$$4 \cdot C_6^4 + 2 \cdot C_4^2 \cdot 2 \cdot C_6^4 + C_4^2 \cdot 2 \cdot C_6^4 = 600.$$

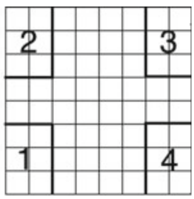
Problem 5. Given 8×8 square grid. Consider among 2×3 rectangular grids only those consisting of the cells of the given square grid. Find the smallest positive integer n, such that the following conditions holds true:

a) There exist n such 2×3 rectangular grids that any 2×3 rectangular grid has at least with one of them a common cell.

b) For any $n-1$ rectangular grids size of 2×3, there exists at least one other 2×3 rectangular grid having no common cell with any of these $n-1$ rectangular grids.

Solution. *From the example provided below, it follows that* $n \geq 4$.





Now, let us prove that $n=4$. *This means that if we choose three* 2×3 *rectangular grids, then we can choose another* 2×3 *rectangular grid that does not have any*

common cells with these three rectangular grids. The proof is straightforward, as there is no 2×3 rectangular grid that intersects simultaneously two among the rectangles 1, 2, 3, 4.

Problem 6. Find the number of all permutations $x_1, x_2, \ldots, x_6$ of numbers 1, $2, \ldots, 6$, such that $3 \nmid x_{i+1} - x_i$, where $i = 1, 2, \ldots, 5$.

Solution. *Denote by a_n, $(n \geq 2)$ the number of all permutations $x_1, x_2, \ldots, x_6$ of numbers $1, 2, \ldots, 6$, such that $3 \nmid x_{i+1} - x_i$, where $i = 1, 2, \ldots, n-1$.*

We have that $a_2 = 2$, $a_3 = 6$, $a_4 = 4! - 2 \cdot 3! = 12$, $a_5 = 5! - 2 \cdot 2 \cdot a_3 - 2 \cdot 2 \cdot a_4 = 48$, $a_6 = 6! - 2 \cdot 2 \cdot 2 \cdot a_3 - 2 \cdot 2 \cdot 3 \cdot a_4 - 2 \cdot 3 \cdot a_5 = 240$.

Hence, we obtain that the number of all such permutations is equal to 240.

Problem 7. Let on every square of 4×4 chessboard be placed a chess knight, such that any black knight attacks only one white knight and any write knight attacks only one black knight. Find the number of all such placements.

Solution. *Note that in squares 2 and 3 are placed knights of different colours.*

8			4
	2	7	
	6	3	
1			5

Therefore, in the squares 1 and 4 are placed knights of different colours. Thus, it follows that in the squares 1, 2, 3, 4, 5, 6, 7, 8 knights can be placed in $2 \cdot 2 \cdot 2 \cdot 2 = 16$ ways and after this the placement is continuous in a unique way.

Hence, the number of all such placements is equal to 16.

Problem 8. At least, at how many points can intersect the diagonals of a convex heptagon? The common endpoint of two or more diagonals is not considered as an intersection point.

Solution. *Let $A_1A_2A_3A_4A_5A_6A_7$ be a given convex heptagon. Consider the convex hexagon $A_2A_3A_4A_5A_6A_7$. We call hexagons $A_1A_3A_4A_5A_6A_7$ and $A_1A_2A_3A_4A_5A_6$ the neighbour hexagons of the hexagon $A_2A_3A_4A_5A_6A_7$. In a similar way, we define the neighbour hexagons of any hexagon with vertices A_i. Note that from two neighbour hexagons only in one hexagon can three diagonals intersect at the same point. Thus, it follows that among seven hexagons at most in three hexagons can three diagonals intersect at the same point. Therefore, the number of the intersection points of a convex heptagon is not less than $C_7^4 - 3 \cdot 2$. We have that $C_7^4 - 3 \cdot 2 = 29$. Below we provide an example of a convex heptagon, such that its diagonals intersect at 29 points.*

Hence, the answer is 29.

Problem 9. Let the entries of 100×100 square grid be integer numbers, such that in each cell is written only one integer. Given that if integers a, b are written in any two cells having a common side, then $|a-b| \leq 1$. At least, how many times can the number, written maximum number of times in the given square grid, be written?

Solution. *Let us enumerate the columns of given* 100×100 *square grid, from left to right, by numbers 1, 2, …, 100. Let m_i be the smallest number of i^{th} column and M_i be the greatest number of i^{th} column.*

Consider the following three cases.

a) If there exists a number i, such that $m_i = M_i$. Thus, it follows that in all 100 cells of i^{th} column are written numbers m_i.

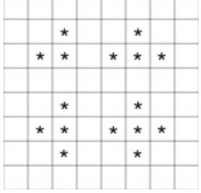
b) If $m_i < M_i$, $i = 1,2,\ldots,100$. Let any two among the line segments $[m_1,M_1]$, …, $[m_{100},M_{100}]$ intersect. Then, the number $\max(m_1,\ldots,m_{100})$ *is written in each column. Therefore, it is written at least 100 times.*

c) There are two line segments among line segments $[m_1,M_1]$, …, $[m_{100},M_{100}]$ that do not intersect. Let $m_1 < M_1 < m_{100} < M_{100}$ and m be such positive integer that $M_1 \leq m \leq m_{100}$. Therefore, m is written in each row. Hence, it is written at least 100 times.

If one writes numbers 1 in the cells of the first column, in the cells of the second column numbers 2 and so on, in the cells of 100th column numbers 100. In this case, we obtain that each number is written 100 times. Hence, the answer is 100.

Problem 10. Let in some cells of 8×8 square grid be placed asterisks, only one asterisk per cell. Given that the total number of asterisks is equal to n. Find the greatest possible value of n, such that for any placement of asterisks there exists 3×3 square grid that has at most two asterisks.

Solution. *Let us provide an example for 16 asterisks, such that any* 3×3 *square grid has at least three asterisks.*

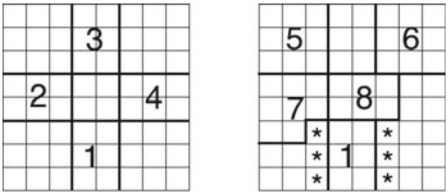


Therefore, the greatest possible value of n is less than 16. Let us prove that it is equal to 15.

It is sufficient to prove that if the given n asterisks are such that 3×3 *square grid has at least three asterisks, then* $n \geq 16$.

Consider the following two cases.

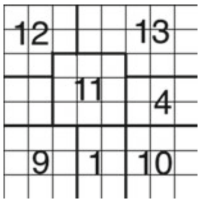
a) If at least in one of the rectangles 1, 2, 3, 4 (see the figure below) there is no asterisks placed. For example, in 1.



Therefore, in any of the figures 5 and 6 there are at least three asterisks. On the other hand, in any of the figures 7 and 8 there are at least two asterisks. Thus, it follows that

$$n \geq 4 \cdot 3 + 2 \cdot 2 = 16.$$

b) If in any of the figures 1, 2, 3, 4 there is at least one asterisks. Hence in any of the figures 9, 10, 11, 13 (see the figure below) there are at least three asterisks and in the figure 12 at least two asterisks.



We deduce that

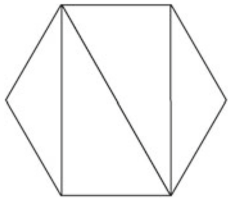
$$n \geq 4 \cdot 3 + 2 + 1 + 1 = 16.$$

Thus, in both cases we have obtained that $n \geq 16$.

This ends the solution.

Problem 11. A triangle is called "beautiful", if its all angles are less than or equal to $120°$. Given that a regular hexagon is divided into triangles. At least, how many of these triangles can be "beautiful"?

Solution. *In the figure below, a regular hexagon is divided into six "beautiful" triangles.*



Now, let us prove that if a regular hexagon is divided into triangles, then at least four of them are "beautiful" triangles. We split the set of the vertices of the triangles into three sets A, B, C. Let A be the set of vertices of the regular hexagon, B be the set of vertices, such that any among them is either an inner point of a side of the regular hexagon or inner point of the side of another triangle and C be the set of the other vertices.

Let $|B| = k$, $|C| = l$, *then the sum of all the angles of all the triangles is equal to* $720° + 180°k + 360°l$. *Thus, it follows that the number of all triangles is equal to* $4 + k + 2l$.

On the other hand, the number of not "beautiful" triangles is not more than $k + 2l$. *Therefore, the number of "beautiful" triangles is not less than 4.*

This ends the proof of the statement.

Problem 12. Let the entries of 100×100 square grid be integer numbers, such that in each cell is written only one integer. Given that if integers a, b are written in any two cells having a common side, then $|a - b| \leq 1$. Let integers c, d be written maximum number of times in the given square grid. Moreover, c is written n times and d is written k times. Find the smallest possible value of $n + k$.

Solution. *In the example provided below, we have that* $n + k = 199$.

If we prove that $n + k \geq 199$, *then it follows that the smallest possible value of* $n + k$ *is equal to 199.*

Let us enumerate the columns of given 100×100 *square grid, from left to right, by numbers 1, 2, ..., 100. Let* m_i *be the smallest number of* i^{th} *column and* M_i *be the greatest number of* i^{th} *column.*

Consider the following three cases.

a) If there exists a number i, such that $m_i = M_i$. *Thus, it follows that in all 100 cells of* i^{th} *column are written numbers* m_i. *In a similar way, we can assume that there exists a row such that in its all cells are written numbers* m_i. *Therefore,* $n \geq 199$. *Hence, we obtain that* $n+k > 199$.

b) If $m_i < M_i$, $i = 1,2,\ldots,100$. *Let any two among the line segments* $[m_i, M_i]$, $i = 1,2,\ldots,100$ *intersect.*

If $m = \max(m_1,\ldots,m_{100}) < \min(M_1,\ldots,M_{100}) = M$, *then the numbers m and* M *are written in each column. Therefore* $n+k \geq 100+100 = 200$.

If $m_i = m = \max(m_1,\ldots,m_{100}) = \min(M_1,\ldots,M_{100}) = M$, *then* $n \geq 100+S-1$, *where S is the number of the cells of* i^{th} *column where is written number m. Hence, the number* $m+1$ *is written at least in* $100-S$ *rows. Thus, it follows that* $k \geq 100-S$. *We deduce that* $n+k \geq 199$.

c) There are two line segments among line segments $[m_1, M_1]$, ..., $[m_{100}, M_{100}]$ *that do not intersect. Let* $m_1 < M_1 < m_{100} < M_{100}$, *then any of the numbers* M_1 *and* m_{100} *is written in all rows. Therefore* $n+k \geq 100+100 = 200$. *This ends the proof.*
Hence $n+k \geq 199$.

Chapter 8
Answers

8.1 Problem Set 1

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	4	53	0	1	64
Problem 2	16	2	19	1	630
Problem 3	1	2	6	7	720
Problem 4	1	1	512	0	15
Problem 5	10	1	5	0	4
Problem 6	8	8	1	7	64
Problem 7	54	11	729	1	999
Problem 8	1	721	1	8	6
Problem 9	6	673	3	1	12

8.2 Problem Set 2

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	29	6	44	15	651
Problem 2	2	96	1	50	210
Problem 3	578	1	5	1	10
Problem 4	8	3	999	3	1
Problem 5	5	1	4	10	98
Problem 6	1	70	1	1	351
Problem 7	512	10	4	710	1
Problem 8	9	0	21	2	128
Problem 9	10	6	8	1	768

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8.3 Problem Set 3

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	10	40	0	0	72
Problem 2	6	120	4	3	204
Problem 3	1	7	75	1	551
Problem 4	26	1	10	20	604
Problem 5	1	0	8	3	44
Problem 6	1	1	8	21	8
Problem 7	37	15	3	51	11
Problem 8	20	15	2	0	88
Problem 9	5	614	15	765	10

8.4 Problem Set 4

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	24	62	1	729	64
Problem 2	90	720	0	1	363
Problem 3	18	0	5	44	252
Problem 4	14	999	6	0	4
Problem 5	25	1	80	18	14
Problem 6	20	210	17	8	503
Problem 7	60	1	1	2	924
Problem 8	24	1	8	300	22
Problem 9	100	999	4	0	12

8.5 Problem Set 5

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	30	32	3	0	139
Problem 2	3	50	2	1	336
Problem 3	8	968	8	2	240
Problem 4	10	941	56	36	7
Problem 5	90	12	0	25	90
Problem 6	11	11	98	3	686
Problem 7	729	105	3	8	255
Problem 8	4	1	4	0	4
Problem 9	2	500	4	1	21

8.6 Problem Set 6

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	15	972	56	999	240
Problem 2	41	72	45	1	432
Problem 3	1	60	1	101	176
Problem 4	841	18	15	2	256
Problem 5	21	17	50	0	570
Problem 6	756	13	108	6	16
Problem 7	35	37	540	2	1
Problem 8	9	1	18	1	882
Problem 9	135	64	117	2	10

8.7 Problem Set 7

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	20	289	105	4	672
Problem 2	8	1	6	2	5
Problem 3	46	2	4	16	294
Problem 4	30	2	560	1	2
Problem 5	4	2	1	1	15
Problem 6	8	499	4	820	9
Problem 7	448	225	13	32	816
Problem 8	16	1	4	12	11
Problem 9	120	22	2	4	300

8.8 Problem Set 8

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	58	5	1	10	40
Problem 2	44	1	3	1	6
Problem 3	5	29	4	50	196
Problem 4	48	95	81	18	672
Problem 5	10	199	26	24	754
Problem 6	10	10	68	32	571
Problem 7	6	964	3	301	12
Problem 8	104	32	5	1	780
Problem 9	1	3	5	1	9

8.9 Problem Set 9

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	25	60	101	1	4
Problem 2	35	2	28	2	512
Problem 3	10	4	3	55	144
Problem 4	13	1	2	2	4
Problem 5	1	90	833	1	7
Problem 6	1	45	1	12	357
Problem 7	90	1	1	6	611
Problem 8	1	1	100	2	147
Problem 9	30	20	1	73	39

8.10 Problem Set 10

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	9	649	5	1	144
Problem 2	400	0	5	0	961
Problem 3	56	40	2	0	40
Problem 4	5	59	8	98	171
Problem 5	1	1	992	1	992
Problem 6	50	12	18	1	216
Problem 7	1	3	272	18	7
Problem 8	390	208	6	5	11
Problem 9	25	4	12	9	700

8.11 Problem Set 11

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	30	1	6	13	128
Problem 2	8	3	3	10	357
Problem 3	100	101	6	16	198
Problem 4	200	203	5	10	1
Problem 5	16	2	9	16	273
Problem 6	8	77	1	1	49
Problem 7	100	2	4	5	751
Problem 8	0	946	1	820	12
Problem 9	1	60	192	2	995

8.12 Problem Set 12

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	15	9	1	2	14
Problem 2	0	4	7	2	264
Problem 3	150	816	5	0	384
Problem 4	336	36	2	1	144
Problem 5	2	0	26	0	1
Problem 6	149	1	3	1	2
Problem 7	2	1	4	12	162
Problem 8	1	450	1	26	3
Problem 9	1	0	2	1	4
Problem 10	12	3	8	1	432
Problem 11	75	3	816	1	8
Problem 12	1	44	400	1	3

8.13 Problem Set 13

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	4	25	3	21	32
Problem 2	7	20	6	192	243
Problem 3	25	7	12	0	3
Problem 4	7	503	17	50	3
Problem 5	1	576	900	4	33
Problem 6	10	1	0	305	203
Problem 7	15	0	12	4	687
Problem 8	16	2	952	25	513
Problem 9	3	325	6	20	8
Problem 10	180	82	999	30	48
Problem 11	32	8	63	150	32
Problem 12	2	480	17	142	38

8.14 Problem Set 14

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	6	8	126	820	60
Problem 2	120	899	5	27	36
Problem 3	13	9	5	4	912
Problem 4	11	665	209	397	5
Problem 5	10	90	9	3	14
Problem 6	0	162	2	3	560
Problem 7	61	3	4	5	10
Problem 8	48	4	145	960	9
Problem 9	100	162	100	625	40
Problem 10	3	23	2	672	506
Problem 11	12	25	63	250	100
Problem 12	21	1	11	3	6

8.15 Problem Set 15

	Geometry and trigonometry	Number theory	Algebra	Calculus	Combinatorics
Problem 1	25	14	3	819	576
Problem 2	18	7	5	6	630
Problem 3	15	55	3	1	12
Problem 4	48	1	122	12	600
Problem 5	40	0	8	7	4
Problem 6	150	72	145	2	240
Problem 7	27	45	3	3	16
Problem 8	50	51	33	50	29
Problem 9	0	8	12	5	100
Problem 10	96	6	25	14	15
Problem 11	25	50	3	10	4
Problem 12	80	950	401	37	199

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